PROCEEDINGS

OF

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1907—JUNE, 1908.

Thursday, November 14th, 1907.

ANNUAL GENERAL MEETING.

Prof. W. BURNSIDE, President, in the Chair.

Present twenty members and two visitors.

Messrs. B. M. Walker, W. J. Harrison, B. B. Ghosál were elected members.

Messrs. T. J. I'A. Bromwich and E. Cunningham were admitted into the Society.

The Treasurer, Prof. J. Larmor, presented his Report. On the motion of Lieut.-Col. Cunningham, seconded by Rev. F. H. Jackson, the report was received.

Dr. J. G. Leathem was appointed Auditor.

Mr. Grace, as Secretary, reported that the number of members at the beginning of the previous Session was 278. During the Session 13 new members had been elected, 6 members had resigned, 2 had died. The number of members at the beginning of the present Session was 283. The American Academy of Arts and Sciences (Boston) had been added to the list of Societies with which the Society exchanges publications.

The Council and Officers for the ensuing Session were elected as follows:—President, Prof. W. Burnside; Vice-Presidents, Prof. A. R. Forsyth and Prof. H. M. Macdonald; Treasurer, Prof. J. Larmor; Secretaries, Prof. A. E. H. Love and Mr. J. H. Grace; other members of the Council, Dr. H. F. Baker, Mr. A. Berry, Mr. T. J. I'A. Bromwich, Mr. A. L. Dixon, Prof. E. B. Elliott, Mr. G. H. Hardy, Dr. E. W. Hobson, Sir W. D. Niven, Mr. H. W. Richmond, Mr. A. E. Western.

The following papers were communicated :—

*The Invariants of a Binary Quintic and the Reality of its Roots: Dr. H. F. Baker.

*Addendum to a Paper on the Inversion of a Repeated Infinite Integral: Mr. T. J. I'A. Bromwich.

†On a Transformation of a certain Hypergeometric Series: Prof. M. J. M. Hill.

*Generalisation of a Theorem in the Theory of Divergent Series: Mr. G. H. Hardy.

*On Hyper-Complex Numbers: Mr. J. H. Maclagan Wedderburn.

*Uniform and non-Uniform Convergence and Divergence of a Series and the Distinction between Right and Left: Dr. W. H. Young.

*Application of Quaternions to the Problem of the Infinitesimal Deformation of a Surface: Mr. J. E. Campbell.

*Nodal Cubics through Eight Given Points: Mr. J. E. Wright.

†On a Transformation of Hypergeometric Series: Rev. Dr. E. W. Barnes.

*A General Theorem on Integral Functions of Finite Order: Mr. J. E. Littlewood.

Thursday, December 12*th*, 1907.

Prof. W. BURNSIDE, President, in the Chair.

Present sixteen members and three visitors.

Messrs. G. N. Watson, D. G. Taylor, F. B. Pidduck, W. E. Dalby were elected members.

Messrs. S. T. Shovelton and G. N. Watson were admitted into the Society.

Prof. Love presented the Report of the Auditor (Dr. J. G. Leathem).

On the motion of Mr. C. S. Jackson, seconded by Mr. Hilton, the Treasurer's Report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

The following papers were communicated :—

A Formula in Finite Differences and its Application to Mechanical Quadrature: Mr. S. T. Shovelton.

*Weierstrass' E-Function in the Calculus of Variations: Prof. A. E. H. Love.

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* Printed in this volume. † See the Note in *Proceedings*, Ser. 2, Vol. 5, p. xxi.

Thursday, January 9th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present fourteen members and two visitors.

Mr. T. J. Garstang was elected a member.

The following resolution was moved by Prof. Love, seconded by Sir W. D. Niven, and carried unanimously :—

That this meeting of the London Mathematical Society records its sense of the profound loss sustained by science in general, and by the London Mathematical Society in particular, through the death of Lord Kelvin, who was a member of the Society during forty years, in two of which he honoured the Society by acting as its President, and whose achievements, springing as they did from the science to which the Society is devoted, rank among the main scientific advances of modern times. That the Secretaries be requested to convey to Lady Kelvin the expression of the respectful and sympathetic condolence of the members.

In moving the resolution Prof. Love said—

Since our last meeting there has died perhaps the greatest of our past Presidents, Lord Kelvin. He became a member of our Society in 1867, within less than two years of its foundation. I find a record of his having attended one of our meetings in 1875, when he made two communications. At that time Henry Smith was President, Maxwell and George Darwin and Cayley were frequent attendants at our meetings. In 1898, when he was at the height of his fame, he became our President. Among the meetings at which he presided was a memorable one in June of 1899, at which Mittag Leffler was present, and Kelvin then made a communication to the Society; at another memorable meeting at which he presided, that of June, 1900, Darboux, Klein, and Poincaré were present and made communications. Lord Kelvin also attended the meeting in November of 1900, at which he inducted Dr. Hobson into the chair and delivered a valedictory address on "The Transmission of Force through a Solid."

It is fitting that on this occasion we should endeavour to appreciate Kelvin's greatness. Whether we have regard to the practical utility of his inventions, to the subtlety of his physical speculations, to the wide range of his contributions to natural knowledge, or to the power of his mathematical methods, we must hold him great, and we admire him not less for the generosity of his character. To his readiness to assist a beginner in mathematical research I can personally testify. To his eagerness to attribute to others their full share of merit for any discovery witness is borne by his published correspondence with Stokes and by the prominence which he gave to citations of the work of Green as soon as he became acquainted with it.

Of his contributions to mathematics it is perhaps fair to say that they consisted rather in the development of new methods, leading to new results, than in the creation of new mathematical theories. He was an acute geometer, but geometry for its own sake did not satisfy him. The method of inversion became in his hands a means of investigating electrical distributions; the partitioning of space into equal and similar portions was a step in the theory of the molecular arrangement of crystals. He was a profound analyst, but his analysis always had an immediate application. In connexion with researches on the constitution of the earth he incidentally revised the whole theory of those functions which previously had been known as Laplace's functions and from that time forward became spherical harmonics. It may be that his most important contributions to mathematical analysis are contained in the short papers in which he gave examples of that method for the solution of partial differential equations which, unknown to him, had been initiated by Green. He showed how to build up the requisite solutions synthetically from simple special ones which involve the existence of singular points. The singular point in electrostatics is a point charge, in magnetism it is a magnetic particle. In the theory of the conduction of

heat it is an instantaneous point source, and the memoir in which he developed the application of the corresponding special solution to obtain all the valuable solutions of the equation of diffusion is well known. In the theory of elasticity the corresponding singularity is a point at which a force is applied. The special solution for this singularity was given by him in a very early paper, and he pointed out its possible applications, though he did not work them out in detail. The corresponding solution for vibratory motion produced in an elastic solid by force applied at a point was given some years later by Stokes, and the paper which Kelvin read to our Society in June, 1899, was occupied with a development of this solution. The subject was the advance of waves into a previously undisturbed elastic medium. In some of the papers which he published within the last year or two he was occupied with the very intractable question of the advance of surface waves into previously still water, and he there emphasized again the importance of a special solution of the equations.

In Physics, on the other hand, he created new theories. The theory of energy, and, in particular, Thermodynamics, is in great part his work. His introduction of the absolute thermodynamic scale of temperature is perhaps the greatest single step in the direction of precision that has ever been taken in that theory. The broad view that he took of natural philosophy led him to extend its traditional boundaries. Geology had been developed very much in independence of Physics; but the results which he obtained by mathematical investigations concerning the diffusion of heat, and concerning the deformation of a planet by the attraction of a satellite, led him to attack the conclusions of geologists as regards the age and constitution of the earth. They have been forced by him to revise their estimates of the duration of geological time, and to accept the view that the earth is a solid body, not a fluid body with a thin solid crust.

It is perhaps too soon to attempt to estimate Kelvin's achievements, to place him as it were in a list in order of merit with Newton at the head; but it is not too soon to attempt to understand the special character of his genius, the qualities that distinguish him from others whom also we honour. I think perhaps we may take this special character to consist in a certain very rare, possibly unique, combination of qualities. He combined the ingenuity and enthusiasm of the inventor, the exact knowledge of the practised experimenter, the trained intellect of the mathematician, with a wonderful gift of imagination. Perhaps nothing short of this combination of qualities would have made submarine telegraphy possible. He made it not only possible, but actual. It is, indeed, difficult to overestimate the value of his inventions, either from the point of view of practical life—his compass is in all iron ships—or from the point of view of the advancement of knowledge—his measuring instruments are used in all our laboratories. His enthusiasm is well shown in the eagerness with which he adopted and developed Stokes's oral explanation of the dark lines in the solar spectrum. He saw at once that it must be right, and taught it in his University lectures for years before it had become part of generally received physical theory. His quality of imagination is nowhere better shown than in his early writings on electrostatics. In this theory he owed something to Poisson, who exerted a profound influence on the English school of mathematical physics; but where Poisson saw an effective formula Kelvin discerned an electric image. We all know the brilliant mathematical investigations to which he was led by this simple intuition; what perhaps is not so well understood is that the concrete interpretation of the details of abstract formulæ, of which type of interpretation this was one of the first examples, has become the ideal of mathematical physics. According to the standard that Kelvin set up it is not sufficient to obtain an analytical result and to reduce it to numerical computation; every step in the process must be associated with some intuition, the whole argument must be capable of being conducted in concrete physical terms. Nothing illustrates this better than the interpretation in terms of circulation and vortex strength of the transformation of line integrals into surface integrals. But the most striking example of Kelvin's simultaneously concrete and imaginative mode of working is to be found in his theory of vortex motion and vortex atoms. Where Helmholtz had found interesting types of motions of air and water, depending upon his new integrals of the equations of hydrodynamics, Kelvin detected a possible interpretation of all nature, consisting in the permanence of vortices. His theory of vortex atoms became the type to which, as we now believe, a dynamical theory of ultimate physical reality must conform, inasmuch as it set forth in a realised example the

doctrine that ether and atoms are one and the same stuff, the difference between matter and non-matter being kinematic. In any region into which he could not carry this concrete and imaginative mode of thinking he worked with less confidence. This was perhaps the reason of the hesitation which he showod in regard to the electromagnetic theory of light and in regard to some recent theories of the constitution of matter. I think he felt that such a theory as the electromagnetic theory of light was incomplete; it could only mean that light and electric oscillations are manifestations of similar activities in the same medium, the nature and properties of which are not disclosed by identifying the two manifestations. The mechanical properties of the medium, the dynamical interactions of ether and matter were for him the problem. Indeed this is well seen in his Baltimore lectures, and in the paper which he read to the British Association last year. He has focussed attention upon that which is in the end the fundamental problem of theoretical physics.

Truly we may be proud that such a man has been numbered among our Presidents.

The following paper was communicated :—

A Formula of Interpolation : Mr. C. S. Jackson.

Informal communications were made as follows :—

Hilbert's Invariant Integral in the Calculus of Variations : Mr. T. J. I'A. Bromwich.

An Operator related to q-Series : Rev. F. H. Jackson.

Thursday, February 13th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present twenty-two members.

Mr. W. E. Dalby was admitted into the Society.

The following papers were communicated :—

Proof that every Algebraic Equation has a Root : Dr. H. A. de S. Pittard.

Note on q-Differences : Rev. F. H. Jackson.

*An Extension of Eisenstein's Law of Reciprocity (Second Paper) : Mr. A. E. Western.

*On the Uniform Approach of a Continuous Function to its Limit : Dr. W. H. Young.

Conformal Representation and the Transformation of Laplace's Equation : Mr. E. Cunningham.

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* Printed in this volume.

Thursday, March 12th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present seventeen members.

Messrs. P. E. Marrack and D. K. Picken were elected members.

The following papers were communicated:—

*On the Projective Geometry of some Covariants of a Binary Quintic: Prof. E. B. Elliott.

The Operational Expression of Taylor's Theorem: Dr. W. F. Sheppard.

*On a Formula for the Sum of a Finite Number of Terms of the Hypergeometric Series when the Fourth Element is Unity (Second Paper): Prof. M. J. M. Hill.

*On the Inequalities connecting the Double and Repeated Upper and Lower Integrals of a Function of Two Variables: Dr. W. H. Young.

*Note on a Soluble Dynamical Problem: Prof. L. J. Rogers.

Thursday, April 30th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present sixteen members and a visitor.

Mr. T. J. Garstang was admitted into the Society.

The following papers were communicated:—

*On a General Convergence Theorem and the Theory of the Representation of a Function by a Series of Normal Functions: Dr. E. W. Hobson.

†On the Ordering of the Terms of Polars and Transvectants: Mr. L. Isserlis.

*Oscillating Successions of Continuous Functions: Dr. W. H. Young.

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* Printed in this volume. † Printed, in abstract, in this volume.

*The Relation between the Convergence of Series and Integrals: Mr. T. J. I'A. Bromwich.

*On the Multiplication of Conditionally Convergent Series: Mr. G. H. Hardy.

Porisms: Mr. H. Bateman.

*The Influence of Viscosity on the Oscillations of Superposed Fluids: Mr. W. J. Harrison.

Informal communications were made as follows:—

†(i.) On Mersenne's Numbers, (ii.) On Quartans with numerous Quartan Factors: Lt.-Col. Allan Cunningham.

Thursday, May 14th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present sixteen members.

The following papers were communicated:—

On the Invariants of the General Linear Homographic Transformation in Two Variables: Major P. A. MacMahon.

On the Order of the Group of Isomorphisms of an Abelian Group: Mr. H. Hilton.

On the Calculation of the Normal Modes and Frequencies of Vibrating Systems (Preliminary Note): Prof. A. E. H. Love.

A Question in Probability: Prof. J. E. A. Steggall.

Thursday, June 11th, 1908.

Prof. W. BURNSIDE, President, in the Chair.

Present seventeen members.

Mr. F. M. Saxelby was elected a member.

The President announced that the Council had awarded the De Morgan Medal for 1908 to Dr. J. W. L. Glaisher for his researches in Pure Mathematics.

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* Printed in this volume. † See "Notes and Corrections," in this volume.

The following papers were communicated:—

*Relations between the Divisors of the First n Natural Numbers: Dr. J. W. L. Glaisher.

Electrical Resonance: Prof. H. M. Macdonald.

*A Form of the Eliminant of Two Binary Quantics: Mr. A. L. Dixon.

Perpetuant Syzygies of the nth Kind: Mr. H. Piaggio.

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* Printed in this volume.

LIBRARY

Presents.

BETWEEN October, 1907, and November, 1908, the following presents were made to the Library, from the respective Authors or Publishers:—

Bachelier, L.—"Calcul des Probabilités," 1908.
Bromwich, T. J. I'A.—"An Introduction to the Study of Infinite Series," London, 1908.
Fehr, H.—"Le 4e Congrès International des Mathématiciens," Rome, 1908.
Fukuzawa, Sampachi.—"Klassifikation et Unstetigkeiten von Funktionen einer Reelen Veränderlichen," Tokyo, 1907.
Heywood, H. Bryon.—"Thèses Présentées à la Faculté des Sciences," Paris, 1908.
Hilton, H.—"An Introduction to the Theory of Groups of Finite Order," Oxford, 1908.
Huygens, Christiaan.—"Œuvres Complètes," Tome 11; La Haye, 1908.
"India, Great Trigonometrical Survey of," vol. 18; Dehra Dun, 1906.
Norman, J. S.—"The Teaching of Arithmetic to Simple Proportion," London, 1908.
"Radio-Coordinates and Carnot's Theorem," London, 1908.
"Royal Society, Catalogue of Scientific Papers, Subject Index," vol. 1, Pure Mathematics, Cambridge, 1908.
Rye, R. A.—"Student's Guide to the Libraries of London," London, 1908.
Shaw, James Byrnie.—"Synopsis of Linear Associative Algebra," Washington, 1907.
Teixeira, F. Gomes.—"Obras Sobra Matematica," vol. 4; Coimbra, 1908.

Calcutta: Indian Engineering, vol. 42, nos. 16–26, 1907; vol. 43, 1908; vol. 44, nos. 1-16, 1908.
Coimbra: Academia Polyt. de Porto, Ann. Scientificos, vol. 2, no. 4, 1907; vol. 3, nos. 1, 2, 1908.
Hamburg: Math. Gesellschaft Mittheilungen, bd. 4, heft 8, 1908.
London: Educational Times, vol. 60, no. 560, 1907; vol. 61, nos. 561–571, 1908.
London: Educational Times, Math. Questions and Solutions, vols. 12, 13, 1907–8.
London: Mathematical Gazette, vol. 4, nos. 66–74, 1907-8.
London: Nautical Almanac for 1911.
London: Scientific Monthly, vol. 1, nos. 2-4, 1908.
Paris: L'Enseignement Math., ann. 9, no. 6, 1907; ann. 10, nos. 1–5, 1908.
Tokyo: Physico-Math. Society, Proceedings, vol. 4, nos. 7-13, 16-18, 1907–8.
Warsaw: Wiadomości Matem., tom 11, zeszyt 5, 6, 1907.

Exchanges.

Between October, 1907, and November, 1908, the following exchanges were received:—

American Journal of Mathematics, vol. 30, 1908.
American Mathematical Society, Transactions, vol. 8, no. 4, 1907; vol. 9, 1908.
American Mathematical Society, Bulletin, vol. 14, nos. 2–10, 1907; vol. 15, nos. 1, 2, 1908.

American Philosophical Society, Proceedings, vol. 46, nos. 186, 187, 1907 ; vol. 47, nos. 188, 189, 1908.
Amsterdam : Nieuw Archiev, deel 8, stuk 2, 3, 1907–8.
Amsterdam : Revue Semestrielle, tome 16, pts. 1, 2, 1908.
Amsterdam : Wiskundige Opgaven, deel 10, stuk 1, 1907.
Belgique : Académie Royale des Sciences, Bulletin, 1907, nos. 9–12 ; 1908, nos. 1–5.
Berlin : Jahrbuch über die Fortschritte, bd. 36, 1907 ; bd. 37, heft 1, 1908.
Berlin : Journal für die Mathematik, bd. 133, heft 1, 1907.
Berlin : Sitzungsberichte der K. Preuss. Akademie, 1907, nos. 39–53 ; 1908, nos. 1–39.
Boston : American Academy of Arts and Sciences, Proceedings, vol. 43, pts. 7–21, 1908.
Bordeaux : Société des Sciences, Procès-Verbaux, 1906–7.
Cambridge Philosophical Society, Proceedings, vol. 14, pts. 3–5, 1907–8.
Cambridge Philosophical Society, Transactions, vol. 20, pts. 14–16, 1907 ; vol. 21, pts. 1–4, 1908.
Cambridge, Mass. : Annals of Mathematics, vol. 9, nos. 2–4, 1908 ; vol. 10, no. 1, 1908.
Catania : Accademia Gioenia, Bollettino, fasc. 1, 2, 1908.
Dublin : Royal Irish Academy, Proceedings, vol. 26, nos. 1, 2, 1906 ; vol. 27, nos. 1, 2, 1907.
Dublin : Royal Irish Academy, Transactions, vol. 33, pt. 1, 1906.
Edinburgh : Mathematical Society, Proceedings, vol. 24, 1906 ; vol. 25, 1907 ; vol. 26, 1908.
France : Société Mathématique, Bulletin, tome 35, fasc. 3, 4, 1907 ; tome 36, fasc. 1–3, 1908.
Göttingen : Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1907, hefte 4, 5 ; 1908, hefte 1–3.
Göttingen : Königl. Gesell. der Wissenschaften, Nachrichten, Geschaft. Mitth., 1907, heft 2 ; 1908 ; heft 1.
La Haye : Archives Néerlandaises, tome 12, liv. 5, 1907 ; tome 13, 1908.
Leipzig : Beiblätter zu den Annalen der Physik, bd. 31, hefte 21–24, 1907 ; bd. 32, hefte 1–21, 1908.
Leipzig : K. Sächsische Gesell., Math. Klasse, Berichte, 1907, nos. 2–4 ; 1908, nos. 1, 2.
Leipzig : K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 30, nos. 1–3, 1908.
Livorno : Periodico di Matematica, anno 23, fasc. 2–6, 1907 ; anno 24, fasc. 1, 1908.
Livorno : Periodico di Matematica, Supplemento, anno 11, fasc. 1–4, 7–9, 1907–8.
London : Royal Society, Proceedings, Series A, vol. 79, nos. 533, 534, 1907 ; vol. 80, 1908 ; vol. 81, nos. 543–546, 548.
London : Royal Society, Proceedings, Series B, vol. 79, nos. 534, 535, 1907 ; vol. 80, no. 536, 1908.
London : Royal Society, Transactions, Series A, vol. 207, 1908.
London : Physical Society, Proceedings, vol. 20, pt. 4, 1907 ; vol. 21, pts. 1, 2, 1908.
London : Institution of Naval Architects, Transactions, 1906.
London : Institute of Actuaries, Journal, vol. 40, pts. 2–4, 1906 ; vol. 41, pt. 1, 1907.
London : Nat. Physical Laboratory, Collected Researches, vols. 3 and 4, 1908.
London : Nat. Physical Laboratory, Report for 1907.
London : Nature, vol. 77, nos. 1984–2009, 1907 ; vol. 75, nos. 2010–2035, 1908.
Manchester Literary and Philosophical Society, Memoirs, vol. 50, pts. 2, 3, 1906 ; vol. 51, pt. 1, 1907.
Milano : Reale Istituto Lombardo, Rendiconti, vol. 40, fasc. 17–20, 1907 ; vol. 41, fasc. 1–16, 1908.
Milano : Reale Istituto Lombardo, Memorie, vol. 20, fasc. 10, 1908.
Modena : Regia Accademia, Memorie, Ser. 3, vol. 7, 1908.
Napoli : Accademia delle Scienze, Rendiconti, vol. 13, fasc. 8–12, 1907 ; vol. 14, fasc. 1–3, 1908.
Napoli : Accademia delle Scienze, Atti, vol. 13, 1908.
Palermo : Rendiconti del Circolo Matematico, tomo 24, fasc. 3, 1907 ; tomo 25, 1908 ; tomo 26, fasc. 1, 2, 1908.
Paris : Bulletin des Sciences Mathématiques, tome 31, Sept.–Dec., 1907 ; tome 32, Jan.–Aug., 1908.

Paris : Journal de l'Ecole Polyt., cah. 12, 1908.
Roma : Reale Accademia dei Lincei, Rendiconti, vol. 16, sem. 2, fasc. 7–12, 1907 ; vol. 17, sem. 1 and sem. 2, fasc. 1–7, 1908.
Roma : Reale Accademia dei Lincei, Rendiconti delle Sedute Solenni, vol. 2, pp. 351–394.
Stockholm : Acta Mathematica, bd. 31, 1908.
Torino : R. Accademia delle Scienze, Atti, vol. 43, 1908.
Torino : R. Accademia delle Scienze, Osservazione Meteorologiche, 1907.
Toulouse : Faculté des Sciences, Annales, tome 9, fasc. 2–4, 1907.
Venezia : Atti del R. Istituto, tomi 65, 66, and 67, disp. 1–5, 1906–8.
Venezia : Atti del R. Istituto, Osservazione Meteorologiche e Geodinamiche, 1906.
Wien : Monatshefte für Mathematik, jahr 19, 1908.
Zurich : Vierteljahrsschrift, hefte 3–4, 1907.

International Catalogue of Scientific Literature.

In the year April, 1907, to March, 1908 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature :—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXV., 1907.

"Transactions of the Institution of Naval Architects," London, 1907.

"Journal of the Institute of Actuaries," Vol. XLI., Pts. 2–4, and Vol. XLII., Pt. 1 ; London, 1907–8.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. LI., Pts. 2, 3, and Vol. LII., Pt. 1, 1907–8.

"Proceedings of the Royal Society of Edinburgh," Vol. XXVII., Pts. 1–5, and Vol. XXVIII., Pts. 1, 2, 1907–8.

The following were also sent especially for the purposes of the Catalogue :—

"Mathematical Gazette," Nos. 63–70 ; London, 1907–8.

"Educational Times," Nos. 552–563 ; London, 1907–8.

"Journal of the Royal Statistical Society," Vol. LXX., Pts. 1–4 ; London, 1907.

"Proceedings of the Royal Irish Academy," Vol. XXVII., Section A, Nos. 3–9, 1907–8.

"Transactions of the Insurance and Actuarial Society of Glasgow," Ser. 6, Nos. 3–5, 1907.

OBITUARY NOTICES

LORD KELVIN

[For this notice the Council is indebted to Prof. H. Lamb.]

William Thomson, afterwards Lord Kelvin, was born at Belfast on June 26, 1824. His father, a distinguished teacher, migrated in 1832 to Glasgow, where he had been appointed Professor of Mathematics. He himself, with his elder brother James, entered as a student of the University in 1834. In 1841 he proceeded to Cambridge, where he graduated in 1845; he was Second Wrangler and first Smith's Prizeman. He was elected soon afterwards to a Fellowship at Peterhouse, and in the following year to the Chair of Natural Philosophy at Glasgow, which was to become the most famous in the world. He held this till 1899. He continued to be formally connected with the University, first (at his own desire) as "research student" and afterwards, from 1904, as Chancellor. He died at his house at Largs on December 17, 1907.

To hardly anyone has it been granted to work so long, with unfailing enthusiasm, with unabated powers, with universally acknowledged success, and with unquestioned authority. From his undergraduate days to almost the last of his long life he was engaged without ceasing in scientific investigation. In addition to his work as a teacher and explorer, he was associated as adviser with many practical enterprises, notably with oceanic telegraphy. Whether in the classroom or in the laboratory, on yachting cruises or on the "Great Eastern," his mind was constantly engaged in trying to unravel the problems presented by Nature. Nothing was too abstruse, nothing too common-place, for his consideration, if only it could be made to contribute to the great end of making the world intelligible and its energies subservient to our use. True to his ancestry, he delighted in work, and every other capable worker, in whatever field or on however limited a scale, could count on his sympathy and respect. He had the most generous admiration for his great predecessors and contemporaries—such as Faraday, Stokes, Joule, Maxwell, Helmholtz, Rayleigh—in his own lines of research, and

he was, of course, not unconscious of the importance of his own achievements, but he never said anything to contribute to the notion, occasionally met with, that scientific work is intrinsically more meritorious or more to be honoured than any other kind of honest endeavour.

To Thomson, more than to anyone else, is to be ascribed the transformation which physical science has undergone during the last century. It would be difficult to appreciate soberly the magnitude of his services in such things as the establishment of the laws of thermodynamics, theoretical and practical electricity, telegraphy, and scientific engineering. In any case, the task cannot be attempted here.* But it is fitting that in the records of our Society, which is proud to number him among her presidents, something should be said of him as a mathematician. In one sense, indeed, it may be claimed that he was above all things a mathematician. His thoughts on even the most concrete subjects were ever cast in a mathematical form, and some of his most valuable practical contrivances, such as his electrometers and other measuring instruments, and his mariner's compass, owed their success to the resolute application of mathematical principles even in the most minute details of construction. Thus, where mechanicians had been wont to accept a compromise, he insisted on scientific thoroughness—as, for instance, in his simple but effective invention of "geometric slides and clamps." Again, when once the principles of a theory were firmly established, he would accept without hesitation the most remote deductions which mathematical analysis could base upon them. Perhaps nowhere was the courage of his mathematical faith more manifest than in the various bold cosmical speculations which he initiated, and which have been the inspiration of many later investigators.

The mathematical equipment with which he started on his career consisted in part of the traditional geometrical training of British Universities, but was in other respects derived mainly from the great school of French analysts who flourished about the end of the eighteenth, or the beginning of the nineteenth, century. Lagrange, Laplace, Fourier, Cauchy, Poisson were familiar to him, and it is interesting to note, in his very latest researches on water-waves, the masterly power with which he wields the trusted weapons. Of the work of later pure mathematicians he assimilated little. Some of it he viewed, indeed, with dislike, if not with distrust; or perhaps it would be fairer to say that he could not bring himself to use tools which he had not time thoroughly to master.

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* For these and many other matters hardly touched upon in this notice reference may be made to the obituary notice by Prof. Larmor, *Proc. Roy. Soc.*, Vol. LXXXI., A., p. iii.

It has been said that he was before everything a mathematician. We might go further and assert that, in his eyes, *the* science of all was dynamics. Probably no one has ever possessed so complete and intuitive a mastery of it, whether on the practical or the analytical side. And certainly no one since Newton has done so much to simplify and to extend it. In particular, his command of the perplexing phenomena of "gyroscopic" systems was unrivalled. It enabled him to make a notable contribution to physics, in the dynamical illustration of Faraday's magnetic rotation of the plane of polarization of light; and it led him to develop the beautiful theory of cyclic motions in hydrodynamics. In his subtlest speculations in optics and electricity his constant aim was to reduce everything to a mechanical explanation; to his mind no theory was complete and satisfactory until it was resolved into the operations of a dynamical system.

Few occurrences have been more important in the history of mathematical physics than the issue of the first volume of Thomson and Tait's *Natural Philosophy* in 1867. It is not difficult to make out, from internal and other evidence, what were Thomson's own principal contributions. In the first place, we have the elegant symmetrical treatment of spherical harmonics by Cartesian coordinates. In its purely analytical aspect this had, to some extent, been anticipated by Clebsch, but it is owing mainly to Thomson and to his interpretative and illustrative skill that the method owes its present recognition. The sections on attractions and elasticity, and on the figure of the Earth, are also to be attributed to his hand. Perhaps most important of all is the exposition of the theory of dynamical systems in general, starting from the method of Lagrange. To quote Maxwell's brilliant description: "The credit of breaking up the monopoly of the great masters of the spell, and making all their charms familiar to us as household words, belongs in great measure to Thomson and Tait. The two northern wizards were the first who, without compunction or dread, uttered in their mother tongue the true and proper names of those dynamical concepts which the magicians of old were wont to invoke only by the aid of muttered symbols and inarticulate equations. And now the feeblest among us can repeat the words of power and take part in dynamical discussions which but a few years ago we should have left for our betters." Throughout the book, indeed, we meet with a characteristic feature of all his work—viz., the endeavour to give a distinct physical or geometrical meaning to every analytical concept, and, if possible, to invest it with an appropriate name. A mere list of technical terms introduced by him, each embodying some valuable idea, is instructive in this respect. A few examples must suffice:

in general dynamics we have *generalized coordinates, velocities, momenta, impulses, normal modes* of vibration, *gyrostatic domination;* in hydrodynamics, *flow, circulation, vorticity, cyclic motion;* in electricity and magnetism, *solenoidal, lamellar, centrobaric, permeability, idiostatic, heterostatic,* and the fruitful notion of *images;* in analysis, *spherical harmonics,* with their varieties *zonal, tesseral, sectorial.* The phrase *dissipation of energy,* again, is a monument of his work in thermodynamics.

The first volume of the *Natural Philosophy* was revised and extended in various ways in a second edition, but the work was never continued. Thomson's scientific interests were so numerous and so keen, and his attention was thereby so constantly diverted into new channels, that many of his literary projects, conceived on a large scale, were destined never to be completed, at all events on the original lines. Among such impressive though unfinished structures we may mention an early memoir on magnetism, begun quite in the classical style, a remarkable paper in which the theory of elasticity is discussed with great generality, and the well known memoir on vortex-motion. As time went on, publication tended more and more to take the form of brief and somewhat fragmentary notes, which often hardly did justice to the important ideas which they contained.

The mention of vortex-motion calls up the brilliant theory of vortex atoms, which has had a powerful and suggestive influence on physical speculation, although its author saw reason afterwards to abandon this particular theory of the constitution of matter. The whole subject of hydrodynamics had a life-long fascination for him. The general mathematical analogies were freely employed in illustration of the relations of electric and magnetic fields; and the theory of water-waves, where he was on common ground with Helmholtz and Rayleigh, interested him keenly to the end. It may be permissible now to quote from a fragment on this subject saved "from his waste-paper basket" in 1904. Speaking of a particular type of waves, and referring to the *ποντίων τε κυμάτων ἀνήριθμον γέλασμα* of Aeschylus, he says: "If sea waves were like these, the eye of the Greek poet, with all its perceptivity for beauty in Nature, would never have seen anything suggesting *countless smilings* to his imagination. We may try, perhaps in vain, to find other cases of unfurrowed waves in water left to itself after an initiating disturbance. We want this very much for waves circling out from a stone thrown into water. I doubt if we can find it. But whether we find it or not, the sea will go on for ever, gaily laughing at the mathematicians."

The extent to which his published writings have inspired the work of

others may perhaps some day be estimated. It will be more difficult to appreciate the influence which he personally, but unconsciously, exercised by the simple directness and sincerity of his character, by his ready accessibility, by his generous and kindly encouragement.

There was hardly any distinction accessible to scientific men, in any part of the world, which was not gladly conferred on him. He was knighted in 1866, and was raised to the peerage, by the style of Baron Kelvin, in 1892. He was laid to rest on December 23, 1907, in Westminster Abbey, near the grave of Newton.

CHARLES TAYLOR

[For this notice the Council is indebted to Prof. A. E. H. Love.]

CHARLES TAYLOR was born in London in 1840. In 1858 he proceeded from King's College School to St. John's College, Cambridge, where he graduated as ninth Wrangler in the Mathematical Tripos of 1862, and gained also, in this and the following years, many academic distinctions for classics and theology. In 1881 he was elected Master of the College, and he held this office till his death in August, 1908.

He was chiefly eminent as a theologian and a master of Rabbinic learning. Of his theological writings and contributions to the study of Hebrew literature, as well as of his generous benefactions to his College and University, and of the services which he rendered to both by his capacity for business, record may be found elsewhere. To mathematicians he is best known as the author of two books: a text-book entitled *The Elementary Geometry of Conics*, first published in 1872, and a larger treatise entitled *An Introduction to the Ancient and Modern Geometry of Conics*, published in 1881. These were preceded in 1863 by a book entitled *Geometrical Conics, including Anharmonic Ratio and Projection.* The later text-book has passed through several editions, each one marked by some improvements and additions, and yet, in spite of the additions, the book has always remained small, for its author had the art of compressing much work into a small compass. The larger treatise contains, under the heading "Prolegomena," a brief but masterly sketch of the early history of geometry and of the development of the geometry of conics from the time of Euclid onwards. In this history Taylor emphasized the importance of the principle of geometrical continuity, usually associated with the name of Poncelet, and he traced this principle back to Kepler. He returned to the subject later in a memoir, "The Geometry of Kepler and Newton," which he contributed to the volume of the *Transactions of the Cambridge Philosophical Society* published in honour of Sir George Gabriel Stokes's jubilee (Vol. XVIII.), and in the article "Geometrical Continuity" which he contributed to the tenth edition of the *Encyclopædia Britannica* (1902).

In 1862 the *Oxford, Cambridge, and Dublin Messenger of Mathematics* was founded by a band of six enthusiasts who acted as editors of the first volume. Among them were numbered John Casey, afterwards

Professor of Higher Mathematics and Mathematical Physics in the Catholic University of Ireland, William Esson, now Savilian Professor of Geometry in the University of Oxford, Charles Taylor, and William Allen Whitworth, afterwards Professor of Mathematics at Queen's College, Liverpool. In subsequent years the composition of the editorial committee was changed, and the name of the periodical was changed to the *Messenger of Mathematics*, but Taylor continued to be an editor until 1884, being the last of the original six to maintain his connexion with the undertaking. In the "Introduction" to the first volume, the editors pointed out that it is often much easier to solve the equations by which a mathematical or physical theory is expressed than to express the theory by analysis; and they stated that one chief object aimed at in founding the periodical was to provide an opportunity for beginners in mathematical research to exercise themselves in the difficult art of translating theories into analysis. In the years that followed the founding of the *Messenger*, a great quickening of interest took place in this country in regard to original mathematical work, and it seems that credit is due to Taylor for no small share in this movement, although with characteristic unobtrusiveness he never claimed any such credit. Symptoms of the reawakening were the foundation of our Society and of the Association for the Improvement of Geometrical Teaching, which afterwards became the Mathematical Association. Taylor joined both in 1872, and was President of the Association in 1892.

Besides the books and memoirs already mentioned, Taylor's mathematical writings include some thirty or forty papers, mostly on geometry, which were published in the *Messenger of Mathematics*, the *Quarterly Journal of Pure and Applied Mathematics*, and the *Proceedings of the Cambridge Philosophical Society*. In his single contribution to our *Proceedings* (Vol. VI.) he gave an account of a method of geometrical transformation called "reversion" which had been used in a neglected treatise by G. Walker, published at Nottingham in 1794. Reversion is an example of a type of projective transformation. A special case of it, developed by Boscovich in a forgotten paper of date 1757 which was unearthed by Taylor, gives the now familiar construction of points on a conic by means of the eccentric circle. This paper, like all Taylor's writings on geometry, is marked by elegance, conciseness, a rare knowledge of the history of the subject, and a veneration for the great geometers of the past.

NOTES AND CORRECTIONS.

Dr. H. F. Baker sends the following Note in completion of his paper (p. 122 of this volume) :—

It should be remarked that the resolution of the syzygy connecting the invariants of a quintic given on p. 123 of this volume, and of many other similar syzygies, is possible by aid of the following remark, which arose in the course of a correspondence between Prof. Burnside and the author.

If an equation $\xi^p = F(\xi_1, \ldots, \xi_n)$, wherein p is an integer and the right side is a polynomial in $\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \ldots, \xi_n, \xi_n^{-1}$ be isobaric when to $\xi, \xi_1, \ldots, \xi_n$ are assigned respective weights $a, a_1, \ldots, a_n$; if d be the highest common factor of $a_1, a_2, \ldots, a_n$, and if p and ap/d be prime to one another, then $\xi, \xi_1, \ldots, \xi_n$ can be expressed rationally by rational functions of themselves.

For putting $\lambda_1 a_1 + \ldots + \lambda_n a_n = d$, wherein $\lambda_1, \ldots, \lambda_n$ are integers without common factor, and

$$X = \xi_1^{\lambda_1} \ldots \xi_n^{\lambda_n}, \quad x_1 = \xi_1 X^{-a_1/d}, \quad \ldots, \quad x_n = \xi_n X^{-a_n/d},$$

we have $$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} = 1, \quad \text{and} \quad \xi^p X^{-ap/d} = F(x_1, \ldots, x_n);$$

taking co-prime integers μ, ν, such that

$$\mu \frac{ap}{d} + \nu p = 1,$$

and putting $$x = \xi^p X^{-ap/d}, \qquad y = \xi^\mu X^\nu,$$

which give $$X = x^{-\mu} y^p, \qquad \xi = x^\nu y^{ap/d},$$

we thus find $$x = F(x_1, \ldots, x_n).$$

Since $\lambda_1, \ldots, \lambda_n$ are co-prime, the equation

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} = 1$$

can be resolved by putting for each of $x_1, \ldots, x_n$ a proper product of integral powers of $n-1$ new variables $t_1, \ldots, t_{n-1}$, these being conversely each a product of integral powers of $x_1, \ldots, x_n$. The original equation is then resolved in terms of $y, t_1, \ldots, t_{n-1}$, which are rational functions of $\xi, \xi_1, \ldots, \xi_n$.

The simplest example after the quintic is perhaps the syzygy for the sextic

$$2R^2 = abc + 2fgh - af^2 - bg^2 - ch^2$$

(Gordan, *Invariantentheorie*, 1887, p. 291), to which the method applies at once. Another example is Gordan's syzygy for a particular ternary quartic given in Weber's *Algebra II.*, 1896, p. 464. It was in consequence of hearing from Prof. Burnside that he had resolved this (by a method which ultimately proved different from the above) that the above general formulation was devised.

At the meeting of April 30th, 1908, Lt.-Col. Allan Cunningham, R.E., reported a new divisor of a Mersenne's number, viz., $2^{163} - 1 \equiv 0 \pmod{150287}$, whereby only eighteen Mersenne's numbers remain unverified; and that these eighteen contain no divisor < 200,000. He also showed how to construct Quartans $N = (x^4 + y^4)$ containing any desired number of algebraic factors of this same form.

Dr. W. H. Young sends the following corrections of his paper (p. 29 of this volume) :—

P. 50, line 8, *for* " $-\dfrac{n^2x+n}{1+n(nx+1)^2}$ " *read* " $\dfrac{n^2x-n}{1+n(nx-1)^2}$."

P. 50, line 12, *for* " unity " *read* " negative unity."

P. 50, line 14, *for* " $\dfrac{n^{\frac{3}{2}}x+n^{\frac{1}{2}}}{1+(n^{\frac{3}{2}}x-n^{\frac{1}{2}})^2}$ " *read* " $\dfrac{n^{\frac{1}{2}}(n^{\frac{3}{2}}x-n^{\frac{1}{2}})}{1+(n^{\frac{3}{2}}x-n^{\frac{1}{2}})^2}$."

Mr. A. E. Western sends the following corrections of his papers (pp. 16 and 265 of this volume) :—

P. 20, on right side of (4), *for* " $F(\zeta)^q$ " *read* " $F(\zeta^q)$."

P. 22, on right side of (11), *for* " $q r_{-t}$ " *read* " $[q r_{-t}]$."

P. 23, line 5, *for* " b_t " *read* " b_t^q."

P. 26, last line of § 18, *for* " b_{v-t} " *read* " b'_{v-t}."

P. 278, line 5, *for* " $\nu_{u+}^{\dagger}$ " *read* " $\nu_{u+f}^{\dagger}$."

PAPERS

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NOTE ON A SET OF SPECIAL CLASSES OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By A. R. Forsyth.

[Received June 11th, 1907.—Read June 13th, 1907.]

In a recent paper,* dealing with partial differential equations of the second order which admit of formal integration in finite terms, I have given three classes of such equations, possessing primitives of the type

$$x,\ y,\ z = \text{function of } u + \text{function of } v.$$

The functions of u and of v in the expression of the primitive may involve an arbitrary function of u, say U, as well as its derivatives up to a finite order, and an arbitrary function of v, say V, as well as its derivatives up to a finite order: and the two arbitrary functions (together with possible derivatives) must occur in the combined expression of the primitive.

The partial equations are of the form

$$ar+bs+ct = 0,$$

where a, b, c involve no derivatives of order higher than the first, and are not necessarily rational functions of x, y, z, p, q; and certain relations (§ 20) must be satisfied by the ratios of a, b, c, in order that the primitive may be of the form stated. The classes of equations obtained in the

* *Proc. London Math. Soc.*, Ser. 2, Vol. 5 (1907), pp. 117–176.

paper which has been quoted are three; they are as follows:—

(A) When b is zero, then a/c must be a pure constant: the equation is simply transformable to

$$r+t=0,$$

and (as is well known) the primitive is given by

$$x=u+v,$$
$$y=i(-u+v),$$
$$z=\Phi(u)+\Psi(v),$$

where Φ and Ψ are arbitrary functions.

When b is not zero, the partial equation can be divided throughout by b, so that it takes the form

$$ar+s+ct=0;$$

and then a, c must satisfy relations in order that the primitive may be of the form stated. The two remaining classes of equations obtained in the paper are:—

(B) The quantities a and c can be functions of x and y alone. In that case, we denote two variable quantities by u and v; and we express x and y in terms of u and v by the relations

$$x=f'(u)+g'(v), \qquad y=uf'(u)-f(u)+vg'(v)-g(v),$$

where f and g are any two functions of their arguments. The partial differential equation is

$$r+(u+v)s+uvt=0;$$

and its primitive is

$$z=\Phi(u)+\Psi(v),$$

where Φ and Ψ are arbitrary functions.

It is easy to verify that the partial equation can be expressed in the form

$$\frac{\partial^2 z}{\partial u\,\partial v}=0.$$

It may, however, be remarked that an equation

$$r+s\theta(x,y)+t\phi(x,y)=0$$

is reducible to this form, only if θ and ϕ are subject to the relations

$$\theta(x,y)=-\frac{\partial\psi(x,y)}{\partial y}, \qquad \phi(x,y)=\frac{\partial\psi(x,y)}{\partial x},$$

where

$$x=f'(u)+g'(v),$$
$$y=uf'(u)-f(u)+vg'(v)-g(v),$$
$$-\psi=u^2f''(u)-2uf'(u)+2f(u)+v^2g''(v)-2vg'(v)+g(v).$$

(C) The quantities a and c can be functions of p and q alone. In that case, we denote two variable quantities by λ and μ, and we express them in terms of p and q by the relations

$$q-\lambda p = F(\lambda), \qquad q-\mu p = G(\mu),$$

where F and G are any two functions of their arguments. The partial differential equation is

$$\lambda\mu r-(\lambda+\mu)\,s+t = 0\,;$$

and the primitive is given by the three equations

$$\begin{aligned} x &= \lambda^2\Phi'(\lambda)+\mu^2\Psi'(\mu), \\ -y &= \lambda\Phi'(\lambda)+\Phi(\lambda)+\mu\Psi'(\mu)+\Psi(\mu), \\ -z &= \int \{\lambda\Phi''(\lambda)+2\Phi'(\lambda)\}\,F(\lambda)\,d\lambda+\int \{\mu\Psi''(\mu)+2\Psi'(\mu)\}\,G(\mu)\,d\mu, \end{aligned}$$

where Φ and Ψ are arbitrary functions.

It is further proved that the coefficients a and c in the equation

$$ar+s+ct = 0,$$

when the primitive is of the specified type, cannot involve x, y, p, q simultaneously and alone. The case when these coefficients may involve x, y, z, p, q simultaneously was not discussed; and certain special cases, associated with zero values of the quantities denoted (§ 6) by A and B, also were not discussed. The aim of this note is to supply the omissions.

1. When we deal with the general case of the equation

$$ar+s+ct = 0,$$

which has a primitive of the form

$$x,\ y,\ z = \text{function of } u + \text{function of } v,$$

the coefficients a and c are subject to the four conditions

$$\frac{\partial'}{\partial' x}\left(\frac{1}{a}\right)+\frac{\partial'}{\partial' y}\left(\frac{c}{a}\right) = 0,$$

$$\frac{\partial'}{\partial' y}\left(\frac{1}{c}\right)+\frac{\partial'}{\partial' x}\left(\frac{a}{c}\right) = 0,$$

$$c^2\frac{\partial a}{\partial p}-ac\frac{\partial c}{\partial p}+a\frac{\partial c}{\partial q} = 0,$$

$$a^2\frac{\partial c}{\partial q}-ac\frac{\partial a}{\partial q}+c\frac{\partial a}{\partial p} = 0,$$

where $$\frac{\partial'}{\partial' x} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z}, \qquad \frac{\partial'}{\partial' y} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z}.$$

In the third and the fourth of the conditions, x, y, z are parametric; hence, by the results of (C), we may take

$$q - \lambda p = F(\lambda, x, y, z),$$

$$q - \mu p = G(\mu, x, y, z),$$

where, so far as these two conditions are concerned, the functions F and G are unlimited, the values of a and c being given by

$$\frac{1}{a} = -\frac{1}{\lambda} - \frac{1}{\mu}, \qquad \frac{1}{c} = -\lambda - \mu, \qquad \frac{a}{c} = \lambda\mu.$$

The values of λ and μ must be such as to allow the first two conditions to be satisfied: substituting the values of a and c, we find

$$\frac{1}{\lambda^2}\frac{\partial'\lambda}{\partial' x} + \frac{1}{\mu^2}\frac{\partial'\mu}{\partial' x} - \frac{1}{\lambda^2\mu}\frac{\partial'\lambda}{\partial' y} - \frac{1}{\lambda\mu^2}\frac{\partial'\mu}{\partial' y} = 0,$$

$$\mu\frac{\partial'\lambda}{\partial' x} + \lambda\frac{\partial'\mu}{\partial' x} - \frac{\partial'\lambda}{\partial' y} - \frac{\partial'\mu}{\partial' y} = 0,$$

and therefore $$\frac{\partial'\lambda}{\partial' y} - \mu\frac{\partial'\lambda}{\partial' x} = 0, \qquad \frac{\partial'\mu}{\partial' y} - \lambda\frac{\partial'\mu}{\partial' x} = 0.$$

Now, from the equation $q - \lambda p = F(\lambda, x, y, z)$,

we have $$-\left(p + \frac{\partial F}{\partial \lambda}\right)\frac{\partial'\lambda}{\partial' x} = \frac{\partial' F}{\partial' x},$$

$$-\left(p + \frac{\partial F}{\partial \lambda}\right)\frac{\partial'\lambda}{\partial' y} = \frac{\partial' F}{\partial' y},$$

and therefore $$\frac{\partial' F}{\partial' y} - \mu\frac{\partial' F}{\partial' x} = 0,$$

that is, $$\frac{\partial F}{\partial y} - \mu\frac{\partial F}{\partial x} + G\frac{\partial F}{\partial z} = 0.$$

Similarly, $$\frac{\partial G}{\partial y} - \lambda\frac{\partial G}{\partial x} + F\frac{\partial G}{\partial z} = 0.$$

These are the two equations for the determination of the forms of F and G, where $$F = F(\lambda, x, y, z), \qquad G = G(\mu, x, y, z).$$

As F does not involve μ, we have, by differentiating the equation

$$\frac{\partial F}{\partial y} - \mu \frac{\partial F}{\partial x} + G \frac{\partial F}{\partial z} = 0$$

twice with respect to μ, the two equations

$$-\frac{\partial F}{\partial x} + \frac{\partial G}{\partial \mu}\frac{\partial F}{\partial z} = 0, \qquad \frac{\partial^2 G}{\partial \mu^2}\frac{\partial F}{\partial z} = 0.$$

Hence either
$$\frac{\partial^2 G}{\partial \mu^2} = 0,$$

that is,
$$G = \beta + \sigma\mu,$$

where β and σ are functions of x, y, z only; or else

$$\frac{\partial F}{\partial z} = 0,$$

and then
$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = 0,$$

that is, F is a function of λ only.

Similarly, from the other equation, we find that either

$$F = \alpha + \rho\lambda,$$

where α and ρ are functions of x, y, z only; or else that G is a function of μ only.

The combination

$$F = \text{function of } \lambda \text{ only}, \qquad G = \text{function of } \mu \text{ only},$$

has already occurred: it is case (C). We therefore have to consider the other possible combinations: these are

(i.) $F =$ a function of λ only, and G has to be obtained;

(ii.) $G =$ a function of μ only, and F has to be obtained;

(iii.) $F = \alpha + \rho\lambda$, $G = \beta + \sigma\mu$, where α, ρ, β, σ are functions of x, y, z, which have to be obtained.

2. In the case (i.), the second of the equations for F and G is

$$\frac{\partial G}{\partial y} - \lambda \frac{\partial G}{\partial x} + F \frac{\partial G}{\partial z} = 0.$$

As F is a function of λ only, and as G does not involve λ, we must have

$$F = a\lambda - b,$$

where a and b are constants; and then

$$G = G_1(\mu,\ z+ax+by),$$

where now G_1 can be any function of its two arguments. Thus

$$q-\lambda p = a\lambda - b,$$
$$q-\mu p = G_1(\mu,\ z+ax+by),$$

and the differential equation is

$$\lambda\mu r-(\lambda+\mu)\,s+t = 0.$$

When we take $$z' = z+ax+by,$$
$$G(\mu,\ z') = G_1(\mu,\ z+ax+by)-a\mu+b,$$

there is no organic change, either in the form of the primitive, or in the form of the equation: and, as will be seen hereafter (§ 7), linear transformations of each of the variables are admissible. Hence, now writing z in place of the new z', the values of λ and μ are given by

$$\lambda = \frac{q}{p}, \qquad q-\mu p = G(\mu,\ z).$$

The differential equation, being

$$\lambda\mu r-(\lambda+\mu)\,s+t = 0,$$

becomes $$q\mu r-(q+\mu p)\,s+pt = 0,$$

that is, $$(qr-ps)\,\mu = qs-pt.$$

We can state the new class of equations as follows:—

(D) The differential equation is

$$\frac{q^2r-2pqs+p^2t}{qr-ps} = G\left(\frac{qs-pt}{qr-ps},\ z\right),$$

where G is any specific function of its two arguments. Its primitive, obtained by Ampère's method, is

$$x = a\Phi''(a)-\Phi'(a)+\beta\Psi'(\beta)-\Psi(\beta),$$
$$y = \Phi''(a)+\Psi'(\beta),$$
$$dz = G(-a,\ z)\,\Phi'''(a)\,da.$$

Thus, as a particular example, the third equation in the primitive of

$$(qr-ps)(q^2r-2pqs+p^2t) = z\,(qs-pt)^2$$

is $$z = e^{a^2\Phi''(a)-2a\Phi'(a)+2\Phi(a)}.$$

In the case (ii.), the first of the equations for F and G is

$$\frac{\partial F}{\partial y} - \mu \frac{\partial F}{\partial x} + G \frac{\partial F}{\partial z} = 0.$$

As G is a function of μ only, and as F does not involve μ, we must have

$$G = a\mu - b,$$

where a and b are constants; and then

$$F = F(\lambda, z + ax + by),$$

where F now is any specific function of its arguments. We thus have the preceding case, provided λ and μ are interchanged; but the partial equation is symmetric in λ and μ, and so case (ii.) provides no new set of equations.

3. In the case (iii.), we have

$$F = \alpha + \rho\lambda, \qquad G = \beta + \sigma\mu,$$

where α, ρ, β, σ are functions of x, y, z only, such as to allow the two equations in F and G to be satisfied. When these values of F and G are substituted, it is necessary and sufficient (to secure this property) that

$$\frac{\partial \alpha}{\partial x} = \sigma \frac{\partial \alpha}{\partial z}, \qquad \frac{\partial \alpha}{\partial y} = -\beta \frac{\partial \alpha}{\partial z},$$

$$\frac{\partial \rho}{\partial x} = \sigma \frac{\partial \rho}{\partial z}, \qquad \frac{\partial \rho}{\partial y} = -\beta \frac{\partial \rho}{\partial z},$$

$$\frac{\partial \beta}{\partial x} = \rho \frac{\partial \beta}{\partial z}, \qquad \frac{\partial \beta}{\partial y} = -\alpha \frac{\partial \beta}{\partial z},$$

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial \sigma}{\partial z}, \qquad \frac{\partial \sigma}{\partial y} = -\alpha \frac{\partial \sigma}{\partial z}.$$

A number of sub-cases will have to be considered: of these, the most extensive is that in which all the quantities α, ρ, β, σ are actually functions of the variables and are not merely constants. We then have

$$\frac{\partial(\alpha, \rho)}{\partial(x, y)} = 0, \qquad \frac{\partial(\alpha, \rho)}{\partial(x, z)} = 0, \qquad \frac{\partial(\alpha, \rho)}{\partial(y, z)} = 0;$$

so that a functional relation exists, involving ρ and α alone and otherwise none of the variables: let it be

$$\rho = f(\alpha),$$

where f denotes some specific function. Similarly, there is a functional relation of the form

$$\sigma = g(\beta),$$

where again g denotes some specific function.

From the first equation in the first column, we have

$$\sigma\frac{\partial^2 a}{\partial z^2}-\frac{\partial^2 a}{\partial x\,\partial z}=-\frac{\partial\sigma}{\partial z}\frac{\partial a}{\partial z}.$$

Differentiating the first equation in the first column with respect to y, and the first equation in the second column with respect to x, and subtracting, we have

$$\sigma\frac{\partial^2 a}{\partial y\,\partial z}+\beta\frac{\partial^2 a}{\partial x\,\partial z}=-\frac{\partial\sigma}{\partial y}\frac{\partial a}{\partial z}-\frac{\partial\beta}{\partial x}\frac{\partial a}{\partial z}$$

$$=-\rho\frac{\partial a}{\partial z}\frac{\partial\beta}{\partial z}+a\frac{\partial a}{\partial z}\frac{\partial\sigma}{\partial z}.$$

From the first equation in the second column, we have

$$\frac{\partial^2 a}{\partial y\,\partial z}+\beta\frac{\partial^2 a}{\partial z^2}=-\frac{\partial a}{\partial z}\frac{\partial\beta}{\partial z}.$$

When these three deduced equations are multiplied by β, 1, $-\sigma$, respectively, and when addition takes place, we find

$$(a-\beta)\frac{\partial a}{\partial z}\frac{\partial\sigma}{\partial z}-(\rho-\sigma)\frac{\partial a}{\partial z}\frac{\partial\beta}{\partial z}=0.$$

Similarly, from the two second equations, we find

$$(a-\beta)\frac{\partial\rho}{\partial z}\frac{\partial\sigma}{\partial z}-(\rho-\sigma)\frac{\partial\rho}{\partial z}\frac{\partial\beta}{\partial z}=0;$$

from the two third equations,

$$(a-\beta)\frac{\partial\beta}{\partial z}\frac{\partial\rho}{\partial z}-(\rho-\sigma)\frac{\partial\beta}{\partial z}\frac{\partial a}{\partial z}=0;$$

and from the two fourth equations,

$$(a-\beta)\frac{\partial\sigma}{\partial z}\frac{\partial\rho}{\partial z}-(\rho-\sigma)\frac{\partial\sigma}{\partial z}\frac{\partial a}{\partial z}=0.$$

Excluding temporarily all the sub-cases in which one or more of the quantities a, ρ, β, σ can be constant, we see that there are two ways in which these equations can be satisfied. In one of these ways, we have

$$a=\beta,\qquad \rho=\sigma:$$

to the sub-case thus given we shall return. In the other of the ways, we have (from the aggregate of equations)

$$\left.\begin{aligned}\frac{\partial \alpha}{\partial x} &= v\sigma(\alpha-\beta)\\ \frac{\partial \alpha}{\partial y} &= -v\beta(\alpha-\beta)\\ \frac{\partial \alpha}{\partial z} &= v(\alpha-\beta)\end{aligned}\right\}, \qquad \left.\begin{aligned}\frac{\partial \beta}{\partial x} &= u\rho(\alpha-\beta)\\ \frac{\partial \beta}{\partial y} &= -u\alpha(\alpha-\beta)\\ \frac{\partial \beta}{\partial z} &= u(\alpha-\beta)\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \rho}{\partial x} &= v\sigma(\rho-\sigma)\\ \frac{\partial \rho}{\partial y} &= -v\beta(\rho-\sigma)\\ \frac{\partial \rho}{\partial z} &= v(\rho-\sigma)\end{aligned}\right\}, \qquad \left.\begin{aligned}\frac{\partial \sigma}{\partial x} &= u\rho(\rho-\sigma)\\ \frac{\partial \sigma}{\partial y} &= -u\alpha(\rho-\sigma)\\ \frac{\partial \sigma}{\partial z} &= u(\rho-\sigma)\end{aligned}\right\},$$

where u and v are two quantities unspecified by the equations thus far considered. Hence

$$\frac{\partial}{\partial x}\log(\alpha-\beta) = \frac{\partial}{\partial x}\log(\rho-\sigma),$$

$$\frac{\partial}{\partial y}\log(\alpha-\beta) = \frac{\partial}{\partial y}\log(\rho-\sigma),$$

$$\frac{\partial}{\partial z}\log(\alpha-\beta) = \frac{\partial}{\partial z}\log(\rho-\sigma),$$

and therefore $$\frac{\rho-\sigma}{\alpha-\beta} = A',$$

where A' is a pure constant. Also

$$\rho = f(\alpha),$$

so that $$\frac{\partial \rho}{\partial x} = f'(\alpha)\frac{\partial \alpha}{\partial x};$$

that is, $$f'(\alpha) = \frac{\rho-\sigma}{\alpha-\beta} = A'.$$

Similarly, $$g'(\beta) = \frac{\rho-\sigma}{\alpha-\beta} = A'.$$

Consequently, $\rho = f(\alpha) = A'\alpha+B', \quad \sigma = g(\beta) = A'\beta+B',$

where B' is a pure constant; and therefore

$$q-\lambda p = F(\lambda) = a+\lambda(A'a+B'),$$

$$q-\mu p = G(\mu) = \beta+\mu(A'\beta+B'),$$

so that
$$\lambda = \frac{q-a}{p+A'a+B'}, \qquad \mu = \frac{q-\beta}{p+A'\beta+B'}.$$

The partial differential equation is

$$\lambda\mu r-(\lambda+\mu)s+t = 0;$$

when we take new variables

$$z' = z+B'x, \qquad x' = -A'x,$$

(which changes do not affect the character of the integral system), the equation is unaltered in form, and the new values of λ and μ are

$$\lambda = \frac{q-a}{p-a}, \qquad \mu = \frac{q-\beta}{p-\beta}.$$

Hence, without loss of generality in character, we can take $B' = 0$, $A' = -1$; and then
$$\rho = -a, \qquad \sigma = -\beta.$$

The equations for the determination of a and β now are

$$\frac{\partial a}{\partial x} = -\beta\frac{\partial a}{\partial z}, \qquad \frac{\partial a}{\partial y} = -\beta\frac{\partial a}{\partial z},$$

$$\frac{\partial \beta}{\partial x} = -a\frac{\partial \beta}{\partial z}, \qquad \frac{\partial \beta}{\partial y} = -a\frac{\partial \beta}{\partial z};$$

it is not difficult to obtain the equations

$$z = a\phi'(a)-\phi(a)+\beta\chi'(\beta)-\chi(\beta),$$

$$x+y = \phi'(a)+\chi'(\beta),$$

(where ϕ and χ are any functions), as giving a and β.

The partial differential equation is thus determinate in form: its primitive can be obtained by Ampère's method, with the result:

(E) The primitive of the equation

$$(q-a)(q-\beta)r-\{2pq-(p+q)(a+\beta)+2a\beta\}s+(p-a)(p-\beta)t = 0,$$

where a and β are given in terms of x, y, z by the relations

$$z = a\phi'(a)-\phi(a)+\beta\chi'(\beta)-\chi(\beta), \qquad x+y = \phi'(a)+\chi'(\beta),$$

ϕ and χ being any specific functions, is

$$x = \Phi(u)+\Psi(v),$$
$$y = \phi'(u)-\Phi(u)+\chi'(v)-\Psi(v),$$
$$z = u\phi'(u)-\phi(u)+v\chi'(v)-\chi(v),$$

where Φ and Ψ are arbitrary functions.

4. We return now to the other of the ways in which the equations for F and G can be satisfied. We then have

$$\alpha = \beta, \qquad \rho = \sigma = -\tau,$$

say. The values of λ and μ are equal, being

$$\lambda = \mu = \frac{q-\alpha}{p-\tau};$$

and the relations determining α and τ are

$$\frac{\partial\alpha}{\partial x} = -\tau\frac{\partial\alpha}{\partial z}, \qquad \frac{\partial\alpha}{\partial y} = -\alpha\frac{\partial\alpha}{\partial z},$$

$$\frac{\partial\tau}{\partial x} = -\tau\frac{\partial\tau}{\partial z}, \qquad \frac{\partial\tau}{\partial y} = -\alpha\frac{\partial\tau}{\partial z}.$$

It is not difficult to obtain the equations

$$\tau = \theta(\alpha), \qquad z-x\tau-y\alpha = \psi(\alpha),$$

where ϕ and ψ are any functions), as giving α and τ.

The partial equation thus has the form

$$(q-\alpha)^2 r-2(q-\alpha)(p-\tau)s+(p-\tau)^2 t = 0;$$

its primitive, obtained by Ampère's method, is given by the elimination of v between the equations

$$y+x\Phi(v) = \Psi(v), \qquad z = x\theta(v)+yv+\psi(v).$$

But this primitive does not possess the postulated form; and therefore we do not include this set of equations among the classes that have been obtained.*

* The form of the equation shews that the two sets of characteristics are coincident, and that therefore, when it is of the first class as defined by Ampère (and this has been secured through the values of α and β), the two arbitrary functions in the primitive have the same argument. Consequently, the primitive is not expressible by means of two quantities u and v, such that arbitrary functions of these quantities should appear.

It may be added that a very special set of cases, to be similarly omitted from the final enumeration, and actually omitted in the construction of the preceding equation, is given by

$$a = \text{constant} = b, \qquad \tau = \text{constant} = a:$$

the primitive is known, being

$$z = x\Phi(ax+by+z)+\Psi(ax+by+z),$$

where Φ and Ψ are arbitrary functions: the primitive is not, and cannot be made, of the required type.

5. It remains to state the results when one, or more than one, of the quantities α, ρ, β, σ can be constant.

When α and ρ are constants, then either (i.) β and σ are constants, and the equation belongs to class (C); or (ii.) one of the two quantities β and σ is a constant while the other is any function of $z+\rho x-\alpha y$, and the equation belongs to class (D); or (iii.) β and σ are any functions of $z+\rho x-\alpha y$, and again the equation belongs to class (D). Similarly for the various alternatives as to α and ρ, when β and σ are constants.

When α and β are constant, then either (i.),

$$\rho = \text{constant}, \qquad \sigma = \text{any function of } z+\rho x-\alpha y;$$

or (ii.),

$$\sigma = \text{constant}, \qquad \rho = \text{any function of } z+\sigma x-\beta y,$$

and in each of these cases the equation belongs to class (D); or $\alpha=\beta=k$, where k is a constant. Owing to the transformations (§ 7) which are admissible, we can take $z-ky$ as a new dependent variable, thus effectively making k zero. We now have an apparently new set of equations, as follows. The equation

$$q^2r-q(2p+\rho+\sigma)s+(p+\rho)(p+\sigma)t = 0,$$

where ρ and σ are functions of x and z, such that

$$-x = \psi'(\rho)+\chi'(\sigma),$$
$$z = \rho\psi'(\rho)-\psi(\rho)+\sigma\chi'(\sigma)-\chi(\sigma),$$

ψ and χ being specific functions, has

$$y = \Phi(u)+\Psi(v),$$
$$-x = \psi'(u)+\chi'(v),$$
$$z = u\psi'(u)-\psi(u)+v\chi'(v)-\chi(v),$$

for its primitive, Φ and Ψ being arbitrary functions. But by interchange of the variables y and z, so as to have y for the dependent variable (and this interchange is only one of the transformations of § 7 which are admissible), the equation becomes equation (B): it thus does not provide any actually new set.

In all other alternatives as to possible constant values of some of the quantities α, ρ, β, σ, it is found that the equations, which emerge, belong to one or other of the classes already settled.

6. All the preceding results are obtained on the supposition that the quantities denoted (in § 6 of the former paper) by A and B do not vanish. The results, when the supposition is not justified by fact, can be stated briefly.

(i.) Suppose that $A = 0$ and that B is not zero: we then have

$$\frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} - q\frac{\partial g}{\partial U_m} = 0, \qquad H_1 - pF_1 - qG_1 = 0.$$

If these are independent equations, p and q are functions of u alone: the integral equations then lead to a couple of differential equations

$$r + ks = 0, \qquad s + kt = 0,$$

where k is a pure constant.

If they are effectively only a single equation, then

$$H_1 = \theta\frac{\partial h}{\partial U_m}, \qquad F_1 = \theta\frac{\partial f}{\partial U_m}, \qquad G_1 = \theta\frac{\partial g}{\partial U_m},$$

where θ is a function of u only, and involves no quantities that do not occur in x, y, z: and the case, when the second equation is evanescent, is covered by a zero value of θ. We have

$$f_1 = (U_{m+1}+\theta)\frac{\partial f}{\partial U_m}, \qquad g_1 = (U_{m+1}+\theta)\frac{\partial g}{\partial U_m}:$$

the factor $U_{m+1}+\theta$, which cannot vanish, divides out from the equation

$$rf_1f_2 + s(f_1g_2 + f_2g_1) + tg_1g_2 = 0,$$

and the analysis in the former paper applies.

(ii.) The results are similar when $B = 0$ and when A is not zero: and they are similar, and more special, when $A = 0$, $B = 0$.

Under no one of the alternatives is a set of equations provided that is not included within the sets already retained.

7. Transformations of the variables are possible: and the differential equations, after such transformations, still admit of formal integration. But the only transformations which leave the primitive of the desired type, viz.,

$$x,\ y,\ z = \text{function of } u + \text{function of } v,$$

are included in the set

$$\left.\begin{aligned} x &= a_1x'+b_1y'+c_1z' \\ y &= a_2x'+b_2y'+c_2z' \\ z &= a_3x'+b_3y'+c_3z' \end{aligned}\right\},$$

where the coefficients a, b, c are constant.

If equations thus transformable into one another are to be regarded as not independent of one another, then equation (A) at the beginning of this note is to be regarded as a special example of (C). When the preceding transformations are applied to equation (A), viz., $r+t=0$, it becomes

$$Lr'-2Ms'+Nt'=0,$$

where

$$\begin{aligned} L &= (b_1+c_1q')^2+(b_2+c_2q')^2, \\ M &= (a_1+c_1p')(b_1+c_1q')+(a_2+c_2p')(b_2+c_2q'), \\ N &= (a_1+c_1p')^2+(a_2+c_2p')^2; \end{aligned}$$

the special forms of $F(\lambda)$ and $G(\mu)$, for inclusion in (C), are

$$F(\lambda) = \lambda\frac{a_1+ia_2}{c_1+ic_2}-\frac{b_1+ib_2}{c_1+ic_2},$$

$$G(\mu) = \mu\frac{a_1-ia_2}{c_1-ic_2}-\frac{b_1-ib_2}{c_1-ic_2};$$

and the known primitives of the two equations can be constructed from each other.

It may be added, for the sake of general reference, that the formulæ of transformation for the derivatives p, q, r, s, t are:—

$$\frac{p}{A_1p'+B_1q'-C_1} = \frac{q}{A_2p'+B_2q'-C_2} = \frac{-1}{A_3p'+B_3q'-C_3},$$

$$r = \frac{\Delta}{(A_3p'+B_3q'-C_3)^3}\left[-(b_2+c_2q')^2r'+2(a_2+c_2p')(b_2+c_2q')s' - (a_2+c_2p')^2t'\right],$$

$$s = \frac{\Delta}{(A_3p'+B_3q'-C_3)^3}\left[(b_1+c_1q')(b_2+c_2q')r'+(a_1+c_1p')(a_2+c_2p')t' - \{(b_1+c_1q')(a_2+c_2p')+(a_1+c_1p')(b_2+c_2q')\}s'\right],$$

$$t = \frac{\Delta}{(A_3p'+B_3q'-C_3)^3}\left[-(b_1+c_1q')^2r'+2(a_1+c_1p')(b_1+c_1q')s' - (a_1+c_1p')^2t'\right],$$

where Δ denotes the non-evanescent determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix},$$

and the quantities A, B, C are the customary minors of a, b, c in Δ.

The applications of the transformations to the sets of equations retained provide a great variety of different forms, and, in particular, they allow for all appropriate changes of the dependent variable.

8. There is one specialised form of primitive, not included in the preceding discussion: it can, after the transformations just indicated, be represented by

$$x = \text{function of } u \text{ only},$$

$$y, z = \text{function of } u + \text{function of } v,$$

arbitrary functions of u and of v occurring in these equations. Manifestly, the primitive does not change its character by any additional transformation of the type

$$x' = \text{function of } x,$$

$$y' = \alpha' x + \beta' y + \gamma' z,$$

$$z' = \alpha'' x + \beta'' y + \gamma'' z,$$

where the quantities α, β, γ are constants. Having regard to this property, it is easy to prove that the partial equation can be reduced to the form

$$s + t = 0,$$

the primitive of which is obvious.

9. Finally, the case when the primitive is of the type

$$x = \text{function of } u, \quad y = \text{function of } v, \quad z = \text{function of } u + \text{function of } v,$$

is a very special instance of the case mentioned in § 33 of the former paper: it merely leads to the partial equation $s = 0$.

AN EXTENSION OF EISENSTEIN'S LAW OF RECIPROCITY

By A. E. WESTERN.

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1. Eisenstein's Law of Reciprocity* is as follows. ζ is a primitive l-th root of unity, where l is a rational prime other than 2. If $\mathfrak{p}$ is a prime ideal of the cyclotomic field† $k(\zeta)$, P is its norm, and a is any number of $k(\zeta)$, then $\{a/\mathfrak{p}\}$ denotes that power of ζ which is congruent to $a^{(P-1)/l}$ (mod $\mathfrak{p}$). And, if $\mathfrak{p}$ is composite, and its prime factors are $\mathfrak{q}, \mathfrak{q}', \ldots$, then $\{a/\mathfrak{p}\} = \{a/\mathfrak{q}\}\{a/\mathfrak{q}'\} \ldots$.

A number ν of $k(\zeta)$ is called semi-primary when it is prime to $1-\zeta$ and congruent to a rational number, mod $(1-\zeta)^2$. Then, ν being a semi-primary number and a being a rational number, prime both to l and ν, the law is

$$\{a/\nu\} = \{\nu/a\}.$$

The object of the present paper is to prove a similar law of reciprocity in the field $k(\zeta)$, where ζ is a primitive l^n-th root of 1, and l is a rational prime number other than 2.

For the laws of divisibility in this field I refer to Hilbert's invaluable Report on Algebraic Numbers (already referred to), and in particular to §§ 94–99 thereof, which relate to the cyclotomic field of l^n-th roots of unity.

2. *Notation.*—Rational numbers will be denoted by *italic* letters, complex numbers of the field $k(\zeta)$ by Greek letters, and ideals of the field $k(\zeta)$ by German letters. Throughout, "number" means "whole number."

$p = ml^n+1$, $p' = m'l^n+1$, ... are rational primes.

R, R', ... are primitive roots to mod p, p', ... respectively.

$$\mathfrak{p} = (p, \zeta - R^{-m}), \quad \mathfrak{p}' = (p', \zeta - R'^{-m'}), \quad \ldots$$

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* *Berlin Monatsberichte* (1850), p. 189. See also H. J. S. Smith, "Report on the Theory of Numbers," Art. 56 (*Collected Papers*, Vol. I., p. 123), and D. Hilbert, *Bericht*, "Die Theorie der Algebraischen Zählkörper," §§ 113–115 (*Deutschen Math. Verein*, Bd. 4, 1897).

† German, *Kreiskörper.*

are prime ideal factors of $p, p', \ldots$ respectively. These ideals are of the first grade, and each of $p, p', \ldots$ has $l^{n-1}(l-1)$ different prime ideal factors.

r is any primitive root mod l^n.

The substitution s changes ζ into ζ^r.

The prime ideal factors of p are $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_u, \ldots$ where

$$\mathfrak{p}_u = s^u \mathfrak{p} = \mathfrak{p}(\zeta^{r^u}).$$

If $f(s) = \Sigma a_t s^t$, $f(s)\mathfrak{p}$ denotes $\Pi(s^t\mathfrak{p})^{a_t}$.*

(x) denotes the least positive residue of x (mod l^n); and $r_t = (r^t)$.

$[x]$ denotes the greatest number not greater than xl^{-n}, so that

$$x = [x]l^n + (x).$$

$\xi, \xi', \ldots$ are p-th, p'-th, ... roots of 1.

$F(\zeta)$, sometimes written $F(\zeta, \xi)$, $F'(\zeta)$, ... are the corresponding Lagrangian resolvents, viz.,

$$F(\zeta, \xi) = \xi + \zeta\xi^R + \zeta^2\xi^{R^2} + \ldots + \zeta^{p-2}\xi^{R^{p-2}}.$$

In particular, $$F(1, \xi) = -1.$$

$\psi_g, \psi'_g, \ldots$ are corresponding reciprocal factors of $p, p', \ldots$, viz.,

$$\psi_g = \frac{F(\zeta)\,F(\zeta^g)}{F(\zeta^{g+1})} \quad (g = 1, 2, \ldots, l^n-2).$$

It is well known† that ψ_g is a function of ζ only and does not contain ξ, and that $\psi_g \cdot \psi_g(\zeta^{-1}) = p$. On account of this property I call ψ_g a *reciprocal factor* of p.

Then ψ_g contains $\frac{1}{2}l^{n-1}(l-1)$ different prime ideal factors of p, and no other prime ideals.

It will be found convenient to use the above definitions of ψ_g for all values of g. We thus obtain $\psi_{l^n-1} = -p$, $\psi_{l^n} = -1$, and $\psi_g = \psi_h$, when $g \equiv h$ (mod l^n).

In the first instance we take ν to be a number of $k(\zeta)$ containing only factors of $p, p', \ldots$, so that $\nu = f(s)\mathfrak{p} \cdot f'(s)\mathfrak{p}' \ldots$.

N is the norm of ν in the field $k(\zeta)$.

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* Hilbert's notation for $f(s)\mathfrak{p}$ is $\mathfrak{p}^{f(s)}$.

† H. J. S. Smith, *op. cit.*, Art. 30, p. 78. P. Bachmann, *Die Lehre von der Kreistheilung*, cap. 8 (Leipzig, 1872). H. Weber, *Lehrbuch der Algebra*, Second Edition, 1898, Bd. I., pp. 610–621.

$$\nu_t = s^t \nu.$$

$$\mathrm{F}(\zeta) = f(s)\,F(\zeta)\,.\,f'(s)\,F'(\zeta)\,\ldots.$$

$$\Psi_g = f(s)\,\psi_g\,.\,f'(s)\,\psi_g'\,\ldots.$$

Then $\Psi_g . \Psi_g(\zeta^{-1}) = N$, and Ψ_g contains $\frac{1}{2}l^{n-1}(l-1)$ different conjugates of ν and no additional factors.

q is a rational prime, different from l, p, p', ...; e is the exponent to which q appertains, mod l^n, so that $q^e \equiv 1 \pmod{l^n}$.

$Q = (q^e - 1)\,l^{-n}$.

Without loss of generality, r may be so chosen that $q \equiv r^f \equiv r_f \pmod{l^n}$, where $ef = l^{n-1}(l-1)$.

$\mathfrak{q}_0$ is any one of the f prime ideal factors of q. The others are

$$\mathfrak{q}_u = s^u\,\mathfrak{q}_0 \quad (u = 1, 2, \ldots, f-1).$$

It is convenient to allow the suffix u in $\mathfrak{p}_u$, $\mathfrak{q}_u$, ... to have any integral values, defining $\mathfrak{p}_u$, $\mathfrak{q}_u$, ... as before. Then $\mathfrak{p}_x = \mathfrak{p}_y$, if

$$x \equiv y \quad [\mathrm{mod}\ l^{n-1}(l-1)],$$

and $\mathfrak{q}_x = \mathfrak{q}_y$, if

$$x \equiv y \pmod{f}.$$

The symbol $\{a/\nu\}$ has the same meaning as in § 1, except that l^n is substituted for l.

3. I define a semi-primary number as in § 1, namely: ν is semi-primary if it is prime to $1-\zeta$ and satisfies

$$\dot{\nu}(1) = \frac{d\nu(\zeta)}{d\zeta} \equiv 0 \pmod{l},$$

where 1 is put for ζ after differentiation.

Since the irreducible equation satisfied by ζ is

$$Z = \zeta^{l^{n-1}(l-1)} + \zeta^{l^{n-1}(l-2)} + \ldots + \zeta^{l^{n-1}} + 1 = 0;$$

the number $\nu + \phi Z$, where ϕ is any number of the field, is the same number as ν in a different form. As $Z(1) = l$ and $\dot{Z}(1) = \frac{1}{2}\,l^n(l-1)$, the semi-primary property is an essential property of ν as a number, and not merely a property of a particular form of ν. It follows at once that, if ν is semi-primary, $\nu \equiv n_0$ [mod $(1-\zeta)^2$], where n_0 is $\nu(1)$, that is, the sum of the coefficients of ν. And, since ν is prime to $1-\zeta$, n_0 is prime to l.

Among the set $\zeta^h\nu$ ($h = 0, 1, \ldots, l-1$), one only is semi-primary; for the condition that $\zeta^h\nu$ should be semi-primary is

$$hn_0+\dot{\nu}(1) \equiv 0 \pmod{l}.$$

Henceforth ν is supposed to be semi-primary. It is easily proved that the conjugates of a semi-primary number are all semi-primary, and that the product of semi-primary numbers is semi-primary.

4. All the numbers $\zeta^{hl}\nu$ ($h = 0, 1, \ldots, l^{n-1}-1$) are semi-primary. For the present purpose it is necessary that some test should be discovered, by which one of the set $\zeta^{hl}\nu$ should be picked out as the standard, and called "primary"; and, at first sight, it is tempting to adopt the simple rule that ν is primary when $\dot{\nu}(1) \equiv 0 \pmod{l^n}$, a condition which is satisfied by one only of the set.

This, however, is open to the objection that, if ν is primary, $\nu+\phi Z$ is not in general primary; so that primariness is a property merely of a form of ν. And this leads to a further difficulty; if $\nu'' = \nu\nu'$, where ν and ν' are primary, ν'' is primary only when

$$\nu'' = \nu\nu'+\phi(\zeta^{l^n-1}+\zeta^{l^n-2}+\ldots+\zeta+1) \tag{1}$$

is an algebraical identity in ζ. When this identity holds, I shall call ν'' *the unreduced product* of ν and ν'.

The suggested definition of primariness must therefore be abandoned. For the present, I postpone dealing with this difficulty; and meanwhile it must be remembered that ν, when given as the product of given prime ideals, is undetermined as regards the unit ζ^{hl} multiplying it.

5. From the definition of $\{\nu/\mathfrak{q}\}$, we have $\{\nu/\mathfrak{q}_{-t}\} \equiv \nu^Q \pmod{\mathfrak{q}_{-t}}$, where $Q = (q^e-1)l^{-n}$. Applying the substitution s^t, and remembering that $\{\nu/\mathfrak{q}_{-t}\}$ is a power of ζ, we obtain

$$\{\nu/\mathfrak{q}_{-t}\}^{r^t} \equiv \nu_t^Q \pmod{\mathfrak{q}_0},$$

and then, raising both sides to the power r_{-t},

$$\{\nu/\mathfrak{q}_{-t}\} \equiv \nu_t^{Qr_{-t}} \pmod{\mathfrak{q}_0}.$$

Therefore $$\{\nu/q\} \equiv \Pi\nu_t^{Qr_{-t}} \pmod{\mathfrak{q}_0},\ (t = 0, 1, \ldots, f-1).$$

Now, applying s^x, and then raising to the power r_{-x} as before, we find that the last congruence is true with any prime factor of q as modulus,

and therefore also with q as modulus; that is,

$$\{\nu/q\} \equiv \Pi\nu_t^{Q r^{-t}} \quad (\text{mod } q),\ (t = 0, 1, \ldots, f-1). \quad (2)$$

Instead of using the substitution s^t, we could use s^{t+hf}, where h is any number. Thus, on the right side of (2), t may represent any complete set of residues mod f.

Applying s^f to the congruence (2), we get

$$\{\nu/q\}^q \equiv \Pi\nu_{t+f}^{Q r^{-t}} \quad (\text{mod } q). \quad (3)$$

We have hitherto supposed that r was so chosen that $q \equiv r^f$ (mod l^n). If, however, r is any primitive root mod l^n, and $q \equiv r^{hf}$ (mod l^n), (2) becomes

$$\{\nu/q\} \equiv \Pi\nu_t^{Q r^{-ht}} \quad (\text{mod } q).$$

6. Let $$K \equiv R^k \quad (\text{mod } p).$$

Then $$F(\zeta^h, \xi^K) = \zeta^{-hk} F(\zeta^h, \xi).$$

But $$\zeta^{-k} \equiv R^{mk} \equiv K^m \equiv \{K/\mathfrak{p}\} \quad (\text{mod } \mathfrak{p}),$$

and so $$\zeta^{-k} = \{K/\mathfrak{p}\} \quad \text{and} \quad F(\zeta^h, \xi^K) = \{K/\mathfrak{p}\}^h F(\zeta^h, \xi).$$

Therefore $$F(\zeta)^q \equiv \xi^q + \zeta^q \xi^{Rq} + \ldots \quad (\text{mod } q)$$

$$\equiv F(\zeta^q, \xi^q) \quad (\text{mod } q),$$

that is, $$F(\zeta)^q \equiv \{q/\mathfrak{p}\}^q F(\zeta)^q \quad (\text{mod } q). \quad (4)$$

It may be noted here that it is at this point that the proof of the law of reciprocity in the present case becomes essentially different from that used by Eisenstein. If we try to follow his method, we get, by repetition of the process by which (4) was obtained,

$$F(\zeta)^{q^e-1} \equiv \{q/\nu\}^e \quad (\text{mod } q).$$

But, in the present case, e in general is divisible by some power of l, which may be l^{n-1}, and this congruence is therefore in general useless. If, however, q is such that e is prime to l, then the law of reciprocity may be proved easily by Eisenstein's method.

7. Now it follows immediately from the definition of ψ_g that

$$F(\zeta)^g = F(\zeta^g)\,\psi_1\psi_2\ldots\psi_{g-1} \quad (\text{for all values of } g).* \quad (5)$$

* This was discovered by Gauss, "Disquisitionum circa Æquationes Puras Ulterior Evolutio," *Werke*, Bd. II., p. 256.

And, in particular, that

$$F(\zeta)^{l^n} = -\psi_1\psi_2\ldots\psi_{l^n-1} = \psi_1\psi_2\ldots\psi_{l^n}.$$

So $$\mathrm{F}(\zeta)^g = \mathrm{F}(\zeta^g)\Psi_1\ldots\Psi_{g-1}, \tag{6}$$

and $$\mathrm{F}(\zeta)^{l^n} = -\Psi_1\ldots\Psi_{l^n-1};$$

(4) therefore becomes $$\{q/\mathfrak{p}\}^q \equiv \psi_1\ldots\psi_{q-1} \pmod{q}.$$

And multiplying together the congruences corresponding to this for all the prime factors of ν, we obtain

$$\{q/\nu\}^q \equiv \Psi_1\ldots\Psi_{q-1} \pmod{q}. \tag{7}$$

Comparing this with (3), we wish to prove that

$$\Psi_1\ldots\Psi_{q-1} \equiv \Pi\nu_{t+f}^{Q\,r_{-t}} \pmod{q}. \tag{8}$$

Now the left side of this is equal to the product of some unit and powers of ν and its conjugates. The congruence will be proved in two stages; first, so far as regards ν and its conjugates; and secondly, I shall show that the unit of $\Psi_1\ldots\Psi_{q-1}$ becomes 1, if ν is "primary," according to a suitable definition of primariness.

8. Kummer* has determined the factorisation of reciprocal factors into prime ideal factors of p. His result is that ψ_g contains $\mathfrak{p}_t$ if, and only if,

$$r_{-t}+(gr_{-t}) > l^n. \tag{9}$$

Since r_{-t} and (gr_{-t}) are both less than l^n, this condition may be written

$$2l^n > r_{-t}+(gr_{-t}) > l^n.$$

Now $$gr_{-t} = [gr_{-t}]\,l^n+(gr_{-t}),$$

so the condition becomes

$$([gr_{-t}]+2)\,l^n > (g+1)\,r_{-t} > ([gr_{-t}]+1)\,l^n,$$

that is, $$[(g+1)\,r_{-t}] = [gr_{-t}]+1.$$

Similarly, if the condition be not satisfied, we get

$$[(g+1)\,r_{-t}] = [gr_{-t}].$$

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* "Theorie der idealen Primfactoren," &c., *Berlin Abhandlungen*, 1856, pp. 42–46. H. J. S. Smith, *op. cit.*, Art. 60, p. 130. Hilbert, *op. cit.*, § 108.

Therefore the index of $\mathfrak{p}_t$ (either 0 or 1) in ψ_g is

$$[(g+1)\,r_{-t}]-[gr_{-t}]. \tag{10}$$

It is easily proved that

$$[(l^n-1)\,r_{-t}] = r_{-t}-1, \quad [l^n r_{-t}] = r_{-t}, \quad \text{and} \quad [(l^n+1)\,r_{-t}] = r_{-t};$$

so (10) remains true for $g = l^n-1$, or l^n. Therefore, for every value of g, the index of $\mathfrak{p}_t$ in $\psi_1 \dots \psi_{g-1}$ is $[gr_{-t}]$, since $[r_{-t}] = 0$.

Now, the index of ν_t in Ψ_g is the same as that of $\mathfrak{p}_t$ in ψ_g, and so

$$\Psi_1 \dots \Psi_{q-1} = \epsilon \Pi \nu_t^{q\,r_{-t}}, \qquad [t = 0, 1, \dots, l^{n-1}(l-1)-1], \tag{11}$$

where ϵ is a unit.

9. It is well known that $\qquad \nu_t^q \equiv \nu_{t+f} \pmod q$,

and hence that $\qquad \nu_t^{q^x} \equiv \nu_{t+xf} \pmod q$.

Consequently each of the factors on the right side of (11) whose suffixes are $\equiv t \pmod f$ may be expressed as a power of $\nu_{t+f} \pmod q$. Having done this, the index of ν_{t+f} in (11) is

$$\Sigma [qr_{-t+xf}]\, q^{e-x-1} \quad (x = 0, 1, \dots, e-1).$$

Now $\qquad [qr_{-t+xf}]\, q^{e-x-1}\, l^n = r_{-t+xf}\, q^{e-x} - r_{-t+(x+1)f}\, q^{e-x-1},$

so $\qquad \Sigma [qr_{-t+xf}]\, q^{e-x-1} = (q^e-1)\, r_{-t}\, l^{-n} = Qr_{-t}.$

Therefore $\qquad \Psi_1 \dots \Psi_{q-1} \equiv \epsilon \Pi \nu_{t+f}^{Q\,r_{-t}} \pmod q, \tag{12}$

and the first stage in the proof of (8) is accomplished.

10. For the second stage of the proof of (8), we must establish some further properties of reciprocal factors. It is known* that

$$\psi_g = \Sigma \zeta^{x+gy},$$

where the summation is taken over all solutions of

$$R^x + R^y \equiv 1 \pmod p. \tag{13}$$

Henceforth reciprocal factors of p are supposed to remain in this form; that is, the identity Z must not be used to alter their form; and all products of reciprocal factors of p, p', ... are supposed to be unreduced products.

* See references in the footnote to § 2. And H. J. S. Smith, *op. cit.*, footnote to Art. 60, p. 131.

Writing* $$\psi_g = \Sigma a_t^g \zeta^t \quad (t = 0, 1, \ldots, l^n - 1);$$

a_t^g is the number of solutions of (13) satisfying

$$x + gy \equiv t \pmod{l^n}.$$

Similarly, let $$\psi_g(\zeta^l) = \Sigma b_t^g \zeta^{tl} \quad (t = 0, 1, \ldots, l^{n-1} - 1);$$

b_t is the number of solutions of (13) satisfying

$$x + gy \equiv t \pmod{l^{n-1}},$$

and so $$b_t^g = \Sigma a_{t+hl^{n-1}}^g \quad (h = 0, 1, \ldots, l-1). \tag{14}$$

$\psi^g(\zeta^l)$ is the g-th reciprocal factor of p of degree l^{n-1}. So we have the remarkable fact that the g-th reciprocal factor of degree l^{n-1} can be derived from the g-th reciprocal factor of degree l^n by changing ζ into ζ^l in the latter.

11. Now† $$\psi_g(1) \equiv -1, \quad \text{and} \quad \dot{\psi}_g(1) \equiv 0 \pmod{l^n}.$$

So ψ_g is semi-primary, and $$\psi_g \equiv -1 \quad [\text{mod } (1-\zeta)^2].$$

So also Ψ_g is semi-primary, and is $\equiv (-1)^v \quad [\text{mod } (1-\zeta)^2]$,

where v is the number of prime ideal factors of ν. Further, since the property $$\dot{\psi}(1) \equiv 0 \pmod{l^n}$$

is invariant for unreduced multiplication (see § 4), it holds for each Ψ and for any product thereof.

$$\psi_{tg}\psi_{tg+1} \ldots \psi_{tg+g-1} = \frac{F(\zeta)^g F(\zeta^{tg})}{F(\zeta^{tg+g})} = \frac{F(\zeta)^g}{F(\zeta^g)} \psi_t(\zeta^g) = \psi_1 \ldots \psi_{g-1} \psi_t(\zeta^g), \tag{15}$$

for all values of t and g.

This relation between the reciprocal factors does not appear to have been previously noticed, though the particular case of it for $g = 2$ is given by Weber,‡ who mentions it as known to Jacobi.

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* The g in a_t^g is not an index, but a suffix.

† H. J. S. Smith, *op. cit.*, footnote to Art. 60, p. 131; dealing only with the modulus l, but the proof is applicable to the modulus l^n.

‡ *Lehrbuch der Algebra*, 2nd ed., 1898, Bd. I., p. 611.

12. Kronecker has shewn that every unit of $k(\zeta)$ is equal to the product of a power of ζ and a real unit.*

So, when g is not $\equiv -1$ or $0 \pmod{l^n}$,

$$\Psi_g = E\zeta^k \Pi\nu_t, \quad t \text{ satisfying (9)},$$

and
$$\Psi_g(\zeta^{-1}) = E\zeta^{-k}\Pi'\nu_t, \quad t \text{ not satisfying (9)},$$

where E is a real unit. Then, since

$$\Psi_g . \Psi_g(\zeta^{-1}) = N = \Pi\nu_t . \Pi'\nu_t,$$

$E^2 = 1$, that is, $E = \pm 1$.

Since Ψ_g and $E\Pi\nu_t$ are semi-primary, so also is ζ^k, that is,

$$k \equiv 0 \pmod{l}.$$

Writing kl for k,
$$\Psi_g = E\zeta^{kl}\Pi\nu_t.$$

Since
$$\nu_t \equiv n_0 \quad [\text{mod } (1-\zeta)^2],$$

$$\Pi\nu_t \equiv n_0^{\frac{1}{2}l^{n-1}(l-1)} \quad [\text{mod } (1-\zeta)^2].$$

But
$$n_0^{\frac{1}{2}(l-1)} \equiv (n_0/l) \pmod{l},$$

so
$$\Pi\nu_t \equiv (n_0/l) \quad [\text{mod } (1-\zeta)^2].$$

Also
$$\zeta^{kl} \equiv 1 \quad \text{and} \quad \Psi_g \equiv (-1)^v \quad [\text{mod } (1-\zeta)^2],$$

so
$$E = (-1)^v (n_0/l). \tag{16}$$

The units of Ψ_{l^n-1} and Ψ_{l^n} are each -1.

13. Here it may be noticed that this method of determining the sign of the unit of Ψ_g breaks down in the case $l = 2$. For, in that case $(1-\zeta)^2$ is a factor of 2, and so

$$\psi_g \equiv -1 \equiv 1 \quad [\text{mod } (1-\zeta)^2],$$

which congruence is of course useless for determining the sign of ψ_g. There is consequently in the case $l = 2$ considerable difficulty in determining the unit in Ψ_g, and a separate investigation for each value of n appears to be necessary. It will be found that this sign is different for different values of g.

14. When q is odd, E occurs in $\Psi_1 \dots \Psi_{q-1}$ to an even power; and when q is 2, $E \equiv 1 \pmod{q}$. In either case E disappears from

* "Ueber complexe Einheiten," *Werke*, Bd. I., p. 109. Hilbert, *op. cit.*, p. 335.

$\Psi_1 \ldots \Psi_{q-1}$, and the unit of this expression is some power of ζ^l. The sign of the unit of any Ψ being therefore immaterial, it may hereafter be ignored, and *the unit* of a Ψ or of a product thereof will mean a power of ζ^l.

15. *The unit of* $F(\zeta)^{l^n}$. We obtain from (5),

$$F(\zeta)^l = F(\zeta^l)\,\Psi_1 \ldots \Psi_{l-1}.$$

Raising this to the l^{n-1}-th power,

$$F(\zeta)^{l^n} = F(\zeta^l)^{l^{n-1}}(\Psi_1 \ldots \Psi_{l-1})^{l^{n-1}}.$$

The unit of $(\Psi_1 \ldots \Psi_{l-1})^{l^{n-1}}$ is 1, so $F(\zeta)^{l^n}$ and $F(\zeta^l)^{l^{n-1}}$ have the same units. But the unit of $F(\zeta^{l^{n-1}})^l$ is 1, so by induction the unit of $F(\zeta)^{l^n}$ is 1. That is, the unit of $\Psi_1 \ldots \Psi_{l^n-1}$, is 1.

It is easy to prove similarly that the unit of $\Psi_1 \ldots \Psi_{vl^h-1}$, where $h < n$, is a power of ζ^{l^h}. This result is not needed for the present purpose.

16. Since $q \equiv r^f \pmod{l^n}$, it follows from the last paragraph that $\Psi_1 \ldots \Psi_{q-1}$ has the same unit as $\Psi_1 \ldots \Psi_{r^f-1}$. Writing

$$X = \Psi_1 \ldots \Psi_{r-1},$$

we obtain from (5),

$$\begin{aligned} F(\zeta)^{r^f} &= X(\zeta)^{r^{f-1}} F(\zeta^r)^{r^{f-1}} \\ &= X(\zeta)^{r^{f-1}} X(\zeta^r)^{r^{f-2}} F(\zeta^{r^2})^{r^{f-2}} \\ &= \ldots \quad \ldots \quad \ldots \\ &= X(\zeta)^{r^{f-1}} X(\zeta^r)^{r^{f-2}} \ldots X(\zeta^{r^{f-1}})\, F(\zeta^{r^f}), \end{aligned}$$

that is,
$$\Psi_1 \ldots \Psi_{r^f-1} = X(\zeta)^{r^{f-1}} X(\zeta^r)^{r^{f-2}} \ldots X(\zeta^{r^{f-1}}). \tag{17}$$

The results of this and the last paragraph may also be proved from (15), without introducing $F(\zeta)$.

If, then, the unit of X is 1, the unit of $\Psi_1 \ldots \Psi_{q-1}$ is also 1.

The proper definition of primariness accordingly is: *ν is primary when* (1) *it is semi-primary, and* (2) *it is such that X has the unit* 1.

17. It only remains to show that, given a number ν, the proper power of ζ^l, by which ν must be multiplied in order to make the unit of X become 1, can be determined.

We have, by (11),

$$X = e\zeta^{kl}\Pi\nu_t^{[rr_{-t}]} \quad [t = 0, 1, \ldots, l^{n-1}(l-1)-1], \tag{18}$$

where
$$e = E^{r-1}.$$

Putting $\nu' = \zeta^{xl}\nu$ in place of ν, the unit of X becomes

$$\zeta^{kl+xl\Sigma r^t[rr_{-t}]}.$$

Now
$$r^t[rr_{-t}]\,l^n = r^{t+1}r_{-t} - r^t r_{-t+1},$$

so
$$\Sigma r^t[rr_{-t}] = r\,(r^{l^{n-1}(l-1)}-1)\,l^{-n}.$$

This expression is prime to l.* Therefore when k is known, x may be so chosen as to make the unit of X become 1.

18. *Lemma.*—Let $\phi = \Sigma a_t\zeta^t$, and $\phi' = \Sigma a'_t\zeta^t$ $(t = 0, 1, \ldots, l^n-1)$ be any two numbers. And let their unreduced product

$$\phi\phi' = \Sigma A_t\zeta^t,$$

so that
$$A_v = \Sigma a_t a'_{v-t},$$

wherein $a'_t = a'_{t'}$ when
$$t \equiv t' \pmod{l^n}.$$

Then writing, as in § 10, $b_t = \Sigma a_{t+hl^{n-1}}$ $(h = 0, 1, \ldots, l-1)$,

$$\begin{aligned} B_v &= \Sigma_k A_{v+kl^{n-1}} = \Sigma_t a_t \Sigma_k a'_{v-t+kl^{n-1}} \\ &= \Sigma_t \Sigma_h a_{t+hl^{n-1}} \Sigma_k a'_{v-t+kl^{n-1}} \quad (t = 0, 1, \ldots, l^{n-1}-1) \\ &= \Sigma_t b_t b_{v-t}. \end{aligned}$$

19. We assume that each ψ and Ψ of degree l^{n-1} is known. Then, from (14), $\Sigma_h a^g_{t+hl^{n-1}}$ is known for each ψ of degree l^n. And so, by the lemma, the similar sums of the coefficients of each Ψ_g and of X are known.

Now multiply out the product $\Pi\nu_t^{[rr_{-t}]}$, and call the result Π; this need not be unreduced multiplication. Then add to Π such multiples of Z, ζZ, ... as to make each $\Sigma_h a_{t+hl^{n-1}}$ of $e\Pi$ equal to the corresponding sum of the coefficients of X. Then, with Π in this form,

$$eX = \zeta^{kl}\Pi + \phi(\zeta^{l^n}-1)$$

must be an identity in ζ.

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* Mathews, *Theory of Numbers*, Part 1, p. 27.

Now differentiating, and then putting $\zeta = 1$, we obtain

$$e\dot{X}(1) \equiv kl\Pi(1) + \dot{\Pi}(1) \quad (\mathrm{mod}\ l^n),$$

but
$$\dot{X}(1) \equiv 0 \quad (\mathrm{mod}\ l^n),$$

so
$$k\Pi(1) + l^{-1}\dot{\Pi}(1) \equiv 0 \quad (\mathrm{mod}\ l^{n-1}),$$

which determines k.

So, beginning with the degree l, for which each ψ is already known, we can proceed to the degree l^2, and so on, up to l^n. Therefore the power of ζ^l by which ν must be multiplied to make it primary, can be found. Henceforth ν is supposed to be primary.

We have now proved the truth of (8), and therefore that

$$\{\nu/q\}^q \equiv \{q/\nu\}^q \quad (\mathrm{mod}\ q),$$

that is,
$$\{\nu/q\} = \{q/\nu\}.$$

The law of reciprocity is therefore proved with these limitations, that ν contains only prime ideals of the first grade, and that q is a rational prime.

20. The law may be extended to any ν and any rational number a, in a similar manner to that employed by Hilbert* for the case of l-th roots of unity.

Thus, finally, we have the law of reciprocity :—

If ν is any primary number of the field $k(\zeta)$ of l^n-th roots of unity, and a is any rational number prime to ν and l, then

$$\{a/\nu\} = \{\nu/a\}.$$

21. Consider now whether the requirement that ν should be primary may be relaxed. Let $\nu' = \zeta^{xl}\nu$; then

$$\{q/\nu'\} = \{q/\nu\}, \quad \text{and} \quad \{\nu'/q\} = \{\zeta^{xl}/q\}\{\nu/q\}.$$

We find from (2) that

$$\{\zeta^{xl}/q\} = \Pi\zeta^{xQlr^t r_{-t}} \quad (t = 0, 1, \ldots, f-1)$$
$$= \zeta^{xQfl}.$$

So, if Qf contains l^v, but no higher power of l, the law holds between q and ν', provided that

$$x \equiv 0 \quad (\mathrm{mod}\ l^{n-v-1}).$$

* Hilbert, *op. cit.*, § 115.

In this case, it suffices that the process of § 19 for determining the unit of X should be carried up to the degree l^{n-r}. In particular, if

$$f \equiv 0 \pmod{l^{n-1}},$$

the law holds whenever ν is semi-primary, as we saw in § 6.

22. It must be admitted that the process above given for making ν primary would be laborious in actual practice; and it is much to be desired that some process should be found applicable directly to ν itself, instead of to the reciprocal factors. I have discovered such a process for the field of ninth roots of unity; and it is possible that this may be capable of generalisation, though up to the present I have failed in this aim.

ON UNIFORM AND NON-UNIFORM CONVERGENCE AND DIVERGENCE OF A SERIES OF CONTINUOUS FUNCTIONS AND THE DISTINCTION OF RIGHT AND LEFT*

By W. H. YOUNG, Sc.D., F.R.S.

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1. Writers on uniform and non-uniform convergence have for the most part contented themselves hitherto with determining the mode of distribution of the points of uniform and non-uniform convergence, without occupying themselves with the various types of non-uniform convergence at a point that may occur, and how far such types are to be regarded as normal or exceptional. In particular, the question whether the character of the non-uniform convergence may be different on the left and on the right has been barely mooted, and the distribution of points at which this is the case has not been discussed at all. For some time the idea of a point of uniform convergence itself was only imperfectly grasped, uniform convergence being thought of as something pertaining to an interval alone; it was not realised that a series could be uniformly convergent at a point, without being uniformly convergent in any interval containing the point; in other words, that a point of uniform convergence may be a limiting point on both sides of points of non-uniform convergence, and this even though all the functions concerned are continuous. Still less was it realised that, when the functions whose sum is considered are discontinuous functions, a point of uniform convergence may be absolutely isolated.†

In the present paper I take as fundamental the definition‡ of uniform convergence at a point I have already employed in previous papers, one which is now coming into general use.§

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* A brief account of the results of this paper was communicated to the British Association at Leicester on Monday, August 5th.

† W. H. Young, "Points of Uniform Convergence . . .," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, pp. 358–360.

‡ W. H. Young, "On non-Uniform Convergence . . . ," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 90.

§ I am not sure who was the first to actually formulate the definition. In Osgood's original paper in the *American Journal*, Vol. XIX., and in Schoenflies's account of it in his *Bericht*, I can

Inevitable as this definition appears, it has several disadvantages:—

(1) It involves the remainder function $R_n(x)$, and therefore the possibly unknown sum, or limiting function $f(x)$, which may be discontinuous even when the functions $f_n(x)$ are all continuous.

(2) It is an "ϵ-definition."

(3) It affords us no means of *classifying* points of non-uniform convergence.

The close connection, or analogy, between the uniform convergence of a series and the continuity of a function is well known. As regards the continuity of a function we may avoid the ϵ-method by introducing* the *associated upper* and *lower right-hand and left-hand limiting*

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find no such formulation. In Townsend's dissertation, "Doppellimes," Göttingen, 1900, I find the following statements:—"Also gleichmässige Convergenz bezieht sich auf das ganze Intervall, dagegen hat der einfache Limes

$$\underset{y=y_0}{\mathrm{Lt}}\ f(x_0, y) = f(x_0, y_0),$$

nur mit einem einzelnen Punkte des Intervalls zu thun," (p. 29); again, on p. 65, "So weit ist diese Bedingung ganz dasselbe wie obige Bedingung für die Stetigkeit von $f(x)$. Sie unterscheidet sich aber davon, indem bei gleichmässiger Convergenz das für jedes n definierte Intervall eine untere Grenze grösser als Null haben muss, wenn n über alle Grenzen wächst. Dagegen braucht bei der Stetigkeitsbedingung . . . , dieses Intervall keine untere Grenze grösser als Null zu haben." These statements suggest that even in 1900 in Göttingen a precise formulation of uniform convergence at a point had not been made. Of course points of uniform convergence play an important part, none the less, in Townsend's dissertation, and Arzelà's paper, which preceded it ("Sulle Serie di Funzioni," Parte 1, *Mem. di Bologna*, Serie 5, Vol. VIII., pp. 131–186, 1899). The phraseology adopted for these points by Townsend is "points at which the series converges regularly," regular convergence being expressed in terms of, and thought of in connection with, the behaviour of the allied functions of two variables introduced by Du Bois Reymond ("Ueber die Integration der Reihen," *Sitzungsbericht d. Berliner Akademie*, 1886, pp. 359–371), and used also by Arzelà. This definition only applies in the case considered by Townsend, which is also that considered in the present paper, when the functions to be summed are continuous.

In Osgood's paper, although the statement is made that "Uniform and non-uniform convergence are conceptions that relate to the behaviour of the variable function throughout an interval," p. 166, the points in question are used, and called ζ-points; their definition as given by Osgood is clear and precise though somewhat complicated (pp. 163–165).

In Hobson's paper ("On Non-Uniform Convergence . . . ," *Proc. London Math. Soc.*, Vol. XXXIV., pp. 254 *et seq.*, Jan., 1902), uniform convergence is defined only for an interval. In his recent book *Functions of a Real Variable* (1907), the definition for a point is implicitly given.

Van Vleck, in a recent paper, "A Proof of some Theorems on Pointwise Discontinuous Functions," *Trans. Amer. Math. Soc.*, Vol. VIII., April, 1907, p. 204, footnote, gives the definition in the form adopted by myself; and I understand that Hilbert has now done the same for some time in his lectures.

* W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," Aug., 1907, *Quarterly Journal of Math.*, Vol. XXXIX., pp. 67–83.

functions ϕ_R, ϕ_L, ψ_R, ψ_L. We are thus naturally led to devise similar functions in the case of convergence of series.

This idea is in embryo in Osgood's paper, above cited, in which he introduces the word "peak" and the term "indices at a point." Osgood deals, however, only with those series of continuous functions whose sum is a continuous function, and his indices have relation to the remainder function; in the case when the sum is zero, his indices are closely connected with the χ and π of the present paper. The case which usually arises in practice is that where the functions of which the sum is considered are continuous, but it is an unnecessary and an undesirable restriction to suppose that their sum is continuous. In what follows I begin by defining four functions π_L, π_R, χ_L, χ_R, which I call *peak* and *chasm functions*, and which are strictly analogous to the associated functions ϕ_L, ϕ_R, ψ_L, ψ_R, above referred to, and I shew that, in the case considered, *the equality of these functions at a point is the necessary and sufficient condition for uniform convergence at the point*, so that we may, if we please, give this equality as a new definition of uniform convergence at a point. This definition is precisely analogous to that referred to of continuity at a point; it is not, however, intended to replace the other, which is indeed fundamental in character, and cannot easily be dispensed with when the functions to be summed are not continuous. It has, however, the advantage of being free from the objections pointed out as inherent in the other. I take occasion to shew how this second definition may be used directly to obtain the well known distribution of the points of non-uniform convergence, viz., that they form an ordinary outer limiting set of the first category.*

The main use, however, that I make of the new definition is to examine the character of the types of non-uniform convergence that may arise, more especially with respect to the distinction of right and left. I shew that *only at a countable number of points, which may, however, be dense everywhere, can the behaviour of a series, as regards non-uniform convergence, be different on the right and on the left of a point.* In particular the points at which the series is uniformly convergent on one side and non-uniformly convergent on the other side of a point, can be, at most, countably infinite.

This discussion would appear to complete the qualitative study of

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* Proofs of this result have been given by Townsend and Hobson, *loc. cit.* Both of these authors deduce it from considerations connected with the theory of functions of two variables. I myself have given a proof of a more general result, including this as a special case, by a method more on the lines of Osgood's classical paper. The proof in the text thus constitutes a fourth proof of the result.

points of non-uniform convergence, in the case when the functions to be summed are all continuous. It should be noted, however, that though many of the theorems obtained apply as they stand, or with slight modifications, to the more general case, the information afforded by the investigations of the paper is not adequate for a complete analysis of the facts of non-uniform convergence except when the functions to be summed are continuous.

It remains to be added that throughout the paper all that has been required from the series of functions is that it should have a definite sum. The possibility of that sum having the value ∞ ($+\infty$, or $-\infty$, as the case may be), is not excluded. In other words, divergence is allowed, provided it be not an oscillatory divergence. Moreover the functions whose sum, or limit, is considered, are not necessarily bounded functions; in other words, their continuity is of the generalised character, in which infinite values are allowed. Thus the new proof, above referred to, relating to the distribution of points of non-uniform convergence, is really wider in scope than any of its predecessors, and the result obtained is of a more general character, having reference moreover to points of "uniform divergence," as we may conveniently call them, as well as to points of uniform convergence.

2. It will be convenient to repeat here the definitions, already referred to, of the associated left- and right-hand upper and lower limiting functions ϕ_L, ϕ_R, ψ_L, ψ_R of a discontinuous function.

Let P be any internal point of a segment throughout which a function $f(x)$ is defined. Take any interval with P as right-hand end-point, then $f(x)$ has, for the points *internal* to this interval, an upper limit; as this interval diminishes, this upper limit cannot increase, and therefore has a limit, which is, at the same time, its lower limit; denote this limit by $\phi_L(P)$.

We thus get for every point P of the segment a function $\phi_L(x)$, or shortly ϕ_L, which may be called *the upper left-hand limiting function of $f(x)$*.

Similarly, changing left into right, we define a function ϕ_R, *the upper right-hand limiting function of f*. Further, interchanging the words "upper" and "lower," "increase" and "decrease," in the definition, we define corresponding *lower limiting functions* which we shall denote by ψ_L and ψ_R.

If at each point P we choose that one of the two upper limiting functions which is not less than the other, we get a new function, which

may be called the (modified) *upper limiting function*, and be denoted by ϕ. Similarly we define the (modified) *lower limiting function* ψ, by taking that one of the two lower limiting functions which is not less than the other.

3. Let $f_1, f_2, \ldots$ be a series of functions, having a definite limiting function f; in other words, at any point P, we have

$$\underset{n=\infty}{\mathrm{Lt}} f_n(P) = f(P). \tag{1}$$

We now define auxiliary numbers at P precisely analogous to the upper and lower left- and right-hand limiting functions of a discontinuous function; these we shall call *the left- and right-hand peak and chasm functions*, and denote them by π_L, π_R, χ_L, χ_R.

We take an interval PQ with P as right-hand end-point, and denote the upper limit of f_n for points x inside this open interval by $M_{n,Q}$. Then for all such points x,

$$f_n(x) \leqslant M_{n,Q}, \tag{2}$$

while either there is such a point x at which

$$f_n(x) = M_{n,Q},$$

or else there is at least a sequence of points passing along which $f_n(x)$ has the limit $M_{n,Q}$.

These numbers $M_{n,Q}$ for the successive integers n, may be conveniently plotted off on the axis of y; they form a countably infinite set* which has therefore at least one limiting point, and, in any case will have a first derived set which may be countable or of potency c. Let the highest of these derived points, or corresponding numbers, be denoted by M_Q, and, though this will be less used in the sequel, the lowest of these derived points, or numbers, be denoted by M'_Q.

Now, if Q_1 and Q_2 are two positions of Q of which Q_2 lies between P and Q_1, it follows from the definitions that

$$M_{n,Q_2} \leqslant M_{n,Q_1}.$$

Hence any limiting point of the points M_{n,Q_2}, for successive values of n, will determine one or more limiting points of the points M_{n,Q_1}, none of which will lie below the former limiting point. It follows that

$$M_{Q_2} \leqslant M_{Q_1},$$

and also

$$M'_{Q_2} \leqslant M'_{Q_1}.$$

Therefore, if we make the point Q approach P as limit, moving along a

* Counting two of the points as distinct whether or no they coincide in position. If there is only a finite number of positions, the highest derived point is, of course, the highest of the points which is repeated an infinite number of times.

sequence, or continuously, or in any manner, the quantities M_Q will have a definite limit, which will be at the same time their lower limit, and which will be denoted by $\pi_L(P)$, and, for all positions of P, be called the *left-hand peak function*. Similarly, the quantities M'_Q have a definite limit, which is their lower limit, and may be denoted by $\pi'_L(P)$.

Here, of course, $$\pi'_L \leqslant \pi_L.$$

Working in like manner on the right of P we obtain the corresponding right-hand quantities, denoted by the subscript R instead of L. Again, interchanging "upper" and "lower," we get the *left- and right-hand chasm functions* χ_L and χ_R, as well as the quantities χ'_L and χ'_R, where

$$\chi'_L \geqslant \chi_L \quad \text{and} \quad \chi'_R \geqslant \chi_R.$$

4. THEOREM 1.—

$$\chi_L(P) \leqslant \chi'_L(P) \leqslant \psi_L(P) \leqslant \phi_L(P) \leqslant \pi'_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, for the right-hand functions.)

For, if x be any point inside the interval PQ, we had as equation (2) of § 3,

$$f_n(x) \leqslant M_{n,Q}. \tag{2}$$

Making n increase indefinitely, $f_n(x)$ has the single limit $f(x)$, which, therefore, cannot lie above any limit of the quantities $M_{n,Q}$; therefore

$$f(x) \leqslant M'_Q.$$

Now letting x describe a suitable sequence with P as limit, we obtain for $f(x)$ the limit $\phi_L(P)$, therefore, Q being still fixed,

$$\phi_L(P) \leqslant M'_Q.$$

Since this is true for all positions of Q,

$$\phi_L(P) \leqslant \pi'_L(P).$$

Similarly $$\psi_L(P) \geqslant \chi'_L(P),$$

which proves the theorem.

THEOREM 2.—*If the functions f_n are continuous at P,**

$$\chi_L(P) \leqslant \chi'_L(P) \leqslant f(P) \leqslant \pi'_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, since $f_n(x)$ is continuous at P, it has the definite limit $f_n(P)$, so

* It follows from Theorem 1, using the results of my paper quoted on p. 30 that this inequality holds whatever the f's are, except at most at a countable set of points.

that, by equation (2) of § 3 or § 4,

$$f_n(P) \leqslant M_{n,Q}.$$

Since this is true for all values of n, the single limit $f(P)$ approached by the left-hand side of the inequality cannot be higher than any limit approached by the right-hand side; therefore

$$f_n(P) \leqslant M'_Q.$$

Since this holds for all positions of Q,

$$f(P) \leqslant \pi'_L(P).$$

Similarly $$f(P) \geqslant \chi'_L(P),$$

which proves the theorem.

5. From Theorem 1 it follows that, if the peak and chasm functions are equal at P, they are equal to the upper and lower associated limiting functions, and therefore, with the possible exception of a countable set of points,* they are all equal to $f(P)$; if, however, the f_n's are continuous, there are no such exceptional points, by Theorem 2.

In other words, *at a point where*

$$\chi(P) = \pi(P) = f(P),$$

f is continuous; if the f_n's are continuous, we may drop the $f(P)$ from this equation.

We see, however, from the enunciations of Theorems 1 and 2 that, though this condition is sufficient for continuity it is not necessary; it is still sufficient if

$$\chi'(P) = \pi'(P) = f(P),$$

but this condition also is not necessary; anyone familiar with examples of non-uniform convergence will recognise that this is so, and that thereby hangs a tale.

6. The definition of uniform convergence at a point P is as follows :—

* *Loc. cit.*, p. 30.

"The series of functions $f_1, f_2, \ldots$ is said to 'converge uniformly to the function f at the point P,' if, given any positive quantity ϵ, however small, an interval d can be described, having P as internal point, so that, for all points x within the interval d,

$$|f(x)-f_n(x)| < \epsilon$$

for all values of $n \geqslant m$, where m is an integer, independent of x, which can always be determined.

"Similarly we may define the expressions right-handed and left-handed uniform convergence* at P; in this case the interval d will have P as end-point."

This definition may easily be adapted so as to give what we may call "uniform divergence" at P, when the value $f(P)$ is infinite with determinate sign; we merely have instead of the above inequality,

$$f_n(x) > A \quad \text{or} \quad f_n(x) < -A,$$

according as the sign of f is $+$ or $-$, A being, like ϵ, preassigned, and being, of course, not "however small" but "however large."

7. The connection of the peak and chasm functions with these definitions is determined by the following theorems.

THEOREM 3.—*If the f_n's are continuous functions, and P a point at which*

$$\chi_L(P) = \pi_L(P),$$

the series $f_1, f_2, \ldots$ is uniformly convergent or divergent, at P on the left.

CASE 1.—Let $\pi_L(P)$ be finite, then, since π_L is the limit of the quantities M_Q, we can choose Q_1 so that

$$M_{Q_1} \leqslant \pi_L(P)+\epsilon.$$

Therefore, since M_{Q_1} is the highest possible limit approached by M_{n, Q_1}, we can determine k_1, so that, for all values of $n \geqslant k_1$,

$$M_{n, Q_1} \leqslant M_{Q_1}+e \leqslant \pi_L(P)+2e.$$

Since M_{n, Q_1} is the upper limit of $f_n(x)$ in the interval PQ_1, it follows that, for all points x of this interval and all values of $n \geqslant k_1$,

$$f_n(x) \leqslant \pi_L(P)+2e.$$

Similarly, working with χ_L instead of π_L, we can find a point Q_2 inside

* It should be noted that in the definitions of right-handed and left-handed uniform convergence or divergence of a series of continuous functions it is immaterial whether the interval with P as end-point be supposed to include P or not, provided we know that the series is actually convergent, or divergent, at P as the case may be.

THEOREM 4.—*If $f_1, f_2, \ldots$ be continuous functions, and the series is uniformly convergent at P on the left (or uniformly divergent), then*

$$\chi_L(P) = \pi_L(P) = f(P).$$

CASE 1.—If the series is uniformly convergent at P on the left, assigning e, we can find an interval PQ, with P as right-hand end-point, and an integer k, such that, for all points x in PQ, and all values of $n \geqslant k$,

$$|f(x) - f_n(x)| < \epsilon. \tag{1}$$

Now, by the definition of $M_{n,Q}$ (§ 1), we can find at least one point, say x_n, inside the open interval PQ, such that

$$0 \leqslant M_{n,Q} - f_n(x_n) < e. \tag{2}$$

Therefore $$|f(x_n) - M_{n,Q}| < 2e. \tag{3}$$

Now, if we let n describe a suitable sequence of constantly increasing integers, $n_1, n_2, \ldots, M_{n,Q}$ will have the limit M_Q.

The countably infinite set of points $x_{n_1}, x_{n_2}, \ldots$ has at least one limiting point, and may have more. Let x_Q be one of these limiting points.

Then we can pick out a subsequence of the set

$$n_1, n_2, \ldots,$$

say $$n'_1, n'_2, \ldots,$$

such that $$x_{n'_1}, x_{n'_2}, \ldots$$

is a sequence having x_Q as limit. In this case, by the definitions of ϕ and ψ, any limit approached by $f(x)$ will lie between

$$\psi(x_Q) \quad \text{and} \quad \phi(x_Q),$$

both inclusive.

But, by (3), any such limit differs from M_Q by at most $2e$, hence

$$\psi(x_Q) - 2e \leqslant M_Q \leqslant \phi(x_Q) + 2e. \tag{4}$$

Now, let us make e describe a sequence with zero as limit, and at the same time, as we may, let us take each interval PQ less than half the length of the preceding one; then Q will describe a sequence having P as limit, and the same will be true of x_Q, which always lies in the interval PQ.

M_Q has then, as we saw in § 1, the definite limit $\pi_L(P)$, while $\phi(x_Q) + 2e$ may, or may not, have a definite limit, but any limit assumed

by it is $\leqslant \phi_L(P)$, by Theorem 1 of my paper quoted on p. 30 of the present memoir. Hence

$$\pi_L(P) \leqslant \phi_L(P).$$

Similarly any limit assumed by $\psi(x_Q)-2e \geqslant \psi_L(P)$, so that

$$\psi_L(P) \leqslant \pi_L(P).$$

Hence
$$\psi_L(P) \leqslant \pi_L(P) \leqslant \phi_L(P). \tag{5}$$

But, the f_n's being continuous and the series uniformly convergent at P, f is continuous at P, so that

$$\psi_L(P) = \phi_L(P) = f(P);$$

therefore, by (5),
$$\pi_L(P) = f(P).$$

Similarly
$$\chi_L(P) = f(P),$$

which proves the theorem in this case.

CASE 2.—Next, let the series be uniformly divergent on the left at P, and first let

$$f(P) = -\infty.$$

Then, by the condition for uniform divergence (§ 6), there is an interval (P, Q) and an integer k, such that, for all points x in that interval, and all integers $n > k$,

$$f_n(x) < -A;$$

and therefore
$$M_{n,Q} \leqslant -A;$$

and therefore
$$M_Q \leqslant -A.$$

Hence, $\pi(P)$ being the lower limit of the quantities M_Q,

$$\pi(P) \leqslant -A.$$

Since this is true for all values of A,

$$\pi(P) = -\infty;$$

and therefore also
$$\chi(P) = -\infty,$$

which proves the theorem in this case.

Similarly, if $f(P) = +\infty$, the result follows, using $\chi(P)$ instead of $\pi(P)$, $>$ for $<$, and replacing M_Q, $M_{n,Q}$ by the corresponding quantities connected with χ.

THEOREM 5.—*If the f_n's are not continuous throughout the interval having P as right-hand end-point, but the series is uniformly convergent (or divergent) at P on the left,*

$$\phi_L(P) = \pi_L(P) \quad \textit{and} \quad \psi_L(P) = \chi_L(P).$$

CASE 1.—First, let the series be uniformly convergent at P on the left. The proof then proceeds as in Case 1 of the preceding theorem down to equation (5).

But, by Theorem 1, $\pi_L(P) \geqslant \phi_L(P)$;

therefore, by (5), $\pi_L(P) = \phi_L(P)$.

Similarly, $\chi_L(P) = \psi_L(P)$,

which proves the theorem in this case.

CASE 2.—Secondly, let the series be uniformly divergent at P on the left. It will be found on examination that the proof given of this case in Theorem 4 holds without modification.

COR.—*At a point of uniform divergence, or at a point of uniform convergence which is also a point of continuity of $f(x)$ on the left,*

$$\chi_L(P) = \pi_L(P) = f(P).$$

8. Making then our usual assumption that the f_n's are all continuous functions, it follows, by the results of the preceding article, that we may take as the definition of uniform convergence at the point P the equality of all the four peak and chasm functions.

Similarly, we can define uniform convergence on the right or left alone by the equality of the corresponding one-sided peak and chasm functions. It should be noticed that from this point of view uniform divergence is merely a special case of uniform convergence.

9. When the series is non-uniformly convergent at the point P, there are several special cases of interest.

(1) Let f be not less than π at P; then the function f *is upper semi-continuous at P.*

For, by Theorem 1, in this case,

$$\phi(P) \leqslant f(P).$$

(2) Similarly, if f be not greater than χ at P, the function f *is lower semi-continuous at P.*

These conditions are again, as in the case of continuity, sufficient but

not necessary; in particular, Theorem 1 shews that it is still sufficient for upper semi-continuity if $f(P) \geqslant \pi'(P)$, and for lower semi-continuity if $f(P) \leqslant \chi'(P)$.

(3) Let $$\chi'(P) = \pi'(P),$$

then the function f is continuous at P, by Theorems 1 and 2.

[It is clear from Theorem 1 that $\chi'(P) \not> \pi'(P)$.]

This condition is sufficient but not necessary.

10. In Case 3, an argument precisely the same as that used in proving Theorem 3 may be used, only that, as M'_Q is not the highest but the lowest possible limit approached by $M_{n,Q}$, we cannot determine k_1 so that for *all* values of $n \geqslant k_1$, $$M_{n,Q} < M'_Q + e;$$

but we can insure that this is true for all values of n *belonging to a certain sequence* of constantly increasing integers

$$n_1, n_2, \ldots.$$

The same is then true for the inequality

$$L_{n,Q} > L'_Q - e,$$

for all values of $n \geqslant k_2$, belonging to a certain sequence

$$n'_1, n'_2, \ldots,$$

$L_{n,Q}$ denoting the lower limit of $f_n(x)$ in the interval PQ, and L'_Q the highest limit of $L_{n,Q}$.

If these two sequences are the same, or if they have a common sequence, the argument used in the proof of Theorem 3 applies; hence it follows that the condition for uniform convergence at P is satisfied, with the restriction of n to values belonging to a certain sequence. When this is the case the convergence at P is said to be *simply uniform*.

This mode of looking at simple uniform convergence shews that there is no essential difference between simple uniform convergence at one point and uniform convergence at that point. If the corresponding sequence is

$$n_1, n_2, \ldots,$$

we only have to take, instead of the given series of functions

$$f_1, f_2, \ldots,$$

the sub-series $$f_{n_1}, f_{n_2}, \ldots$$

having the same limiting function f, and the convergence at the point under consideration will be uniform.

11. If the given series is simply uniformly convergent at every point of an interval or of a closed set, it may be, but this is not necessarily the case, that the same sequence will serve at every point of the interval, or closed set. By the extended Heine-Borel theorem, it then follows that a finite number of the intervals which are defined at each point by the simple uniform convergence will suffice to contain every point of the interval, or closed set. To these intervals correspond a finite number of integers k; thus, if k' denote the sum of these, it will be true that, for all points x of the interval, or closed set, and for all integers $n \geqslant k'$ belonging to the sequence in question,

$$|f(x)-f_n(x)| < A.$$

A being the quantity with which we started to determine the intervals and integers.

Conversely, if this is true, there is a sequence of integers which will serve at every point.

In this case it is customary to say that the given series is *simply uniformly convergent throughout the interval or closed set*, and it has been pointed out,* by the same argument as that used above, that by properly picking out the functions f_n, we can reduce this to uniform convergence at every point of the interval, or closed set.

It is clear, however, that this is only a particular case of simple uniform convergence at every point of the interval, or closed set, and it by no means follows that in the general case of simple uniform convergence at every point of an interval, or closed set, we can so pick out the functions as to make the convergence uniform at all the points in question simultaneously.

12.† We now proceed to prove for the π's and χ's the same theorems as those obtained in the paper already quoted for the ϕ's and ψ's.

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* Arzelà, *loc. cit.*

† In this article the assumption that f_n is continuous need not be made.

THEOREM 6.*—*Any limit approached by $\pi(x)$, $\pi_L(x)$, or $\pi_R(x)$ as x approaches a point P as limit on the right $\leqslant \pi_L(P)$, and, as x approaches P as limit on the left $\leqslant \pi_R(P)$.*

If $\pi_L(P) = +\infty$, this is certainly the case; if not, we can find a finite quantity $A > \pi_L(P)$.

Then, by the definition of $\pi_L(P)$, we can find an interval d with P as right-hand end-point, such that, Q being any point of this interval,

$$M_Q < A.$$

Therefore, by the definition of M_Q, we can find an integer k, such that, for all integers $n \geqslant k$,

$$M_{n,\,Q} < A.$$

Therefore, $M_{n,\,Q}$ being the upper limit of $f_n(x)$ in the interval (P, Q),

$$f_n(x) < A,$$

for all integers $n \geqslant k$, and all points of (P, Q).

Now, let P_1 and Q_1 be any two points internal to PQ, and let us consider the quantities M_{Q_1} and $M_{n,\,Q_1}$ referred, not to the interval PQ_1 but to P_1Q_1. The preceding inequality (3) gives us then

$$M_{n,\,Q_1} < A.$$

We then have

$$M_{Q_1} \leqslant A.$$

Now, let Q_1 move up to P_1 as limit, we get

$$\pi_L(P_1) \leqslant A, \quad \text{or} \quad \pi_R(P_1) \leqslant A,$$

according as Q_1 lay on the left or the right of P_1.

Since A is at our disposal, provided only it is $> \pi_L(P)$, this proves the first part of the theorem; similarly, working on the right, the second part follows.

COR.—*π_L is upper semi-continuous on the left and π_R on the right, while π is an upper semi-continuous function, and, as such, at most pointwise discontinuous.*

The proofs of the succeeding theorems in my paper quoted, depending, as they do, solely on the theorem correlative to the above theorem, can now be transferred verbatim, changing only the symbol ϕ into π. It is therefore only necessary to give the enunciations here.

THEOREM 7.—*At every point of continuity of π,*

$$\pi_L = \pi_R = \pi,$$

and both π_L and π_R are continuous.

COR.—*π_L and π_R, as well as π, are at most pointwise discontinuous.*

THEOREM 8.—*The only points at which both π_L and π_R are continuous are the points of continuity of π.*

* The correlative of Theorem 1, *loc. cit.*

THEOREM 9.*—*The points, if any, at which π_R differs from π_L, are countable.*

Similar results, interchanging the signs $>$ and $<$, hold, of course, for the χ's.

13. Thus we see that, except possibly at a countable set of points,

$$\pi_L = \pi_R = \pi \quad \text{and} \quad \chi_L = \chi_R = \chi,$$

while

$$\chi \leqslant \psi \leqslant f \leqslant \phi \leqslant \pi.$$

Thus *the distinction of right and left with respect to peaks and chasms exists at most at a countable set of points.*

When the functions f_n are continuous, it is, as we saw, the same to say that there is no distinction of right- and left-handed non-uniform convergence, except at most at a countable set of points.

14. THEOREM 10.—*At any point P where the peak and chasm functions are equal, both these functions are continuous; conversely, at any point where they are both continuous, provided the f_n's are continuous, the peak and chasm functions are equal.*

To prove the first part of the theorem, we only need Theorem 6. For, let P be a point at which

$$\chi(P) = \pi(P).$$

Since, as x approaches P as limit,

$$\chi(P) \leqslant \text{Lt}\, \chi(x) \leqslant \text{Lt}\, \pi(x) \leqslant \pi(P),$$

we must have the sign of equality throughout, which proves that both χ and π are continuous at P.

To prove the second part of the theorem, we proceed as follows.

Suppose, if possible, that there were a common point of continuity of the peak and chasm functions at which the peak and chasm functions were not equal, let this point be P, then

$$\pi(P) > \chi(P), \tag{1}$$

by Theorem 1.

By the sense of this equation $\pi(P)$ cannot be $-\infty$, nor $\chi(P) + \infty$; therefore we can find two numbers α and β, such that

$$\chi(P) < \beta, \quad \alpha < \pi(P); \tag{2}$$

* The more specialised results denoted by Theorem 5a and 5b *loc. cit.*, are not given here; they hold, of course, and the proofs only require the change of ϕ into π.

The correlative of Theorem 6, viz., *the points, if any, at which $f > \pi$ are countable*, follows, of course, at once from our Theorem 1, using the results of the former paper. This was already referred to in the footnote on p. 34.

further we can so choose these numbers that

$$\beta < \alpha. \tag{3}$$

Since P is a common point of continuity of the peak and chasm functions, we can find a whole interval (A, B), at every internal point x of which

$$\chi(x) < \beta, \qquad \alpha < \pi(x), \tag{4}$$

the point P being internal to this interval.

From the definition of the peak function, it now follows from (2) that we can find a point Q in (A, B), such that

$$\alpha < M_Q;$$

and therefore we can find a value n_1 of n, greater than any assigned integer, such that

$$\alpha < M_{n_1, Q}.$$

Since $M_{n_1, Q}$ is the upper limit of the values of $f_{n_1}(x)$ in the interval (P, Q), there is certainly a point of this interval where $f_{n_1}(x) > \alpha$; hence, since f_{n_1} is continuous, there is a whole interval (A_1, B_1) internal to (A, B), at every point x of which

$$\alpha < f_{n_1}(x), \tag{5}$$

while, of course, the relations (4) hold throughout the interval.

By precisely the same argument, using χ instead of π, and interchanging the signs $>$ and $<$, we shew that there is an interval (A_1', B_1') inside (A, B), such that at every point x of it

$$f_{n_1'}(x) < \beta, \tag{6}$$

n_1' being also an integer greater than the assigned integer.

We now take the interval (A_1, B_1) and shew, by the same argument, that there is inside it an interval (A_2', B_2') at every point of which

$$f_{n_2'}(x) < \beta,$$

n_2' being a certain integer greater than n_1.

Similarly, inside this we get an interval (A_3, B_3), at every point of which

$$\alpha < f_{n_3}(x).$$

Proceeding thus we get a countably infinite set of intervals, each lying inside the preceding,

$$(A_1, B_1),\ (A_2', B_2'),\ (A_3, B_3),\ (A_4', B_4'),\ \ldots,$$

and a corresponding series of constantly increasing integers

$$n_1,\ n_2',\ n_3,\ n_4',\ \ldots,$$

such that for the intervals and integers denoted by dashed letters we have relations of the form (6), and for the others of the form (5).

Now, there is at least one point internal to all these intervals and at this point we shall have both

$$f(x) = \underset{i=\infty}{\mathrm{Lt}}\, f_{n_i}(x) \geqslant a,$$

and, using (3), $$f(x) = \underset{i=\infty}{\mathrm{Lt}}\, f_{n'_i}(x) \leqslant \beta < a,$$

which is impossible, $f(x)$ being, by hypothesis, determinate. Thus the supposition made at the beginning is untenable, which, by a *reductio ad absurdum*, proves the theorem.

COR. 1.—*The f_n's being continuous, the points at which*

$$\pi > \chi$$

*form a set of the first category; in other words, the points of non-uniform convergence and divergence form a set of the first category; this set is none other than the set of points at which one at least of the functions χ and π is discontinuous, and is therefore an ordinary outer limiting set.**

It should be noted that in an interval in which there are no points of uniform divergence, $\pi = \chi$ is a null function whose zeros are its points of continuity and are the points of uniform convergence of the series in that interval.

COR. 2.—*In an interval throughout which the series converges, the points at which*

$$\pi = +\infty$$

form a closed set nowhere dense; the same is true of the points at which

$$\chi = -\infty.$$

First, either of these sets must be closed because the peak function is upper semi-continuous and the chasm function is lower semi-continuous.† Hence, unless nowhere dense, either of these sets would fill up an interval, and therefore throughout that interval π would be greater than f or f greater than χ, that is, in either contingency π would be greater than χ throughout that interval, contrary to Cor. 1.

* That is, the outer limiting set of a series of closed sets.

† See my paper in the *Quarterly Journal* already cited.

15. THEOREM 11.—*The points of uniform divergence of a series of continuous functions form an inner limiting set.*

At a point of uniform divergence either

$$\chi = \pi = +\infty \quad \text{or} \quad \chi = \pi = -\infty.$$

Consider the first of these sets; since χ is a lower semi-continuous function, the points $\chi = +\infty$ form an inner limiting set.* But, if

$$\chi = +\infty, \qquad \pi = +\infty;$$

since

$$\pi \geqslant \chi.$$

Thus the first set is an inner limiting set; similarly, the second set is an inner limiting set, and therefore the sum of the two sets is an inner limiting set.

COR.—*If a series of continuous functions diverges at a set of points dense everywhere in an interval, it diverges uniformly at points which form a set of the second category.*

16. In § 13 it was proved that the points at which there can be a distinction of right and left with respect to non-uniform convergence are at most countable. It remains to shew, by an example, that this is the utmost that can be said. In the one we proceed to give, it will be noticed that all four peak and chasm functions are different from one another and from f at every point of a countable set which is everywhere dense in the segment (0, 1).

EX.—Let us construct the continuous function f_n as follows:—

Divide the segment (0, 1) of the x-axis into ten parts. At the first point of division, let the value assigned to f_1 be any chosen quantity A_0; at the second point of division ·2, let the value be any chosen quantity B_0; at the last point of division ·9, let the value be any chosen quantity A_1; and at ·8, any chosen quantity B_1. At the remaining points of division the values assigned are as follows:—At ·5 the value of f_1 is ·5, and this value will remain fixed for every f_n; at ·3 it is ·8; at ·4 it is ·2; at ·6 it is ·4; at ·7 it is ·6.

We then draw a polygonal line starting from the point (0, 0) and passing in order from left to right through the points whose ordinates are the values of f_1 at the points of division, and ending at the point

* *Loc. cit.* in the preceding note.

(1, 1); in the figure, the first two and the last two straight lines are not drawn; the values of the A's and B's may be such that they do not lie entirely within the square. The equation to this polygonal line is to be

$$y = f_1(x).$$

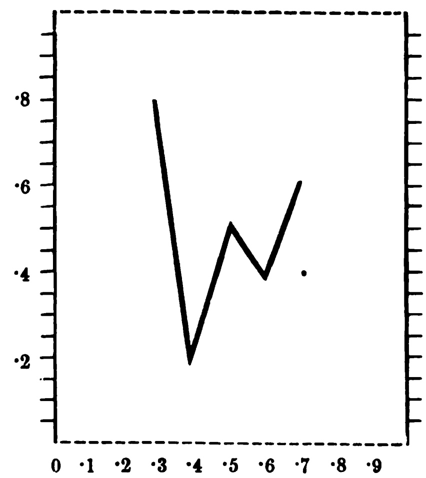


FIG. 1.

Everywhere outside the interval (0, 1), f_1 is to be zero. We may conveniently write symbolically

$$f_1(x) = F(x\,;\ 0,\ 1\,;\ A_0,\ B_0,\ A_1,\ B_1),$$

and a function constructed on the same principle, but to a different scale, the interval being (a, b) instead of (0, 1), and the left-hand bottom corner of the square the point (a, a) instead of (0, 0), will be denoted by

$$F(x\,;\ a,\ b\,;\ A_a,\ B_a,\ A_b,\ B_b).$$

The function f_2 is now the sum of two such functions, corresponding to the two halves of the interval (0, 1), which determined f_1. The quantities A_0, B_0, A_1, B_1 are those used in constructing f_1; the quantities $A_{\cdot 5}$, $B_{\cdot 5}$ corresponding to each interval are determined by the values assigned at the four points of division nearest to ·5 in constructing f_1; we have, in fact,

$$f_2 = F(0,\ \cdot 5\,;\ A_0,\ B_0,\ \cdot 2,\ \cdot 8) + F(\cdot 5,\ 1\,;\ \cdot 4,\ \cdot 6,\ A_1,\ B_1).$$

In each half interval we now repeat the construction, and so get f_3 defined, and then f_4, and so all the f_n's in succession. The value of f_n at the point $\frac{1}{2}$ will then always be $\frac{1}{2}$, and the corresponding peaks and chasms will always be ·8 and ·2 on the left and ·6 and ·4 on the right. Thus

f will be $\frac{1}{2}$ at the point $\frac{1}{2}$, π_L will be $\cdot 8$, χ_L will be $\cdot 2$, π_R will be $\cdot 6$, and χ_R will be $\cdot 4$.

The mode of construction shews that the same will be true with different numbers at all the rational points whose denominators are powers of 2. The difference between any two of the five functions at any point is the same as the same difference at any other point with the same power of 2 as denominator, while this difference is halved when the power of 2 is increased by 1; the value of the limiting function f at any such point x will then itself be x.

If x be any number other than one of the fractions whose denominators are powers of 10, it follows therefore that the values of the peak and chasm functions corresponding to the end-points of the interval (a, b), to which x is internal at each successive stage, will differ by less and less and have the same limiting values. In other words, the segments of the broken line f_n corresponding to this interval will become shorter and shorter without limit. Taking, therefore, any little interval with our point x as end-point, the upper and lower limits of f_n will more and more nearly coincide the greater n is, and this will be still more so the smaller the little interval; there is no positive lower limit to this limit, hence the peak and chasm functions at x coincide at x. Thus x is a point of uniform convergence of the series, and therefore a point of continuity of the limiting function f; it follows that throughout the interval (0, 1) considered

$$f(x) = x.$$

17. We now proceed to give examples showing that the results obtained in Cor. 2 of § 14 are the utmost that can be stated with regard to the points in question. We content ourselves with giving a series for which π_L and π_R are both $+\infty$ and χ_L and χ_R both $-\infty$ at any arbitrarily selected closed set of points nowhere dense. It is at once obvious how the construction may be modified so as to give a series for which χ, for example, is everywhere finite and π everywhere infinite at the points of such a set.

Ex. 2.—We only need to shew how to construct a function uniformly convergent throughout an open interval and having at both end-points

$$\chi = -\infty, \qquad \pi = +\infty,$$

f_n having at the end-points any the same assigned value which has a definite limit as n increases indefinitely.

If we can do this, we can do it for every black interval of any given closed set; we can then ascribe to f_n the above selected value at every

remaining point of the closed set. We thus have a series of continuous functions defined for the whole segment whose peak and chasm functions are respectively $+\infty$ and $-\infty$ at every end-point of a black interval, and therefore at all the limiting points of the set of these end-points, that is, at all the points of the given closed set. In this example, f will be usually discontinuous, but it is easy to arrange it otherwise if we please.

Take the interval (0, 1) and let

$$f_n = \frac{n^2x+n}{1+n(nx+1)^2}$$

from 0 to $\frac{1}{2}$ both inclusive, while from $\frac{1}{2}$ to 1 we use the same expression, changing x into $(1-x)$.

Here f is zero except at the extremities of the interval where it is unity.

Or, again, put

$$f_n = \frac{n^{\frac{3}{2}}x+n^{\frac{1}{2}}}{1+(n^{\frac{3}{2}}x+n^{\frac{1}{2}})^2}$$

from 0 to $\frac{1}{2}$ both inclusive, while from $\frac{1}{2}$ to 1 we use again the same expression, changing x to $(1-x)$.

Here f is always zero, and therefore continuous. At the ends of the interval the peak and chasm functions are in both cases $+\infty$ and $-\infty$ respectively.

18. Can $\pi = +\infty$ throughout an interval?

If so, the series must, as we have seen, be divergent at a set of points dense everywhere in that interval. With this *proviso* it is easy to construct such a series. Such a series, for example, is that given by Borel,*

$$\frac{A_1}{r_1} + \frac{A_2}{r_2} + \ldots,$$

where $r_1, r_2, \ldots$ are the distances, taken positively, of the point x from the rational points of the segment (0, 1) arranged in countable order, and the A's are constants suitably chosen, viz., in such a way that the series whose general term is A_n converges.

Notice further that it follows from Theorem 11 that this series diverges uniformly at a set of the second category.

19. For the sake of completeness we shew how to form a series in which the peak and chasm functions as well as the limiting functions are

* Borel, *Comptes Rendues*, Vol. CXVIII., p. 540; Schoenflies's *Bericht*, p. 243. See also my paper in the *Messenger of Mathematics*, Vol. XXXVII., "On a New Proof of a Theorem of Baire's."

all infinite, say $= +\infty$, throughout an interval, so that the series diverges uniformly at every point of the interval. We have merely to take any series which converges uniformly to zero positively, say

$$u_0+u_1+\dots,$$

the required series is $\qquad v_0+v_1+\dots,$

where
$$v_n = \frac{1}{u_0+u_1+\dots+u_{n+1}} - \frac{1}{u_0+u_1+\dots+u_n}.$$

We content ourselves with the examples already given; it is not difficult to construct others illustrating more fully the various theorems above given; for instance, the fundamental one with respect to the distinction of right and left. Thus we could arrange that the series diverges uniformly on the right and non-uniformly on the left at every point of a countable set nowhere dense.

20. All these theorems still hold *mutatis mutandis* if we consider the behaviour of the series at points of a perfect set of points contained in the segment in which the functions are defined, instead of at all the points of that segment itself. We shall have, of course, to replace "uniform convergence at a point" by "uniform convergence at a point with respect to the perfect set." The new peak and chasm functions relative to the perfect set will be obtained by letting the variable point Q describe not the continuum, but the perfect set in the neighbourhood of a point P of the set.

Thus, for instance, the new and extended form of Theorem 10 gives us the following statement:—

The f_n's being continuous, the series is only pointwise non-uniformly convergent (including divergent) with respect to every perfect set.

Using the result of § 5 relative to a perfect set, this includes Baire's theorem that *the limiting function f of a series of continuous functions f_n is only pointwise discontinuous with respect to every perfect set.*

NODAL CUBICS THROUGH EIGHT GIVEN POINTS

By J. E. WRIGHT.

[Received August 1st, 1907.—Read November 14th, 1907.]

IN the pencil of cubics that passes through eight given points there are in general twelve with nodes. In certain special cases there may be two or more on one particular cubic of the pencil, and then that one is reducible. If, for instance, two lie on the same cubic, that cubic consists of a straight line and a conic. In other cases the number of nodal cubics may be reduced, due to coincidences of nodes. We wish to consider particularly the general pencil from which these particular cases are excluded. The questions with which we are concerned as regards their nodes are those of *analysis situs*; if a certain number of the eight points are real, we wish to know how many of the nodes are real crunodes, how many real acnodes, and how many imaginary. The case that lends itself most easily to algebraic treatment is that where all the eight base points, and therefore the ninth, are real, but it will be obvious from the first portion of the paper that pure algebra involves too heavy work for one to hope for much progress along that line. The results obtained are got by first mapping the plane containing the cubics on a cubic surface in space, by then projecting this cubic surface on a plane from a point on it, and finally by consideration of the tangents of the quartic thus obtained.

First, however, we consider a purely algebraic method in the case when the nine base points are real. Assume in this case that the vertices of the fundamental triangle are nodes of three of the nodal cubics through nine real points. These cubics are

$$C_1 \equiv (a_0 a_1 a_2 a_3 \between yz)^3 + 3\,(b_0 b_1 b_2 \between yz)^2\, x = 0,$$

$$C_2 \equiv (a_0' a_1' a_2' a_3' \between zx)^3 + 3\,(b_0' b_1' b_2' \between zx)^2\, y = 0,$$

$$C_3 \equiv (a_0'' a_1'' a_2'' a_3'' \between xy)^3 + 3\,(b_0'' b_1'' b_2'' \between xy)^2\, z = 0,$$

and since they have nine points common we may take, without loss of generality

$$C_1 + C_2 + C_3 \equiv 0.$$

It is immediately clear from this condition that if we write

$$P \equiv x(a_1x+b_1y+c_1z)+m_1yz,$$
$$Q \equiv y(a_2x+b_2y+c_2z)+m_2zx,$$
$$R \equiv z(a_3x+b_3y+c_3z)+m_3xy,$$

and if we change the notation, we may write C_1, C_2, C_3, in the form

$$C_1 \equiv yQ-zR, \quad C_2 \equiv zR-xP, \quad C_3 \equiv xP-yQ.$$

At a common point of C_1, C_2, C_3, we have

$$xP = yQ = zR.$$

We assume that this point (x, y, z) is not on a side of the fundamental triangle, and write

$$xP = yQ = zR = \lambda xyz.$$

The value of λ is then determined by eliminating x, y, z from $P = \lambda yz$, $Q = \lambda zx$, $R = \lambda xy$ and solving the equation thus obtained. This equation is of the ninth order and to a real value of λ corresponds a real point (x, y, z) and to a real point (x, y, z) a real value of λ. Hence, if all the intersections of C_1, C_2, C_3 are real, all the roots of the equation in λ are real. We proceed to find this equation.

Firstly, $$x(a_1x+b_1y+c_1z)+(m_1-\lambda)yz = 0, \tag{1}$$

$$y(a_2x+b_2y+c_2z)+(m_2-\lambda)zx = 0, \tag{2}$$

$$z(a_3x+b_3y+c_3z)+(m_3-\lambda)xy = 0. \tag{3}$$

From (2) and (3),

$$(a_2x+b_2y+c_2z)(a_3x+b_3y+c_3z) = (m_2-\lambda)(m_3-\lambda)x^2, \tag{4}$$

and there are two equations similar to this.

Write $m_1-\lambda = \lambda_1, \ldots,$ and assume $m_1+m_2+m_3 = 0$. This assumption involves no loss of generality. (4) becomes

$$(a_2a_3-\lambda_2\lambda_3)x^2+b_2b_3y^2+c_2c_3z^2+(b_2c_3+b_3c_2)yz+(c_2a_3+c_3a_2)zx$$
$$+(a_2b_3+a_3b_2)xy = 0.$$

By means of (2) and (3) this may be reduced to

$$(a_2a_3-\lambda_2\lambda_3)x^2-b_3\lambda_2zx-c_2\lambda_3xy+(b_2c_3-b_3c_2)yz+c_3a_2zx+a_3b_2xy = 0,$$

or, from (1),

$$(a_2a_3-\lambda_2\lambda_3)\left\{\frac{-\lambda_1yz-b_1xy-c_1xz}{a_1}\right\}-(b_3\lambda_2-c_3a_2)zx-(c_2\lambda_3-a_3b_2)xy$$
$$+(b_2c_3-b_3c_2)yz = 0,$$

or
$$\{(a_2a_3-\lambda_2\lambda_3)\lambda_1-(b_2c_3-b_3c_2)a_1\}\,yz$$
$$+\{(a_2a_3-\lambda_2\lambda_3)c_1+a_1(b_3\lambda_2-c_3a_2)\}\,zx$$
$$+\{(a_2a_3-\lambda_2\lambda_3)b_1+a_1(c_2\lambda_3-a_3b_2)\}\,xy=0.$$

From this and two similar equations we have a three row determinant equated to zero as the equation for λ, by eliminating yz, zx, xy.

This equation when multiplied out is
$$-\lambda_1^3\lambda_2^3\lambda_3^3+2\lambda_1^2\lambda_2^2\lambda_3^2(a_2a_3\lambda_1+b_3b_1\lambda_2+c_1c_2\lambda_3)$$
$$+\text{terms involving } \lambda^6 \text{ and lower powers of } \lambda=0,$$
or
$$\lambda^9+\{3(m_2m_3+m_3m_1+m_1m_2)-2(a_2a_3+b_3b_1+c_1c_2)\}\lambda^7$$
$$+\text{lower powers of } \lambda=0.$$

Since this equation has all its roots real, by Sturm's theorem
$$3(m_2m_3+m_3m_1+m_1m_2)-2(a_2a_3+b_3b_1+c_1c_2) \text{ is negative.}$$

Now $C_1\equiv y^2(a_2x+b_2y+c_2z)-z^2(a_3x+b_3y+c_3z)+(m_2-m_3)xyz=0$ has a crunode or an acnode at $y=0$, $z=0$, according as $(m_2-m_3)^2$ is greater or less than $-4a_2a_3$. If the three nodes considered are all acnodes
$$(m_2-m_3)^2+(m_3-m_1)^2+(m_1-m_2)^2 \text{ is } <-4(a_2a_3+b_3b_1+c_1c_2).$$

Now $\Sigma(m_2-m_3)^2=2\Sigma m_1^2-2\Sigma m_2m_3=2(\Sigma m_1)^2-6\Sigma m_2m_3=-6\Sigma m_2m_3$.

Hence for three acnodes
$$3\Sigma m_2m_3 \text{ is } >2(a_2a_3+b_3b_1+c_1c_2).$$

But this is impossible, by the result just obtained. Hence there cannot be more than two real acnodes. We therefore have the theorem—

If a pencil of cubics have nine real points common, it cannot include more than two acnodal cubics.

As further progress on these lines seems difficult, we start afresh and consider four independent cubics through six of the points. We use Clebsch's transformation* by means of the linear system thus determined, that is to say, if C_1, C_2, C_3, C_4 are the four independent cubics we take coordinates in space x_1, x_2, x_3, x_4 proportional to C_1, C_2, C_3, C_4.

The points in the plane are now mapped on a cubic surface in space, and the correspondence between the two is (1, 1) and is real. The correspondence breaks down for each of the six base points, and each of

* Clebsch, *Crelle's Journal*, Vol. LXV. (1866), p. 359.

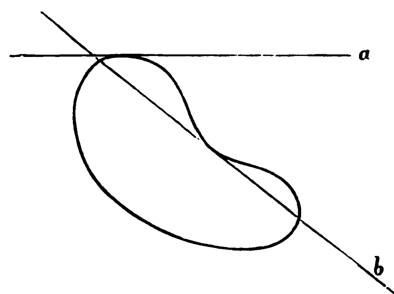
these corresponds to a real straight line on the surface. The other 21 lines on the surface are in this case all real, and six of them correspond to the six conics through five of the six base points. The remaining 15 correspond to the 15 lines joining the six points in pairs. If 2, 4, or 6 of the base points are pairs of conjugate imaginaries, the correspondence exists as before, and in these cases it is clear that the cubic surface has on it 15, 7, or 3 real straight lines respectively. Further, if any non-singular cubic surface is given, a real transformation of this kind can always be found between it and a plane, provided the surface contains two real non-intersecting straight lines. The only case of failure arises therefore when all the real straight lines on the surface lie in a plane. This case can only occur when there are not more than three such lines and these lie in a plane. Hence incidentally we see that of the lines on a cubic surface either 27, 15. 7, or 3 or less are real. It is clear that in the cases where the correspondence can be established the cubic surface has only one sheet (is unipartite) for the correspondence with the plane is (1, 1) and is real. But, obviously, bipartite cubic surfaces can exist, and hence if such a surface be bipartite it can have at most three real lines on it and these must lie in a plane. There are, of course, some surfaces with only three lines on them, lying in a plane, for which the correspondence exists, and these are unipartite. It does not follow that the correspondence can be established for every such unipartite surface. [This is, however, true.—*January 4th*, 1908.]

Now consider any cubic of the pencil. It corresponds to a section of the cubic surface by a plane. Two cubics intersect in three points outside the base points, and these clearly correspond to the three points in which the common line of the two planes in space meets the cubic surface. Hence the pencil of cubics through nine real points corresponds to sections of the surface by planes through a line which meets the surface in three real points. If only seven base points are real the correspondence is of the same kind, except that the axis of the planes meets the surface in one real point.

Now, take one of the three points on the axis and project the surface on to any plane. The result will be that the plane is divided into portions, for one of which all projectors meet the surface in two other real points, while for the remainder the projectors meet the surface in no other real points.

The bounding curve is known to be a quartic. Now consider any nodal cubic in the original plane. It is clear from the nature of the correspondence that the plane section of the cubic surface will also be a nodal cubic, and further that a crunodal cubic will give a crunodal cubic, and an

acnodal an acnodal. The nodal cubics thus correspond to the tangent planes through the given line, and thus finally to tangents to the quartic through a given point. Now the lines on the cubic surface are known to be double tangents of the quartic, and there is one other which is always real. Further, a quartic curve possesses four and only four double tangents called by Zeuthen *of the first kind*, and the remainder are of the second kind. Those of the second kind touch two different branches of the quartic, and have, of course, real contacts. The four of the first kind either have imaginary contacts or their two contacts are on the same branch of the quartic. Now each pair of external ovals of the quartic gives rise to four double tangents of the second kind, and hence the quartic has 1, 2, 3, or 4 external ovals according as the cubic surface has 3, 7, 15, or 27 real lines on it. Hence in the case we are considering at present the quartic has four ovals, all external. Hence, if we imagine an eye placed at the centre of projection of the cubic surface, the surface will appear to have four holes through it. If one hole be that given, a tangent plane such as a will obviously give rise to a crunodal section, whilst one



such as b will give rise to an acnodal section. Also the point of intersection of the tangents is certainly in the region external to the four ovals.

The following results are immediately obvious :—

(1) There are eight tangents of the type a, two to each oval.

(2) There may be one or two tangents of the type b, but in this case the quartic must have at least two or four inflexions.

(3) Each tangent of the type b carries an additional tangent of the type a, for then four tangents may be drawn from the point to that oval.

(4) As there cannot be more than twelve tangents altogether, there cannot be more than two of the type b.

The case when the axis of the planes meets the cubic surface in one real point may be similarly discussed. In this case the point from which

the tangents to the quartic are drawn is inside one of the ovals. The cases now are—

(5) Six of type a.

(6) One of type b and seven of type a.

(7) Two of type b and eight of type a, and this case implies the existence of two inflexions on the oval in which the point lies.

(8) Possibly three of type b and nine of type a, and the existence of four inflexions is implied on the oval in which the point lies.

By taking two of the six original base points as a pair of conjugate imaginaries we may treat this case by means of a quartic with three ovals and a point external to them. In a similar manner we may deal with cases where 4, 6, or 8 of the intersections of a pencil of cubics are imaginary, by means of quartics with three, two, or one external ovals, and we have the following final results.

Let R denote the number of real base points, C the number of real crunodes, A the number of real acnodes, then

(i.) $$C - A = R - 1.$$

(ii.) *If R is 9, A may be 1 or 2.*

(iii.) *If R is > 1, the question is exactly the same as that of the number of real tangents that may be drawn to a quartic with 2, 3, or 4 ovals.*

(iv.) *If $R = 1$, the number of real crunodes is equal to the number of real acnodes, but we cannot give an upper or lower limit to the number by this method.*

The correspondence established leads to some other results. We see that if two of the nine base points of two cubics coincide, we have a point on the quartic as the point from which tangents are drawn, and it follows that the node at the double base point counts for two in the twelve. Further, a cusp arises for an inflexional tangent, and therefore a cusp counts for two of the twelve nodes. It is also clear that a cusp arises from the coincidence of a crunode and an acnode.

[*Added January 4th*, 1908.—In view of the addition on p. 55, it is clear that, if $R = 1$, the possible number of real double points is the same as the possible number of real tangents that can be drawn to a quartic consisting of a single oval, from a point inside that oval.]

VARIOUS EXTENSIONS OF ABEL'S LEMMA

By T. J. I'A. BROMWICH.

[Received June 5th, 1907.—Read June 13th, 1907.]

THE following paper contains a collection of various inequalities which are all, in a certain sense, extensions of Abel's lemma, that if the sequence of factors (v_n) is real, positive and decreasing, then

$$hv_1 < \sum_1^p a_n v_n < Hv_1,$$

where H, h are the upper and lower limits of

$$a_1+a_2+\ldots+a_n,$$

as n varies from 1 to p.

These results do not seem to have been published in a general form hitherto, although no doubt special cases have been used by many authors. A systematic use of them has enabled me to shorten the proofs of a number of known theorems on limits, and to obtain various extensions of such theorems. Some of these applications are given in connexion with each of the inequalities obtained below; of these the only actual novelties appear to be the theorems on divergent series given in §§ 1 and 4–7, and some of the results on double series in § 5.

1. *Real, Decreasing Positive Factors.*

Suppose that the sequence (v_n) consists of positive terms only, and *never increases*, then by the familiar transformation (due to Abel) we have

$$(1) \qquad \sum_1^p a_n v_n = s_1(v_1-v_2)+s_2(v_2-v_3)+\ldots+s_{p-1}(v_{p-1}-v_p)+s_p v_p,$$

where $$s_n = a_1+a_2+\ldots+a_n.$$

Let m be any index less than p and take H, h to denote the upper and lower limits of $s_1, s_2, \ldots, s_{m-1}$, while H_m, h_m denote those of $s_m, s_{m+1}, \ldots, s_p$.

Then the sum on the right of (1) is increased if we put H in place of

$s_1, s_2, \ldots, s_{m-1}$, and H_m in place of $s_m, s_{m+1}, \ldots, s_p$, because all the factors $v_1 - v_2, v_2 - v_3, \ldots, v_{p-1} - v_p, v_p$ are positive. Thus

$$\sum_1^p a_n v_n \leqslant H[(v_1 - v_2) + (v_2 - v_3) + \ldots + (v_{m-1} - v_m)]$$
$$+ H_m[(v_m - v_{m+1}) + (v_{m+1} - v_{m+2}) + \ldots + (v_{p-1} - v_p) + v_p]$$

or $$\sum_1^p a_n v_n \leqslant H(v_1 - v_m) + H_m v_m.$$

By a similar argument with regard to h, h_m, we establish the complete inequality

(2) $$h(v_1 - v_m) + h_m v_m \leqslant \sum_1^p a_n v_n \leqslant H(v_1 - v_m) + H_m v_m,$$

which is *the extended form of Abel's inequality.** We get back to Abel's result by taking $m = 1$.

Applications.—The inequality (2) leads at once to the cases of chief practical interest of the generalized form of Abel's theorem given by Mr. Hardy.† Suppose, in fact, that the factor v_n is a function of a variable x, and that $v_n(x)$ tends to the limit 1 as x tends to 1, while $v_0 \geqslant v_1 \geqslant v_2 \geqslant \ldots$, for values of x less than 1.

Then, *if Σa_n converges to a sum s*, we can choose m so that h_m, H_m lie between $s - \epsilon, s + \epsilon$, however small ϵ may be, and however great p may be. Thus (2) leads to

$$h(v_0 - v_m) + (s - \epsilon) v_m \leqslant \sum_0^x a_n v_n \leqslant H(v_0 - v_m) + (s + \epsilon) v_m.$$

Now, as x tends to 1, the right and left sides of the last inequality tend respectively to $(s - \epsilon)$ and $(s + \epsilon)$, since v_0 and v_m both tend to 1. We have therefore

$$s - \epsilon \leqslant \varliminf_{x \to 1} \Sigma a_n v_n \leqslant \varlimsup_{x \to 1} \Sigma a_n v_n \leqslant s + \epsilon.$$

Since ϵ is arbitrarily small, these inequalities cannot be true unless

$$\lim_{x \to 1} \sum_0^x a_n v_n = s.$$

But *when Σa_n is divergent*, m can be found so that $h_m > N$, however great N is; and so

$$\sum_0^x a_n v_n \geqslant h(v_0 - v_m) + N v_m.$$

Repeating the foregoing argument we see that

$$\varliminf_{x \to 1} \Sigma a_n v_n \geqslant N.$$

Hence $$\lim_{x \to 1} \sum_0^\infty a_n v_n = \infty,$$

a result which appears to be novel, although an immediate extension of one due to Abel. As a simple example we note that

$$\Sigma \frac{x^n}{1 + x^n}, \quad \Sigma \frac{1}{n} \frac{x^n}{1 + x^n},$$

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* If $\Sigma a_n v_n$ is separated into two parts, from 1 to $m - 1$, and from m to p, Abel's inequality can be applied to each part; but the limits obtained are not so close as in (2).

† *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, p. 249 (especially § 3).

tend to infinity as x tends to 1. Of course this conclusion is verified at once by the obvious inequalities

$$\Sigma \frac{x^n}{1+x^n} > \tfrac{1}{2}\Sigma x^n = \tfrac{1}{2}\,\frac{1}{1-x};$$

$$\Sigma \frac{1}{n}\cdot\frac{x^n}{1+x^n} > \tfrac{1}{2}\Sigma \frac{x^n}{n} = \tfrac{1}{2}\log\left(\frac{1}{1-x}\right).$$

The inequality (2) can also be used to establish the comparison theorems for divergent series to which we shall be led later (see § 6).

There is an inequality corresponding to (2) in the case of *increasing* factors, but this seems to be of less practical importance; we record the result without proof beyond the remark that the factors v_1-v_2, v_2-v_3, $\ldots$, $v_{p-1}-v_p$ are *negative* in (1). We then find

$$Hv_1-(H-H_m)\,v_m-(H_m-h_m)\,v_p < \sum_1^p a_n v_n$$
$$< hv_1+(h_m-h)\,v_m+(H_m-h_m)\,v_p.$$

In particular, with $\qquad h_m = h \quad$ and $\quad H_m = H,$

we find $\qquad Hv_1-(H-h)\,v_p < \sum\limits_1^p a_n v_n < hv_1+(H-h)\,v_p.$

2. *Inequalities for Integrals corresponding to* § 1.

The analogy between Abel's inequality and the so-called *second theorem of the mean* at once suggests the following theorem:—

If the function $v(x)$ *never increases with* x, *but is always positive in an interval* (a, b), *then*

$$(3)\quad h\left[v(a)-v(c)\right]+h_c v(c) \leqslant \int_a^b v(x)\,f(x)\,dx \leqslant H\left[v(a)-v(c)\right]+H_c v(c),$$

where H, h *are the upper and lower limits of the integral*

$$\int_a^\xi f(x)\,dx$$

as ξ *ranges from* a *to* c, *while* H_c, h_c *are those found as* ξ *ranges from* c *to* b. *Here* $v(a)$ *and* $v(c)$ *are used to denote the limits* $v(a+0)$ *and* $v(c-0)$ *respectively.*

If the function $v(x)$ is supposed differentiable the inequality (3) is most easily proved by integration by parts (compare p. 65 below); but, in the general case, the inequality can be obtained by a simple modification of Pringsheim's proof* for the case $c = b$.

Let the interval (a, b) be divided into n sub-intervals by inserting

* *Münchener Sitzungsberichte*, Bd. xxx., 1900, p. 209.

points $x_1, x_2, \ldots, x_{n-1}$, and let $x_0 = a$, $x_n = b$; write further $v_r = v(x_r)$, or if $v(x)$ is discontinuous at x_r, we take v_r as the limit* of $v(x)$ as x approaches x_r from *smaller* values of x.

Then, if $J = \int_a^b v(x) f(x)\, dx, \quad J_r = \int_{x_r}^{x_{r+1}} v(x) f(x)\, dx,$

and
$$K_r = \int_{x_r}^{x_{r+1}} f(x)\, dx,$$

we find
$$J = \sum_{r=0}^{n-1} J_r,$$

and
$$J_r - v_{r+1} K_r = \int_{x_r}^{x_{r+1}} [v(x) - v_{r+1}] f(x)\, dx.$$

In the last integral the bracket is positive and less than $v_r - v_{r+1}$, in virtue of the decreasing property of $v(x)$; thus

$$| J_r - v_{r+1} K_r | < (v_r - v_{r+1}) \int_{x_r}^{x_{r+1}} |f(x)|\, dx.$$

Consequently if μ is the maximum value of

$$\int_{x_r}^{x_{r+1}} |f(x)|\, dx$$

for all the sub-intervals, we find

$$| J_r - v_{r+1} K_r | < \mu (v_r - v_{r+1}).$$

Hence
$$\left| J - \sum_{r=0}^{n-1} v_{r+1} K_r \right| < \mu v_0.$$

Now, if we take x_m to coincide with c, we see from the inequality (2) of § 1 that

$$h[v(x_1) - v(c)] + h_c v(c) < \sum_{r=0}^{n-1} v_{r+1} K_r < H[v(x_1) - v(c)] + H_c v(c),$$

because
$$K_0 + K_1 + \ldots + K_{r-1} = \int_a^{x_r} f(x)\, dx.$$

Consequently we have

$$h[v(x_1) - v(c)] + h_c v(c) - \mu v(a) < J < H[v(x_1) - v(c)] + H_c v(c) + \mu v(a).$$

---

* That this limit exists follows from the monotonic property of $v(x)$.

Now, let all the sub-intervals tend uniformly to zero, then μ also tends to zero, provided that the integral

$$\int_a^b |f(x)|\,dx$$

is convergent; and $v(x_1)$ tends to the limit $v(a+0)$, or $v(a)$ in our present notation. Then, since J is independent of the mode of choosing the sub-intervals, we find

$$h[v(a)-v(c)]+h_c v(c) \leqslant J \leqslant H[v(a)-v(c)]+H_c v(c).$$

Pringsheim has shewn, however, that the absolute convergence of $\int_a^b f(x)\,dx$ is superfluous; and that the convergence of this integral together with that of $\int_a^b v(x)f(x)\,dx$ will suffice to establish the result.

In fact, under these conditions, we can find a *finite* number (p) of intervals enclosing all the discontinuities of $f(x)$, and such that $|L_s| < \epsilon$ and $|L'_s| < \epsilon$, where L_s, L'_s denote the integrals of $f(x)$ and of $f(x)\,v(x)$ respectively taken over the s-th of these intervals.

For any part of the range (a, b) outside these p intervals we can argue as above, and deduce that the corresponding contribution to $(J-\Sigma v_{r+1}K_r)$ is less than μv_0, where μ can be made as small as we please. But for these special intervals, the difference is numerically less than

$$|\Sigma v_{s+1}L| + |\Sigma L'_s| < p\epsilon + |\Sigma v_0 L_s| < p\epsilon(1+v_0),$$

and so we arrive finally at the same inequality as before.

Applications.—The arguments of § 1 need no further alteration in order to establish such theorems as the following:—

If $v(x, t)$ is a decreasing function of x ($t > 0$) which tends to the limit 1, *as t tends to* 0, *then*

$$\lim_{t\to 0}\int_a^\infty v(x, t)\,f(x)\,dx = \int_a^\infty f(x)\,dx,$$

if the latter is convergent. Also $\lim_{t\to 0}\int_a^\infty v(x, t)\,f(x)\,dx = \infty$, *if* $\int^\infty f(x)\,dx$ *diverges to infinity.*

As another application, we consider Jordan's theorem:*—

Let $v(x)$ be a function decreasing† *as x increases from a to b; and let $f(x, t)$ be a function of x, t, such that*

(1) *The integral* $\left|\int_a^\xi f(x, t)\,dx\right| < K$, *where ξ lies between a, b and K is independent of ξ and t.*

(2) *The limit* $\lim_{t\to\infty}\int_a^\xi f(x, t)\,dx$ *is independent of ξ and equal say to L, provided that ξ belongs to any sub-interval (a', b'), from which a is excluded; and the convergence to the limit is uniform in the sub-interval.*

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* *Cours d'Analyse*, t. II., 2me éd., 1894, p. 228.

† By taking the difference of two such functions we pass at once to Jordan's *fonction à variation bornée;* and since the operation of subtraction will not affect the final result, there is no real loss of generality in restricting the function at the start.

Then
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx = Lv(a),$$
where $v(a)$ *denotes the limit of* $v(x)$ *as* x *approaches* a *through larger values.*

For, suppose c to be any number greater than a, then we have, from (3),
$$\int_a^\xi v(x)f(x,t)\,dx \leqslant H[v(a)-v(c)]+H_c v(c) = (H-H_c)[v(a)-v(c)]+H_c v(a),$$
where, for brevity, we suppress the left-hand sides of the inequalities.

Now, in virtue of condition (1), $H-H_c < 2K$, and choose c so as to make $2K[v(a)-v(c)]<\epsilon$, then, since $\lim_{t\to\infty} H_c = L$, we have
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx \leqslant Lv(a)+\epsilon.$$
Similarly the other sides of the inequalities give
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx \geqslant Lv(a)-\epsilon.$$
Thus
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx = Lv(a).$$

Clearly in the foregoing $f(x,t)$ may be complex, since the argument can be applied to the real and imaginary parts separately. Thus we have, for example,
$$\lim_{t\to\infty} t\int_0^\xi e^{-tx}\,dx = \lim_{t\to\infty}(1-e^{-t\xi}) = 1, \quad \text{and so} \quad \lim_{t\to\infty} t\int_0^\xi e^{-tx}v(x)\,dx = v(0), \quad (\xi>0),$$
where t is complex and tends to infinity along any path which makes its real part tend to positive infinity (compare Picard, *Traité d'Analyse*, t. II., 1er éd., p. 171).

3. *Complex Factors.*

If the factors v_n are complex, we assume (following Dirichlet) that the series
$$\sum_1^\infty |v_n - v_{n+1}|$$
is convergent. It follows that the series $\sum_1^\infty (v_n - v_{n+1})$ converges, and therefore v_n tends to a definite limit as n tends to infinity. Write then
$$V_n = \{|v_n-v_{n+1}|+|v_{n+1}-v_{n+2}|+\dots \text{ to } \infty\}+\lim_{\nu\to\infty}|v_\nu|,$$
and it follows that $\quad V_n - V_{n+1} = |v_n - v_{n+1}|.$

Hence $\quad V_n - V_p \geqslant |v_n - v_p|, \quad \text{if } p>n,$

and so $\quad V_n \geqslant |v_n|,$

by making p tend to infinity.

It follows from (1) of § 1 that, if σ is any number (real or complex)
$$(4)\quad \left|\sum_1^p a_n v_n - \sigma v_1\right| < \sum_1^{m-1}\eta(V_n - V_{n+1}) + \sum_m^{p-1}\eta_m(V_n - V_{n+1}) + \eta_m V_p$$
$$= \eta(V_1 - V_m) + \eta_m V_m,$$
where η, η_m are the upper limits to $|s_n - \sigma|$ as n ranges from 1 to $m-1$, and from m to p respectively.

Applications.—We can extend the argument of the small type on p. 59 to this case, *provided that* Σa_n *is convergent.*

For suppose that $\sum_0^\infty a_n = \sigma$, and that $\lim_{x\to 1} v_n = 1$, so that

$$\lim_{x\to 1} (V_0 - V_m) = \lim_{x\to 1} \{|v_0 - v_1| + |v_1 - v_2| + \ldots + |v_{m-1} - v_m|\} = 0.$$

Then, if $\lim_{x\to 1} V_0$ *is finite, we have*

$$\lim_{x\to 1} \sum_0^\infty a_n v_n = \sigma.$$

For we can choose m so as to make $\eta_m V_1$ less than ϵ, and when m is fixed, since η is finite, we have
$$\lim_{x\to 1} \eta (V_0 - V_m) = 0\,;$$

thus we find
$$\overline{\lim_{x\to 1}}\, |\, \Sigma a_n v_n - \sigma \,| < \epsilon,$$

which gives the desired result.

The only fresh condition introduced is that $\lim_{x\to 1} V_0$ must be finite. Thus, for example, with $v_n = x^n$, we find that $\lim_{x\to 1} \dfrac{|1-x|}{1-|x|}$ must be finite, which implies that the path by which x tends to 1 must lie within the inner loop of a certain limaçon.

For, if we write $$x = 1 - \rho e^{i\phi},$$

we find from the condition $$|1-x| \leqslant k\{1-|x|\} \quad (k > 1),$$

the equivalent form $$\rho(k^2-1) \leqslant 2k(1-k\cos\phi),$$

which represents the inner loop of a limaçon, with a node at $\rho = 0$ (*i.e.*, $x = 1$). Stolz and Gmeiner have used the limaçon $k\rho = 2(1-k\cos\phi)$, which is similar to the above curve, but of smaller linear dimensions.

In Pringsheim's paper* the area used is bounded by a circle and two lines which intersect at the point $x = 1$; it will be seen that this area falls within the limaçon.

Similarly, if $$v_n = r^n P_n(\cos\theta),$$

it is proved in my paper just quoted (see § 2, p. 206) that

$$V_0 \leqslant \sqrt{(1 - 2r\cos\theta + r^2)}/(1-r),$$

and so the path of approach to the point $r = 1$, $\theta = 0$ must lie within an area of the unit-circle which is bounded in the same way as for a power-series.

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* *Münchener Sitzungsberichte*, Bd. XXXI., 1901, p. 514. Pringsheim's figure is given also in my paper (Fig. 1), on "Series of Zonal Harmonics" (*Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, p. 204). The limaçon used here is drawn on p. 211 of my book on *Infinite Series*.

On the other hand, *when Σa_n is divergent, we cannot infer that*

$$\lim_{x\to 1} \sum_0^\infty a_n v_n = \infty.$$

In fact the argument of § 1 obviously depends on the fact that v_n is real, and in the simplest case ($v_n = x^n$) Pringsheim has proved that, even when a_n is real and positive and Σa_n diverges, the limit may depend on the path by which x approaches 1. Pringsheim gives as an example the series obtained by rearranging in powers of x the series

$$\exp\left\{\frac{1}{(1-x)^2}\right\} = 1+\frac{1}{(1-x)^2}+\frac{1}{2!}\frac{1}{(1-x)^4}+\frac{1}{3!}\frac{1}{(1-x)^6}+\dots.$$

If this series is denoted by $\Sigma a_n x^n$, it is clear that a_n is positive; and Σa_n diverges, because, if x tends to 1 along the real axis, $1/(1-x)^2$ tends to infinity, so that

$$\lim_{x\to 1} \Sigma a_n x^n = \infty \quad (0<x<1).$$

Now, since a_n is positive, Σa_n must either converge or diverge; and if convergent we should have, by the familiar form of Abel's theorem,

$$\lim_{x\to 1} \Sigma a_n x^n = \Sigma a_n \quad (0<x<1),$$

but this limit is infinity, so that Σa_n must diverge.

But yet, if we write $1-x = \rho e^{i\phi}$, as above (p. 64), we find

$$\left|\exp\left\{\frac{1}{(1-x)^2}\right\}\right| = \exp\left(\frac{1}{\rho^2}\cos 2\phi\right),$$

which tends to zero with ρ, if $\cos 2\phi$ is negative, or if $\phi > \frac{1}{4}\pi$.

It is perhaps natural to enquire if the inequality (4) cannot be modified so as to apply to a *complex integral;* in this case the result is obtained most rapidly by the method of integration by parts. This is permissible here because the function $v(x)$ is supposed analytic and $v(x)$ is therefore differentiable. If we write

$$g(z) = \int_a^z f(x)\,dx,$$

it follows that $$\int_a^b f(x)\,v(x)\,dx = g(b)\,v(b) - \int_a^b g(x)\,v'(x)\,dx,$$

and so if H is the upper limit of $|g(x)|$ on the path of integration, we have

$$\left|\int_a^b f(x)\,v(x)\,dx\right| < HV,$$

where $$V = \int_a^b |v'(x)|\,.\,|dx| + |v(b)|.$$

This method has been recently used by Mr. Berry* to prove that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix}\frac{dx}{x} = 0,$$

* *Messenger of Mathematics*, Vol. XXXVII., 1907, p. 61.

when the path of integration is a semicircle joining the points $-R$, R, and passing through the upper half of the complex plane.

In fact, if $f(x) = e^{ix}$ and $v(x) = 1/x$, we find that

$$V = (\pi+1)/R,$$

and
$$\left| \int_R^x f(x)\,dx \right| = \left| \frac{1}{i}(e^{iz}-e^{iR}) \right| < 2,$$

because
$$|e^{iz}| \leqslant 1,$$

so that
$$\left| \int_{-R}^{R} e^{ix}\frac{dx}{x} \right| < \frac{2(\pi+1)}{R},$$

which gives the desired result.

The same method will give (for the same path)

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix}\frac{P(x)}{Q(x)}\,dx = 0,$$

if $P(x)$ and $Q(x)$ are polynomials in x of which the first is of degree one less than the second.

4. *Inequalities corresponding to those of § 1 for Double Series.*

Suppose that $v_{m,n}$ is a real positive sequence which decreases with respect to *both* indices, in the sense that

$$v_{m,n}-v_{m+1,n} \geqslant 0, \qquad v_{m,n}-v_{m,n+1} \geqslant 0,$$

$$\Delta_{m,n} = v_{m,n}-v_{m+1,n}-v_{m,n+1}+v_{m+1,n+1} \geqslant 0.$$

Then it is known that*

(5) $$\sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n}v_{m,n} = \sum_{m=1}^{p-1}\sum_{n=1}^{q-1}\Delta_{m,n}s_{m,n} + \sum_{m=1}^{p-1}\Delta_m s_{m,q} + \sum_{n=1}^{q-1}\Delta_n s_{p,n} + s_{p,q}v_{p,q},$$

where
$$\Delta_m = v_{m,q}-v_{m+1,q}, \qquad \Delta_n = v_{p,n}-v_{p,n+1}.$$

Here, using the ordinary geometrical representation, $s_{m,n}$ denotes the sum of all the terms contained within a rectangle whose sides are m and n. It should, perhaps, be remarked that (5) is an algebraical *identity*, and does not depend on the preceding inequalities.

Now suppose that for all values of m and n between 1, p and 1, q respectively, the upper and lower limits of $s_{m,n}$ are H, h; then since $\Delta_{m,n}$, Δ_m, Δ_n, $v_{p,q}$ are all positive it follows at once from (5) that

(6) $$hv_{1,1} < \sum_1^p\sum_1^q a_{m,n}v_{m,n} < Hv_{1,1},$$

which is the immediate extension to double series of the ordinary form of Abel's lemma. To see that (6) is correct, we need only note that to put

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 124; from the results given there it is easy to infer the truth of our inequalities for any number of variables of summation.

$s_{m,n} = H$ is equivalent to writing H in place of $a_{1,1}$ and 0 in place of all the other a's.

To obtain the inequality corresponding to (2) of § 1, let us suppose that H_ν, h_ν are the upper and lower limits of $s_{m,n}$ when $m \geqslant \nu$, $n \geqslant \nu$; H, h being the upper and lower limits for $s_{m,n}$ if *either* suffix is less than ν. We then obtain

$$(7) \qquad h(v_{1,1}-v_{\nu,\nu})+h_\nu v_{\nu,\nu} < \sum_1^p \sum_1^q a_{m,n} v_{m,n} < H(v_{1,1}-v_{\nu,\nu})+H_\nu v_{\nu,\nu},$$

since, to obtain the right-hand side, we have to write H_ν for $s_{m,n}$ if $m, n \geqslant \nu$, and otherwise H. But this is equivalent to writing $a_{1,1} = H$, $a_{\nu,\nu} = H_\nu - H$, which gives the right-hand side of (7). Similarly for the left-hand side.

It is possible to extend (7) to complex factors by a method similar to that of § 3.

Applications.—The inequality (7) enables us to give a new proof and extension of results already communicated to the Society.*

Suppose, in fact, that the series $\sum_0^\infty \sum_0^\infty a_{m,n}$ is convergent in Pringsheim's sense and satisfies the condition of finitude,† then if $v_{m,n}$ is a function of x, y which satisfies the inequalities prescribed at the beginning of this article, and tends to the limit 1 as x, y tend to 1, we have

$$\lim_{x,y\to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = s,$$

where s is Pringsheim's sum of the double series $\Sigma\Sigma a_{m,n}$.

For, in fact, we can find ν, so that

$$s-\epsilon \leqslant h_\nu < H_\nu \leqslant s+\epsilon,$$

and

$$-C < h, \qquad H < C,$$

by the condition of finitude.

Thus (7) yields‡

$$-C(v_{0,0}-v_{\nu,\nu})+(s-\epsilon)v_{\nu,\nu} < \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} < C(v_{0,0}-v_{\nu,\nu})+(s+\epsilon)v_{\nu,\nu}.$$

Since $v_{0,0}$ and $v_{\nu,\nu}$ both tend to 1 as x, y tend to 1, we find

$$s-\epsilon \leqslant \overline{\lim_{x,y\to 1}} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \leqslant s+\epsilon.$$

Since ϵ is arbitrarily small, these inequalities can only be true if

$$\lim_{x,y\to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = s.$$

* Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, 1904, p. 161 (see § 3, p. 164); the case discussed there is given by writing $v_{m,n} = x^m y^n$ and supposing $\Sigma\Sigma a_{m,n}$ convergent.

† So that $|s_{m,n}| < C$, where C is independent of m, n.

‡ The convergence in Pringsheim's sense of the double series $\Sigma\Sigma a_{m,n} v_{m,n}$ follows from Hardy's paper quoted on p. 66 above, or can be proved by a direct application of the inequality (6).

Pass next to the case of divergence, say to $+\infty$; *it will be assumed that the divergence is not due to the presence of any singly divergent row or column.* Thus, when ν is fixed we can determine a constant C_ν, such that

$$|s_{m,n}| < C_\nu,$$

provided that *either* of m, n is less than ν; thus, for example, we may have m increasing without limit, provided that $n < \nu$.

Let ν be now found so that

$$s_{m,n} > N, \quad \text{if} \quad m, n \geqslant \nu;$$

this is possible in view of the divergence of the double series $\Sigma\Sigma a_{m,n}$; thus $h_\nu \geqslant N$. Also

$$h \geqslant -C_\nu,$$

and so we have
$$\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} > N v_{\nu,\nu} - C_\nu (v_{0,0} - v_{\nu,\nu}).$$

Thus repeating the former argument, we find

$$\varliminf_{x,y \to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \geqslant N,$$

and so we must have
$$\lim_{x,y \to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = \infty.$$

5. *Inequalities for a Quotient.*

We consider the quotient $X_p = R_p/Q_p$,

where
$$R_p = \sum_1^p b_n v_n, \qquad Q_p = \sum_1^p a_n v_n.$$

For brevity write
$$A_n = a_1 + a_2 + \ldots + a_n,$$
$$B_n = b_1 + b_2 + \ldots + b_n.$$

Then, as in § 1, we find

$$R_p = B_1(v_1 - v_2) + B_2(v_2 - v_3) + \ldots + B_{p-1}(v_{p-1} - v_p) + B_p v_p.$$

Now, suppose that $a_1, a_2, \ldots, a_n, \ldots$ are all positive and consider the sequence of quotients
$$B_1/A_1, \; B_2/A_2, \; \ldots, \; B_p/A_p.$$

Let H, h be the upper and lower limits of the *whole** set of quotients, while H_m, h_m are those for which the suffix is not less than m; so that

$$H \geqslant H_m, \quad \text{and} \quad h \leqslant h_m.$$

Thus, *if the sequence* (v_n) *is positive and decreasing*, we find

$$R_p < H[A_1(v_1 - v_2) + A_2(v_2 - v_3) + \ldots + A_{m-1}(v_{m-1} - v_m)]$$
$$+ H_m[A_m(v_m - v_{m+1}) + A_{m+1}(v_{m+1} - v_{m+2}) + \ldots + A_{p-1}(v_{p-1} - v_p) + A_p v_p].$$

Thus
$$R_p < H_m Q_p + (H - H_m)(Q_m - A_m v_m),$$

* Note the distinction between this case and that of § 1.

and since $H \geqslant H_m$, we find (on including the corresponding expression with h, h_m),

$$h_m Q_p - (h - h_m) Q_m < R_p < H_m Q_p + (H - H_m) Q_m.$$

That is

$$h_m - (h - h_m) \frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < H_m + (H - H_m) \frac{Q_m}{Q_p}. \tag{8}$$

Again, *if the sequence* (v_n) *is positive and increasing*, we find that

$$\begin{aligned} R_p < h[A_1(v_1 - v_2) + A_2(v_2 - v_3) + \ldots + A_{m-1}(v_{m-1} - v_m)] \\ + h_m[A_m(v_m - v_{m+1}) + A_{m+1}(v_{m+1} - v_{m+2}) + \ldots + A_{p-1}(v_{p-1} - v_p)] \\ + H_m A_p v_p, \end{aligned}$$

because here all the differences are *negative*, but v_p is still positive.

Hence, as before, we get

$$R_p < h_m Q_p + (H_m - h_m) A_p v_p + (h_m - h)(A_m v_m - Q_m);$$

and since $h_m - h$ and Q_m are *positive*, we may omit the last term in the last bracket. Thus, summing up, we find

$$\begin{cases} H_m - (H_m - h_m) \dfrac{A_p v_p}{Q_p} - (H - H_m) \dfrac{A_m v_m}{Q_p} < \dfrac{R_p}{Q_p}, \\ \dfrac{R_p}{Q_p} < h_m + (H_m - h_m) \dfrac{A_p v_p}{Q_p} + (h_m - h) \dfrac{A_m v_m}{Q_p}. \end{cases} \tag{9}$$

Finally, *if the sequence* (v_n) *first increases to a maximum* v_μ *and afterwards steadily decreases*, there is no difficulty in modifying the foregoing work to prove that, if $m < \mu$,

$$\begin{cases} h_m - (H - H_m) \dfrac{A_m v_m}{Q_p} - (H_m - h_m) \dfrac{A_\mu v_\mu}{Q_p} < \dfrac{R_p}{Q_p}, \\ \dfrac{R_p}{Q_p} < H_m + (h_m - h) \dfrac{A_m v_m}{Q_p} + (H_m - h_m) \dfrac{A_\mu v_\mu}{Q^p}. \end{cases} \tag{10}$$

We note that the method of § 2 can be at once applied to deduce inequalities for the quotient of two integrals from (8)–(10). Thus, if $f(x)$ is a positive function from a to b and $v(x)$ decreases in the same interval, we can obtain limits for the quotient

$$\int_a^b g(x)\, v(x)\, dx \Big/ \int_a^b f(x)\, v(x)\, dx$$

in terms of those of $\displaystyle \int_a^\xi g(x)\, dx \Big/ \int_a^\xi f(x)\, dx.$

I do not stay to write these out, as the reader should have no difficulty in recognizing the necessary changes in (8)–(10); and up to the present I have not made any practical use of these inequalities.

Applications.—*Comparison Theorem for Divergent Series.*

Suppose that Σa_n is a divergent series of positive terms, and that (v_n) is a decreasing sequence of functions of x, such that

$$\lim_{x\to 1} v_n = 1.$$

Then, if
$$\lim_{n\to\infty} (B_n/A_n) = l,$$

we have also
$$\lim_{x\to 1} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) = l.$$

For then we can choose m so that

$$l-\epsilon \leqslant h_m < H_m < l+\epsilon,$$

and then (8) gives

$$l-\epsilon-(h-l+\epsilon)\frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < l+\epsilon+(H-l-\epsilon)\frac{Q_m}{Q_p}.$$

If $\sum_0^\infty a_n v_n$ is divergent, Q_p will tend to infinity with p, and then the inequality becomes
$$l-\epsilon \leqslant \overline{\underline{\lim}}_{p\to\infty} \frac{R_p}{Q_p} \leqslant l+\epsilon,$$

and since these limits are independent of m, we must have

$$\lim_{p\to\infty} \left(\sum_0^p b_n v_n \Big/ \sum_0^p a_n v_n\right) = l,$$

so that $\sum_0^\infty b_n v_n$ is also divergent, and the quotient of $\sum_0^p b_n v_n$ by $\sum_0^p a_n v_n$ tends to the limit l.

On the other hand, if (as happens in the most interesting special cases) $\sum_0^\infty a_n v_n$ converges, it follows from § 1 that

$$\lim_{x\to 1} \left(\sum_0^\infty a_n v_n\right) = \infty,$$

so that
$$\lim_{x\to 1} \left(\sum_0^m a_n v_n \Big/ \sum_0^\infty a_n v_n\right) = 0.$$

If we apply this result to the inequality for R_p/Q_p, first allowing p to tend to infinity, we find that

$$l-\epsilon \leqslant \overline{\lim_{x\to 1}} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) \leqslant l+\epsilon,$$

or

$$\lim_{x\to 1} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) = l.$$

This is an extension of the well known result, due to Cesàro, that

$$\lim_{x\to 1} \left(\sum_0^\infty b_n x^n \Big/ \sum_0^\infty a_n x^n\right) = l,$$

when b_n, a_n are related as already specified.

As another simple example, we take

$$\lim_{x\to 1} \left(\sum_0^\infty b_n \frac{x^n}{1+x^n} \Big/ \sum_0^\infty a_n \frac{x^n}{1+x^n}\right) = l.$$

Another simple application is to establish a result given recently by Mr. Hardy.* In fact, if we write

$$b_n = a_n \sigma_n, \quad v_n = c_n/a_n,$$

and suppose that $\sum_0^\infty a_n$, $\sum_0^\infty c_n$ are both divergent, we find

$$R_p = c_0\sigma_0 + c_1\sigma_1 + \dots + c_p\sigma_p,$$

$$Q_p = c_0 + c_1 + \dots + c_p,$$

$$B_p = a_0\sigma_0 + a_1\sigma_1 + \dots + a_p\sigma_p.$$

Suppose that B_p/A_p has a definite limit l as p tends to infinity, then we can choose m so that

$$l-\epsilon \leqslant h_m < H_m \leqslant l+\epsilon.$$

Thus, *if c_n/a_n is a decreasing sequence*, we have, from (8),

$$l-\epsilon-(h-l+\epsilon)\frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < l+\epsilon+(H-l-\epsilon)\frac{Q_m}{Q_p}.$$

Thus, since Q_p tends to infinity with p, we find as in the last piece of work, that

$$\lim_{p\to\infty} (R_p/Q_p) = l.$$

This result is due to Cesàro;† but Hardy has succeeded in extending it to the case when *c_n/a_n is an increasing sequence subject to the condition*

$$(a_0+a_1+\dots+a_p)/a_p < K(c_0+c_1+\dots+c_p)/c_p$$

for all values of p.

For the last condition gives $\qquad A_p v_p < KQ_p,$

and so the inequality (9) leads to

$$l-(2K-1)\epsilon-(H-l-\epsilon)\frac{A_m v_m}{Q_p} < \frac{R_p}{Q_p} < l+(2K-1)\epsilon+(l-h-\epsilon)\frac{A_m v_m}{Q_p},$$

from which we get as before $\qquad \lim_{p\to\infty} (R_p/Q_p) = l.$

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* *Quarterly Journal*, Vol. XXXVIII., 1907, p. 269.

† *Bulletin des Sciences mathématiques*, (2), t. XIII., 1889, p. 51.

6. *Extension of § 5 to the Case of Complex Factors.*

If the factors v_n are complex we suppose, as in § 3, that the series

$$\sum_{1}^{\infty} | v_n - v_{n+1} |$$

is convergent, and we write again

$$V_n = \{ | v_n - v_{n+1} | + | v_{n+1} - v_{n+2} | + \dots \text{ to } \infty \} + \lim_{\nu \to \infty} | v_\nu | .$$

We suppose that the terms a_n which appear in the denominator Q_p are all real and positive, though the terms b_n may be complex; then write η for the upper limit to the differences

$$| B_1/A_1 - \sigma |, \quad | B_2/A_2 - \sigma |, \quad \dots, \quad | B_p/A_p - \sigma |,$$

and η_m for the upper limit when the suffixes are not less than m.

We get at once, since $V_n - V_{n+1} = | v_n - v_{n+1} |$, $V_n \geqslant | v_n |$ (see p. 63),

$$| R_p - \sigma Q_p | < \eta [A_1 (V_1 - V_2) + A_2 (V_2 - V_3) + \dots + A_{m-1} (V_{m-1} - V_m)]$$
$$+ \eta_m [A_m (V_m - V_{m+1}) + \dots + A_{p-1} (V_{p-1} - V_p) + A_p V_p].$$

Now, let us write

$$M_n = a_1 V_1 + a_2 V_2 + \dots + a_n V_n$$
$$= A_1 (V_1 - V_2) + A_2 (V_2 - V_3) + \dots + A_{n-1} (V_{n-1} - V_n) + A_n V_n,$$

and then $$| R_p - \sigma Q_p | < \eta M_m + \eta_m (M_p - M_m).$$

Thus

$$\left| \frac{R_p}{Q_p} - \sigma \right| < \eta_m \frac{M_p}{| Q_p |} + (\eta - \eta_m) \frac{M_m}{| Q_p |}. \tag{11}$$

Application.—*The Theorem of Comparison for Complex Divergent Series.*

The direct application of (11) is not so easy as that of (8), owing to the fact (already mentioned on p. 65) that we cannot infer the divergence of $\lim \sum_{0}^{\infty} a_n v_n$ from that of $\sum_{0}^{\infty} a_n$. To avoid this difficulty we introduce the idea of *uniform divergence*, as suggested by Pringsheim; this implies that for all points x under consideration

$$\lim_{x \to 1} \left\{ \left(\sum_{0}^{\infty} a_n V_n \right) \Big/ \left| \sum_{0}^{\infty} a_n v_n \right| \right\} < K,$$

where K is fixed.

Making this hypothesis, it follows that

$$\lim_{x\to 1} \left| \sum_0^\infty a_n v_n \right| = \infty,$$

because

$$\lim_{x\to 1} \sum_0^\infty a_n V_n = \infty,$$

in virtue of § 1.

Then (11) yields at once

$$\left| \left\{ \left(\sum_0^\infty b_n v_n \right) \Big/ \left(\sum_0^\infty a_n v_n \right) - \sigma \right\} \right| < K \left\{ \eta_m + (\eta - \eta_m) \left(\sum_0^m a_n V_n \right) \Big/ \left(\sum_0^\infty a_n V_n \right) \right\},$$

and by the usual argument this can be proved to tend to zero as x approaches 1, provided that η_m tends to zero as m tends to infinity. Thus

$$\lim_{x\to 1} \left(\sum_0^\infty b_n v_n \right) \Big/ \left(\sum_0^\infty a_n v_n \right) = \lim_{n\to\infty} (B_n / A_n).$$

This result includes Pringsheim's for the case of power series, and also the result proved in § 6 of my paper on "Zonal Harmonics," quoted above.

Thus for power series $v_n = x^n$, and

$$V_n = |x|^n \, |1-x| / \{1-|x|\},$$

so that the above test for uniform divergence gives

$$\lim_{x\to 1} \frac{|1-x|}{1-|x|} \frac{\Sigma a_n |x|^n}{|\Sigma a_n x^n|} < K,$$

which in Pringsheim's treatment is divided into two separate conditions

$$\frac{|1-x|}{1-|x|} < K, \qquad \frac{\Sigma a_n |x|^n}{|\Sigma a_n x^n|} < K.$$

Similarly for zonal harmonics, we get

$$v_n = r^n P_n(\cos\theta) \quad \text{and} \quad V_n = \rho r^n / (1-r),$$

where

$$\rho^2 = 1 - 2r\cos\theta + r^2.$$

Then the condition becomes $\displaystyle \lim \frac{\rho}{1-r} \frac{\Sigma a_n r^n}{|\Sigma a_n r^n P_n(\cos\theta)|} < K,$

which was also split up into two separate conditions in my paper (see pp. 205, 213).

7. *Extension of § 5 to Quotients of Double Series.*

Let us consider the quotient

$$R_{p,q} / Q_{p,q},$$

where

$$Q_{p,q} = \sum_{m=1}^{p} \sum_{n=1}^{q} a_{m,n} v_{m,n},$$

$$R_{p,q} = \sum_{m=1}^{p} \sum_{n=1}^{q} b_{m,n} v_{m,n},$$

and $a_{m,n}$ is positive, while $v_{m,n}$ is positive and decreasing with respect to both indices (in the sense defined at the beginning of § 4).

We shall now use the notation $A_{m,n}$ and $B_{m,n}$ to denote the sums to m, n terms $\Sigma\Sigma a_{m,n}$ and $\Sigma\Sigma b_{m,n}$; so that $A_{m,n}$ is what was denoted by $s_{m,n}$ in § 4. Then Hardy's equation [see (5), § 4] gives

$$R_{p,q} = \sum_{m=1}^{p-1} \sum_{n=1}^{q-1} \Delta_{m,n} B_{m,n} + \sum_{m=1}^{p-1} \Delta_m B_{m,q} + \sum_{n=1}^{q-1} \Delta_n B_{p,n} + B_{p,q} v_{p,q}.$$

Suppose that H, h are the upper and lower limits of $B_{m,n}/A_{m,n}$ for all values of m, n between 1, p and 1, q respectively, while H_ν, h_ν are those when both m, n are greater than ν. Then we see that $R_{p,q}$ will be increased by writing $HA_{m,n}$ or $H_\nu A_{m,n}$ in place of $B_{m,n}$; thus we find

$$R_{p,q} < HQ_{p,q} - (H-H_\nu)(Q_{p,q} + Q_{\nu,\nu} - Q_{p,\nu} - Q_{\nu,q}),$$

or

$$R_{p,q} < H_\nu Q_{p,q} + (H-H_\nu)(Q_{p,\nu} + Q_{\nu,q} - Q_{\nu,\nu}).$$

Now $H-H_\nu$ is positive and so is $Q_{\nu,\nu}$; thus $Q_{\nu,\nu}$ may be omitted from the last inequality, and we find (on including the corresponding lower limit)

$$(12) \qquad h_\nu - (h_\nu - h)\frac{Q_{p,\nu}+Q_{\nu,q}}{Q_{p,q}} < \frac{R_{p,q}}{Q_{p,q}} < H_\nu + (H-H_\nu)\frac{Q_{p,\nu}+Q_{\nu,q}}{Q_{p,q}},$$

which is the extension of (8) given above. The inequalities corresponding to (9) and (10) are necessarily more complicated; and at present I do not see that they are likely to prove of much use in practical applications. I do not, therefore, write them out here.

Application.—*The Theorem of Comparison of Two Divergent Double Series.*

It is evident that (with the same interpretation of $v_{m,n}$ as we have used in § 4) we can infer from (12) the theorem

$$\lim_{(x,y)} \left\{ \left(\sum_0^\infty \sum_0^\infty b_{m,n} v_{m,n}\right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n}\right) \right\} = \lim_{(m,n)} (B_{m,n}/A_{m,n}),$$

provided that for any given value of ν,

$$\lim_{(x,y)} \left\{ \left(\sum_{m=0}^\infty \sum_{n=0}^\nu a_{m,n} v_{m,n}\right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n}\right) \right\} = 0,$$

and

$$\lim_{(x,y)} \left\{ \left(\sum_{m=0}^\nu \sum_{n=0}^\infty a_{m,n} v_{m,n}\right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n}\right) \right\} = 0.$$

There does not seem to be any way of avoiding these two conditions, nor any way of dividing them into simpler forms in general.

Consider now the specially interesting case $v_{m,n} = x^m y^n$, and suppose further that the coefficients $a_{m,n}$ are also divisible into factors; so that

$$a_{m,n} = f_m g_n,$$

where Σf_m, Σg_n are two divergent series of positive terms. Then

$$\sum_0^\infty \sum_0^\infty a_{m,n} x^m y^n = \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\infty g_n y^n\right),$$

and
$$\sum_{m=0}^{\infty} \sum_{n=0}^{\nu} a_{m,n} x^m y^n = \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\nu g_n y^n\right),$$

so that our first condition reduces to

$$\lim_{y \to 1} \left(\sum_0^\nu g_n y^n\right) \Big/ \left(\sum_0^\infty g_n y^n\right) = 0,$$

which is certainly satisfied since

$$\lim_{y \to 1} \left(\sum_0^\infty g_n y^n\right) = \infty$$

(a result proved in § 1).

Similarly the second condition is satisfied.

Thus, if we write

$$F_m = f_0 + f_1 + \ldots + f_m, \qquad G_n = g_0 + g_1 + \ldots + g_n,$$

we find $\displaystyle \lim_{(x,y)} \left\{ \left(\sum_0^\infty \sum_0^\infty b_{m,n} x^m y^n\right) \Big/ \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\infty g_n y^n\right) \right\} = \lim_{(m,n)} (B_{m,n}/F_m G^n).$

This enables us to give an immediate proof of the extension of Frobenius's theorem to double series,* by writing

$$f_m = 1, \qquad g_n = 1.$$

Then
$$\sum_0^\infty f_m x^m = (1-x)^{-1}, \qquad \sum_0^\infty g_n y^n = (1-y)^{-1},$$

and so, if
$$b_{m,n} = s_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j},$$

we have
$$\sum_0^\infty \sum_0^\infty b_{m,n} x^m y^n = (1-x)^{-1}(1-y)^{-1} \sum_0^\infty \sum_0^\infty c_{m,n} x^m y^n,$$

and then
$$\lim_{(x,y)} \left(\sum_0^\infty \sum_0^\infty c_{m,n} x^m y^n\right) = \lim_{(m,n)} s_{m,n}^{(1)},$$

if
$$(m+1)(n+1)\, s_{m,n}^{(1)} = \sum_{i=0}^{m} \sum_{j=0}^{n} s_{m,n},$$

using the notation of the paper quoted.

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* Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 161 (see § 8, p. 173).

Similarly we can extend the theorem to cases of greater complexity by writing

$$\sum_0^\infty f_m x^m = (1-x)^{-\alpha}, \qquad \sum_0^\infty g_n y^n = (1-y)^{-\beta},$$

where α, β are positive integers; this gives a kind of extension of Hölder's theorem, although the means employed will correspond to those used for the summation of single series by Cesàro, rather than those introduced by Hölder.* Thus, taking $\alpha = 2 = \beta$, we get

$$\lim_{(x,\, y)} \left(\sum_0^\infty \sum_0^\infty c_{m,\,n}\, x^m y^n\right) = \lim_{(m,\, n)} \frac{(2\,!)^2 \sum\limits_{i=0}^{m} \sum\limits_{j=0}^{n} (i+1)(j+1)\, s_{i,\,j}}{(m+1)(m+2)(n+1)(n+2)}.$$

The analogue to Hölder's theorem would have on the right the limit

$$\lim_{(m,\, n)} s^{(2)}_{m,\, n},$$

where
$$(m+1)(n+1)\, s^{(2)}_{m,\, n} = \sum_{i=0}^{m} \sum_{j=0}^{n} s^{(1)}_{i,\, j},$$

the sums $s^{(1)}_{m,\, n}$ being themselves defined by arithmetic means.

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* Compare the form of the theorem given in Art. 123 of my book on *Infinite Series*.

ON HYPERCOMPLEX NUMBERS

By J. H. MACLAGAN WEDDERBURN.

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INDEX OF TERMS.

THE object of this paper is in the first place to set the theory of hypercomplex numbers on a rational basis. The methods usually employed in treating the parts of the subject here taken up are, as a

rule, dependent on the theory of the characteristic equation, and are for this reason often valid only for a particular field or class of fields. Such, for instance, are the methods used by Cartan in his fundamental and far-reaching memoir, *Sur les groupes bilinéaires et les systèmes complexes*. It is true that the methods there used are often capable of generalisation to any field; but I do not think that this is by any means always the case.

My object throughout has been to develop a treatment analogous to that which has been so successful in the theory of finite groups. An instrument towards this lay to hand in the calculus developed by Frobenius, and used by him with great effect in the theory of groups. This calculus is, with slight additions, equally applicable to the theory of hypercomplex number-systems, or, as they will be called below, algebras. Although a short account of this calculus has already been given, it was thought advisable to give a more detailed account in the present paper.

A word or two on the nomenclature adopted will perhaps not be out of place. At Professor Dickson's suggestion I have used the word *algebra* as equivalent to Peirce's *linear associative algebra* which is too long for convenient use. An algebra which is composed of only a part of the elements (or numbers) of an algebra is called a *sub-algebra* of that algebra. It is assumed throughout that a finite basis can be chosen for any algebra which is under discussion, that is, we suppose that it is always possible to find a finite number of elements of the algebra which are linearly independent with regard to some given field, and are such that any other number of the algebra can be linearly expressed in terms of them. This excludes from the present paper an interesting class of algebras which I hope to discuss in a subsequent communication.

Most of the results contained in the present paper have already been given, chiefly by Cartan and Frobenius, for algebras whose coefficients lie in the field of rational numbers; and it is probable that many of the methods used by these authors are capable of direct generalisation to any field. It is hoped, however, that the methods of the present paper are, in themselves and apart from the novelty of the results, sufficiently interesting to justify its publication.

The greater part of Sections 1, 2, 4–6 was read in the Mathematical Seminar of the University of Chicago early in 1905, and owe much to Professor Moore's helpful criticism.

A list of memoirs referred to is given at the end of the paper, and these memoirs are quoted throughout by their number in this list.

1. *The Calculus of Complexes.*

The definition of the term *algebra* or *hypercomplex number-system* is now so well known that it is unnecessary to give here a formal set of postulates.*

Let $x_1, x_2, \ldots, x_n$ be a set of elements which are linearly independent in a given field F. The set of all elements of the form

$$x = \sum_{r=1}^{n} \xi_r x_r,$$

the ξ's being any marks of F, is said to form an *algebra*, if

(i.) $\Sigma \xi_r x_r + \Sigma \xi'_r x_r = \Sigma (\xi_r + \xi'_r) x_r$.

(ii.) The product of any two x's is linearly dependent on $x_1, x_2, \ldots, x_n$ in F, in such a way that the multiplication so defined is associative.

(iii.) For any three elements x, y, z of the algebra

$$x(y+z) = xy + xz, \qquad (y+z)x = yx + zx.$$

The algebra is said to be of order n with respect to F. In what follows the term "linearly independent" will always be understood to be with respect to a given field F which is supposed to be constant throughout but otherwise arbitrary.

The *complex* $A = x_1, x_2, \ldots, x_a$ is defined as the set of all quantities linearly dependent on $x_1, x_2, \ldots, x_a$. The greatest number of linearly independent elements which can be simultaneously chosen, is called the *order* of the complex.

If A and B are two complexes, the complex formed by all elements of A and B and those linearly dependent on them, is called the *sum* of A and B, and is denoted by $A+B$. The operation of addition so defined is evidently associative and commutative.

If a complex B is contained in a complex A, we write $B < A$ or $A > B$. Similarly, if x is an element of a complex A, we write $x < A$. This amounts to representing a complex of order one by one of its elements, and will be found to lead to no confusion if certain obvious precautions are observed.

If $B < A$, we can always find C such that $B+C = A$. C is called the *supplement* of B with regard to A. It is obviously not uniquely

* The reader is referred to the following papers on this subject:—Dickson 2, 3.

determined, but if $B+C' = B+C$, any element of C' can be expressed as the sum of an element of B and an element of C. This is conveniently denoted by writing $C' = C \pmod{B}$.

The elements common to two complexes evidently also form a complex. The greatest complex common to A and B is denoted by $A \frown B$. Thus the statement that A and B have no element in common is equivalent to $A \frown B = 0$.

If A and B are any two complexes, and if x and y are any elements of A and B respectively, the complex of elements of the form xy and those linearly dependent on them, is called the *product* of A and B and is written AB. For instance, if $A = x_1, x_2 \ldots x_a$ and $B = y_1, y_2 \ldots y_b$, then

$$AB = \ldots, x_r y_s, \ldots \quad (r = 1, 2, \ldots, a\,;\ s = 1, 2, \ldots, b).$$

AB of course is not in general the same as BA. The operation of multiplication so defined is associative, and it is also distributive with regard to addition.

The following is a summary of the laws of the calculus described above:—

(i.) $A+B = B+A$.

(ii.) $A+(B+C) = (A+B)+C$.

(iii.) $A\,.\,BC = AB\,.\,C$.

(iv.) $A\,(B+C) = AB+AC, \quad (B+C)\,A = BA+CA$.

(v.) $A \frown (B \frown C) = (A \frown B) \frown C$.

(vi.) $A \frown B = B \frown A$.

(vii.) $A\,(B \frown C) \leqslant AB \frown AC$.

Integral powers of a complex are defined by the methods usually employed in hypercomplex numbers, *e.g.*, $A\,.\,A^m = A^{m+1} = A^m\,.\,A$. A necessary and sufficient condition that a complex A be an algebra is then obviously $A^2 \leqslant A$.

The above definitions will perhaps be made clearer by a special example. Consider the algebra (quaternions) formed by four units e_0, e_1, e_2, e_3, where

$$e_r e_s = -e_s e_r \quad (r, s \neq 0),$$

$$e_0 e_r = e_r \quad \text{and} \quad -e_0^2 = e_1^2 = e_2^2 = e_3^2 = -e_0.$$

If Greek letters are used to denote marks of the given field, elements of the form $\xi e_0 + \xi_1 e_1$ form a complex $A = e_0, e_1$. If $B = e_1, e_2$, then

$A \frown B = e_1$; we have also $A^2 = A$ and $B^2 = e_0, e_1, e_2, e_3 = A$. Again,

$$AB = B^2 = A \quad \text{and} \quad B(A \frown B) = e_0, e_3,$$

but
$$BA \frown B^2 = B^2 > B(A \frown B).$$

2. *The Theory of Invariant Sub-algebras.*

A sub-complex B of a complex A, which is such that $AB \leqslant B$ and $BA \leqslant B$, is called an *invariant** sub-complex of A. If B is contained in no other sub-complex of A which has this property, it is said to be maximal. B is necessarily an algebra, since $B^2 \leqslant BA \leqslant B$. An algebra which has no invariant sub-complex is said to be *simple*.†

The theory of invariant sub-algebras is of great importance, as will be seen in the succeeding sections. As most of the present section has already appeared elsewhere‡ it is given here in a somewhat condensed form.

THEOREM 1.—*If $AB \leqslant B$ and $A^2 \leqslant A$, either $BA = A$ or BA is an invariant sub-algebra of A.*

For $BA.A \leqslant BA$ and $A.BA \leqslant BA$. This theorem is frequently applied in the sequel.

We may also notice that $B+BA$ is also an invariant sub-algebra, unless it is identical with A.

THEOREM 2.—*If B_1 and B_2 are invariant sub-algebras of an algebra A, B_1+B_2 is also an invariant sub-algebra, unless $A = B_1+B_2$*

For
$$A(B_1+B_2) = AB_1+AB_2 \leqslant B_1+B_2,$$
$$(B_1+B_2)A = B_1A+B_2A \leqslant B_1+B_2.$$

COROLLARY.—If B_1 is maximal, then either $A = B_1+B_2$ or $B_2 < B_1$. Hence, if B_1 and B_2 are two different maximal invariant sub-algebras, we must necessarily have $B_1+B_2 = A$.

THEOREM 3.—*If B is an invariant sub-algebra of an algebra A, a new algebra can be derived from A by regarding as identical those elements of A which differ only by an element of B.*§

* Molien [illegible]; Frobenius [illegible], p. 523; Cartan [illegible], p. 57.

† Cartan [illegible], p. 57.

‡ Epsteen and Maclagan Wedderburn [illegible].

§ This fundamental theorem is due to Molien.

The set of elements defined by regarding as identical those elements of A which differ only by an element of B, is evidently closed under the operations of addition and multiplication, and the distributive law holds. The only law that is not evidently satisfied is the associative law for multiplication. This law is shown to hold as follows.

Let $A = B+C$, and let elements of B and C be respectively denoted by x and y with subscripts attached. If, then, y_p, y_q and y_r are any three elements of C,

$$y_p \cdot y_q y_r = y_p (y_{qr}+x_{qr}) = y_p y_{qr} \pmod{B},$$

since $y_p x_{qr} < B$. Similarly,

$$y_p y_q \cdot y_r = (y_{pq}+x_{pq})\, y_r = y_{pq} y_r \pmod{B};$$

therefore, since $$y_p \cdot y_q y_r = y_p y_q \cdot y_r,$$

we have $$y_p y_{qr} = y_{pq} y_r \pmod{B},$$

which shows that multiplication is associative.

The algebra defined in this way is called the *difference* algebra of A and B, and, on the analogy of the symbolism used for the quotient group in the theory of finite groups, it is conveniently denoted by $(A-B)$. $(A-B)$ is said to *accompany* A and to be *complementary** to B.

THEOREM 4.—*If B_1 and B_2 are invariant sub-algebras of an algebra A, and $B_1 > B_2$, $(A-B_2)$ has an invariant sub-algebra which is simply isomorphic with (B_1-B_2) and conversely.*

To show this, let $$A = B_1+C, \quad B_1 \frown C = 0,$$
$$B_1 = B_2+D, \quad B_2 \frown D = 0;$$

then $$A = B_2+D+C.$$

If D' is the complex of $(A-B_2)$, which corresponds to D, we have

$$(A-B_2)\, D' \leqslant D',$$

since $$(D+C)\, D \leqslant D \pmod{B_2}.$$

Similarly $$D'(A-B_2) \leqslant D'.$$

Now D' is derived from D by regarding those elements as equal which differ only by an element of B_2. Hence

$$D' \equiv (B_1-B_2).$$

* Molien (10), p. 92; Frobenius (6), p. 523.

Conversely, if $(A-B_2)$ has an invariant sub-algebra D', and if, as before, D is a complex of A which corresponds to D', then since

$$AD \leqslant D \quad (\text{mod } B_2),$$

we have $$A(B_2+D) \leqslant B_2+D, \quad (B_2+D)A \leqslant B_2+D.$$

Hence B_2+D is an invariant sub-algebra of A.

COROLLARY.—An immediate consequence of this theorem is that $(A-B)$ is simple, if B is a maximal invariant sub-algebra.

THEOREM 5.—*If B_1 and B_2 are two different maximal invariant sub-algebras of an algebra A, then $D = B_1 \frown B_2$ is a maximal invariant sub-algebra of both B_1 and B_2. Further $(A-B_1)$ and $(A-B_2)$ are simply isomorphic with (B_2-D) and (B_1-D) respectively.*

Let $$B_1 = D+C_2, \qquad B_2 = D+C_1,$$

where $$D \frown C_1 = 0, \qquad D \frown C_2 = 0;$$

and therefore, since

$$D = B_1 \frown B_2 \quad \text{and} \quad A = B_1+B_2,$$

$$A = D+C_1+C_2, \quad C_1 \frown C_2 = 0.$$

If we denote simple isomorphism* by the symbol $\sim$, we have

$$(A-B_1) \sim C_1 \quad (\text{mod } B_1),$$

and $$(B_2-D) \sim C_1 \quad (\text{mod } D), \qquad \sim C_1 \quad (\text{mod } B_1),$$

since $C_1 < B_1$, and therefore any two elements of C_1 which are equal modulo B_1, are also equal modulo D. We have therefore

$$(A-B_1) \sim (B_2-D),$$

i.e., (B_2-D) is simple since $(A-B_1)$ is simple. Hence D is a maximal invariant sub-algebra of B_2. In exactly the same way it can be shown that it is a maximal invariant sub-algebra of B_2, and

$$(A-B_2) \sim (B_1-D).$$

If $A_1, A_2, \ldots, A_r$ is a series of algebras such that A_r is a maximal invariant sub-algebra of A_{r-1}, the series is called a *composition* series of A_1. The series $(A_1-A_2), (A_2-A_3), \ldots, (A_{r-1}-A_r), \ldots$ is said to be a *difference* series of A_1. An algebra can of course have many composition series.

* *I.e.*, isomorphism with regard both to addition and multiplication.

Let (i.) $A_1, A_2, A_3, \ldots,$ (ii.) $A_1, B_1, B_2, \ldots,$

be two composition series of A for which $A_2 \neq B_1$. Then, if $A_2 \frown B_1 = D$,

$$\text{(iii.)}\ A_1, A_2, D, D_1, \ldots, \qquad \text{(iv.)}\ A_1, B_1, D, D_1, \ldots,$$

where $D, D_1, \ldots$ is a difference series for D, are two composition series for A_1, and, by Theorem 5, the corresponding differences are identical apart from the order of their terms. If we now assume that all possible difference series of the same algebra are equivalent for all algebras of order less than the order of A, (i.) and (ii.) are respectively equivalent to (iii.) and (iv.) and hence to each other. For algebras of one unit, there is only one difference series possible, hence we have by induction the following theorem.

THEOREM 6.—*Any two difference series of the same algebra are identical apart from the order of their terms.*

If in forming the series $A_1, A_2, \ldots$ we make each term the largest sub-algebra of the preceding algebra which is an invariant sub-algebra of A_1, the corresponding difference series is called a *principal* difference series. It can be shown by a method analogous to that used above, that the principal series is also independent of the particular composition series from which it is formed.

3. *Reducibility.*

If an algebra A is expressible as the sum of two algebras A_1 and A_2, which are such that $A_1 A_2 = 0 = A_2 A_1$, A is said to be *reducible*, and to be the *direct* sum of A_1 and A_2. It was in this sense that the word sum was first used by Scheffers. To avoid circumlocution, we shall in this section call A_1 an *integral* sub-algebra of A, if there is another sub-algebra A_2 such that $A = A_1 + A_2$, and $A_1 A_2 = 0 = A_2 A_1$. This term is not used except in this section. An integral sub-algebra is always invariant.

THEOREM 7.—*If B is an invariant sub-algebra of A, and both A and B have a modulus,* then A is reducible.*

Let $$A = B + C', \qquad B \frown C' = 0,$$

* An algebra is said to have a modulus e, if e is an element such that $ex = x = xe$ for every element x of A.

and let e and e_1 be the moduli of A and B respectively, then

$$C \equiv (e-e_1)\, C'\, (e-e_1) = C' \pmod{B},$$

and
$$(e-e_1)\, B = 0 = B\,(e-e_1),$$

since, if $y < B$, then $ey = y = e_1 y$. Hence $BC = 0 = CB$; and $C^2 = C$, since $A^2 = A$. $e-e_1$ is evidently the modulus of C.

COROLLARY.—If B is an integral sub-algebra of A and both A and B have a modulus, A is expressible uniquely as the direct sum of B and an algebra C. For e and e_1 being as above, we have

$$C = (e-e_1)\, A\, (e-e_1).$$

THEOREM 8.—*If A_1 and A_2 are two different maximal integral sub-algebras of A, then $A = A_1+A_2$.*

Let $\quad A = A_1+B_1, \quad A_1B_1 = 0 = B_1A_1, \quad A_1 \frown B_1 = 0,$

$$= A_2+B_2, \quad A_2B_2 = 0 = B_2A_2, \quad A_2 \frown B_2 = 0.$$

Every element of A_2 can be expressed in the form $x+y$, where $x < A_1$ and $y < B_1$, and the complex of y's so defined forms a sub-algebra C_2 of B_1 which does not vanish.

Similarly, any element of B_2 can be expressed in the form $x+y$, the y's defining a sub-algebra D_2 of B_1. But

$$A_1B_1 = B_1A_1 = 0 = A_2B_2 = B_2A_2;$$

therefore
$$C_2D_2 = 0 = D_2C_2.$$

Now $A = A_2+B_2$ and $A_1 \frown B_1 = 0$, hence we must have

$$B_1 = C_2+D_2.$$

But, since A_1 is maximal, B_1 must be irreducible; from which there results $D_2 = 0$. Hence B_2 is contained in A_1 and $A = A_1+A_2$. It follows also that B_1 is an integral sub-algebra of A_2. For, if the elements of A_2 are expressed in the form $x+y$ as before, the x's compose a sub-algebra D of A_1, which is also a sub-algebra of A_2, since the y's have been shown to be elements of A_2. Since

$$A_2 = D+B_1 \quad \text{and} \quad A_1 \frown B_1 = 0,$$

we must evidently have $\qquad D = A_1 \frown A_2.$

If $A_1, A_2, \ldots$ be a series of algebras such that A_r is a maximal integral sub-algebra of A_{r-1}, the series $(A_1-A_2), (A_2-A_3), \ldots$ is said to

form a *reduction* series of A_1. It then follows exactly as in Theorem 6, that—

THEOREM 9.—*Any two reduction series of an algebra are identical except as regards the order of their terms.**

There are evidently sub-algebras of the given algebra which are isomorphic with the terms of the reduction series, but, as Hölder has noticed, these sub-algebras are not in general uniquely defined. The following theorem is a slight extension of one by Scheffers† dealing with this point.

THEOREM 10.—*An algebra A can be uniquely expressed as the direct sum of irreducible algebras which have each a modulus, and an algebra which has no modulus.*

Let $$A = B+C, \quad BC = 0 = CB, \quad B \frown C = 0,$$

where B has a modulus e_1, and C has (1) no modulus, (2) no integral sub-algebra which has a modulus. A has then no integral sub-algebra which contains B, and at the same time has a modulus.

We can form an algebra A' by adjoining a modulus e' to the basis of A; and if e_1 is the modulus of B, and

$$C' = C+(e'-e_1),$$

then

$$A' = B+(e'-e_1)\,C'\,(e'-e_1)$$
$$= B+C'.$$

Hence C', and therefore C, is unique for a given B by Theorem 7. Suppose there is another algebra B_1 satisfying the same conditions as B. As in Theorem 8, we can express B_1 as the direct sum of two algebras $B_2 < B$ and $C_2 < C$, where B_2 and C_2 have both moduli, unless one is zero, seeing that B has a modulus. Now

$$B_1 \leqslant AB_1 = BB_2+CC_2;$$

therefore $CC_2 = C_2$, and similarly $C_2C = C_2$; and therefore C_2 is an integral sub-algebra of C which has a modulus, contrary to the conditions previously laid down for C. Hence we must have $C_2 = 0$, from which it follows that $B = B_1$, *i.e.*, B is unique.

Let $$B = B_1+B_2+\ldots+B_n, \qquad (1)$$
$$= B_1'+B_2'+\ldots+B_m',$$

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* Epsteen (4), p. 444.

† Scheffers (13).

be two expressions of B as the direct sum of irreducible algebras. From Theorem 9 we have $m = n$. Again, since B has a modulus, we have

$$B_p' = BB_p'B = \sum_{r,s} B_r B_p' B_s = \sum_r B_r B_p' B_r,$$

remembering that $B_r B_p' B_s$ $(r \neq s)$ is contained in both B_r and B_s, and that $B_r \frown B_s = 0$. But, since B_p' is irreducible, $B_r B_p' B_r$ must vanish except for some particular value r_p of r which is necessarily different for each value of p. We may therefore, by rearranging the terms, set $r_p = p$. But $B_p B_p' B_p = B_p$, since B_p is invariant. Hence $B_p = B_p'$.

4. *Nilpotent Algebras.*

It was mentioned in § 1 that a necessary and sufficient condition, that a complex A shall be an algebra, is that $A^2 \leqslant A$. If A has a modulus, *i.e.*, an element e such that $ex = x = xe$ for any element x of A, we must evidently have $A^2 = A$. In general, since we are dealing only with algebras which have a finite basis, we must have $A^{a+1} = A^a$ for some integer a. The smallest integer a for which this is the case is called the *index** of the algebra. For instance, in the algebra whose multiplication table is

	e_1	e_2
e_1	e_2	e_2
e_2	e_2	e_2

we find $A^2 = e_2 = A^3$. Hence its index is 2.

It may, of course, happen that some power of A vanishes as in the algebra

	e_1	e_2
e_1	e_2	0
e_2	0	0

where $A^3 = 0$.

If for some integer a, $A^a = 0$, A is said to be *nilpotent.* Nilpotent algebras are of great importance in the discussion of the structure of algebras.

THEOREM 11.—*If a is the index of A, the elements of A can be divided into $a-1$ complexes $B_1, B_2, \ldots, B_{a-1}$, such that*

$$B_p B_q \leqslant B_{p+q} + B_{p+q+1} + \ldots + B_{a-1},$$

* The index might also be suitably defined as the least integer a for which $(A^a)^2 = A^a$.

i.e., such that the product of two elements, belonging to complexes with subscripts p and q respectively, lies entirely in the sum of the complexes with subscripts greater than $p+q-1$.

For let
$$A = B_1+A^2 = B_1+B_2+A^3 = \dots$$
$$= B_1+B_2+\dots+B_{\alpha-1},$$
where
$$A^\beta = B_\beta+A^{\beta+1}, \quad A^{\alpha-1} = B_{\alpha-1};$$
then
$$B_p B_q \leqslant A^p A^q \leqslant A^{p+q},$$
which proves the theorem.

This theorem is evidently considerably stronger than the similar theorems enunciated by Scheffers* and others.

COROLLARY.—Since $A = B_1+A^2$, we have on squaring
$$A^2 = B_1^2+B_1A^2+A^2B_1+A^4 = B_1^2+A^3;$$
hence
$$B_1^2 = B_2 \pmod{A^3},$$
and similarly
$$B_1^n = B_n \pmod{A^{n+1}}.$$

From this we readily derive the interesting result
$$A = B_1+B_1^2+\dots+B_1^{\alpha-1}+A^\alpha.$$

If $A^\alpha = 0$ is zero, A is said to be generated by B_1. In this case A is reducible if B_1 is reducible, and conversely.

If α is the index of a nilpotent algebra, we have $A^{\alpha-1} \neq 0$, $A^\alpha = 0$; and hence the product of any element of A and any element of $A^{\alpha-1}$ is zero. This is a simple proof of a theorem by Cartan† to the effect that there is at least one element in a nilpotent algebra whose product with any other element is zero. It must be noticed, however, the above definition of a nilpotent algebra is not verbally identical with Cartan's. The identity of the two definitions will be shown in the next section.

An algebra in which the product of any two elements is zero, may be called a *zero-algebra*. For example, if $A^2 < A$, A^2 is an invariant subalgebra of A, and $(A-A^2)$ is a zero algebra. Let $A = B+A^2$, where
$$B = y_1, y_2, \dots, y_m, \quad A^2 = x_1, x_2, \dots, x_n,$$
and $m+n$ is the order of A. $A' = y_2, y_3, \dots, y_m, x_1, \dots, x_n$ is evidently

* Scheffers (12).
† Cartan (1), p. 31.

an invariant sub-algebra of A, such that $(A-A')$ is a zero algebra of order 1. This gives the following theorem regarding the difference series of such an algebra.

THEOREM 12.—*If a is the index of an algebra A, and if the difference of the orders of A and A^a is n, the difference series of A can be so arranged that the first n terms are zero algebras of order* 1.

The following theorem also simplifies the study of the difference series considerably.

THEOREM 13.—*If N is a maximal nilpotent invariant sub-algebra of an algebra A, all other nilpotent invariant sub-algebras of A are contained in N.*

Let N_1 be any nilpotent invariant sub-algebra of A, then, by Theorem 2, $N+N_1$ is also an invariant sub-algebra of A. It is, however, nilpotent. For, if $N_2 = N \frown N_1$, then

$$(N+N_1)^2 \leqslant N^2+N_2+N_1^2,$$

since $NN_1 \leqslant N_2$ and $N_1 N \leqslant N_2$. Similarly,

$$(N+N_1)^a \leqslant N^a+N_1^a+N_2,$$

whence, if a is greater than the indices of N and N_1,

$$(N+N_1)^a \leqslant N_2.$$

But N_2 is nilpotent and therefore also $N+N_1$. Hence, since N is maximal, we must have $N_1 \leqslant N$.

An immediate deduction from this theorem is that $(A-N)$ has no nilpotent sub-algebra. This theorem is very important, its importance lying in the fact that, in studying the difference series, it enables us to confine our attention to algebras which have no nilpotent invariant sub-algebra. Such algebras are called *semi-simple.*

5. *Potent Algebras.*

An algebra which is not nilpotent is called a *potent* algebra. If the index of a potent algebra is a, the index of A^a is 1. It is therefore sufficient in many investigations to consider only algebras with unit index.

Let A be an algebra such that $A^2 = A$. There will in general be some complex $C < A$, such that $AC = A$. In fact, if A has a modulus e, it is possible to find elements x, such that $Ax = A$. Let us suppose,

however, that $Ax_1 < A$ for every $x_1 < A$. Again, suppose that $Ax_1x_2 < Ax_1$ for every $x_2 < Ax_1$, and so on. We thus derive a series of algebras each one containing the preceding one, and, as we are dealing with algebras with a finite basis, this process must terminate at some stage. This may happen in either of two ways. After, say $r-1$ steps, we must find either

$$Ax_1x_2 \dots x_{r-1}x_r = 0 \tag{1}$$

for every $x_r < Ax_1x_2 \dots x_{r-1}$, or

$$Ax_1x_2 \dots x_{r-1}x_r = Ax_1x_2 \dots x_{r-1} \tag{2}$$

for some $x_r < Ax_1x_2 \dots x_{r-1}$. In the first case, if $B = Ax_1x_2 \dots x_{r-1}A$, then

$$B^2 \leqslant (Ax_1x_2 \dots x_{r-1})^2 A = 0,$$

and $AB \leqslant B$, $BA \leqslant B$, *i.e.*, B is an invariant sub-algebra of A, unless $B = 0$ when $Ax_1 \dots x_{r-1}$ is an invariant sub-algebra of A. The first case then cannot arise if A is simple.

In the second case, if $A' = Ax_1 \dots x_{r-1}$, there is an element x, such that $A'x = A'$. Hence every element of A' can be put in the form $y = zx$. Here z is unique. For were $zx = z'x$, then $(z-z')x = 0$, and the order* of the basis of $A'x$ would be less than the order of the basis of A'. In particular we have $x = yx$, hence $yx = y^2x$ and therefore $y = y^2$. Such an element is said to be *idempotent*, and the result we have obtained may be stated in the form that a simple algebra always contains an idempotent element. By means of this result we can now establish the following important theorem:—

THEOREM 14.—*Every potent algebra contains an idempotent element.*

For, let B be a maximal invariant sub-algebra of A^a, where $A^{a+1} = A^a$. $(A^a - B)$ is simple and has 1 as its index.† A has therefore a non-nilpotent element x, namely any element which corresponds to an idempotent element of the simple algebra $(A^a - B)$ Now for some value of n, we must have

$$Ax^{2n+1} = Ax^n,$$

for otherwise we should have

$$A > Ax > Ax^3 > \dots > Ax^n > Ax^{2n+1} > \dots,$$

* In other words, if $e_1, e_3, \dots, e_a$ is a basis of A, $e_1x, e_2x, \dots, e_ax$ are necessarily independent if $Ax = A$.

† Since, if $A^a = B + C$, then $B + C^2 = A^{2a} = A^a = B + C$, and therefore $C = C^2 \pmod{B}$.

which as before is impossible. Ax^n, and *a fortiori* A, must therefore contain an idempotent element.*

The converse of this theorem is that an algebra, every one of whose elements is nilpotent, is itself nilpotent. This shows that the definition of a nilpotent algebra which was given in § 4, is identical with the one given by Cartan and others.

COROLLARY.—If x is nilpotent, then $Ax < A$.

The following extension of a theorem due to Peirce,† is easily deduced from the results obtained above.

THEOREM 15.—*If an algebra A possesses only one idempotent element e, every element which does not possess an inverse‡ with respect to e, is nilpotent.*

This is shown as follows. If for a given x there is no y, such that $xy = e$, the same is true of all elements of the form xz. For were $xzz' = e$, it would suffice to put $y = zz'$. It follows that e is not contained in xA, which is therefore nilpotent by Theorem 14. Hence $x^n = 0$ for some integer n.

An obvious corollary to this theorem is that if an algebra A contains only one idempotent element e and no nilpotent element, then every element possesses an inverse with respect to e. Further, e is the modulus of A. For, since $Ae = A$, every element x can be put in the form $x = ye$, and hence $xe = x$. Similarly $ex = x$. Such an algebra is said to be *primitive*. Also, if e is the only idempotent element of an algebra A, which is contained in eAe, e is said to be a *primitive idempotent* element of A.

THEOREM 16.—*Every algebra A, which does not possess a modulus, has a nilpotent invariant sub-algebra.*

If A is nilpotent, the theorem is obvious, and it may therefore be assumed that this is not the case. Under this assumption A has at least one idempotent element e_1. If $Ae_1 < A$, there must be elements x such that $xe_1 = 0$. All such elements form a sub-algebra B_1 of A; because, if $x_1e_1 = 0$, $x_2e_1 = 0$, then $(x_1+x_2)e_1 = 0$ and $x_1x_2e_1 = 0$. Let $A = B_1 + C$,

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* In most proofs of this theorem, the idempotent element which is found, is in general irrational. This objection does not apply to the proof given by Hawkes (7), p. 320.

† Peirce (11), p. 112.

‡ x is said to possess an inverse with respect to e, if there exist elements x_1 and x_2, such that $xx_1 = e = x_2x$.

where $B_1 \frown C = 0$. C can be chosen so that $Ce_1 = C$. For

$$Ce_1 \leqslant C \quad (\text{mod } B_1),$$

and, if

$$Ce_1 < C \quad (\text{mod } B_1),$$

there would be an element x of C such that $xe_1 < B$, which is impossible since $B_1 e_1 = 0$ and $xe_1 \neq 0$. $Ce_1 = Ae_1$ can therefore take the place of C, and $Ce_1 . e_1 = Ce_1$.

We have then $$A = B_1 + Ae_1, \quad B_1 e_1 = 0, \tag{1}$$

and similarly $$A = B_2 + e_1 A, \quad e_1 B_2 = 0. \tag{2}$$

From (1) follows $$e_1 A = e_1 B_1 + e_1 A e_1, \tag{3}$$

and, from (2), $$Ae_1 = B_2 e_1 + e_1 A e_1. \tag{4}$$

Now $e_1 B_1 \frown B_2 e_2 = 0$, since $B_1 e_1 = 0$ and $e_1 B_2 = 0$, hence

$$e_1 A \frown Ae_1 = e_1 A e_1,$$

and if $B = B_1 \frown B_2$, we find similarly that

$$B_1 = B + e_1 B_1, \qquad B_2 = B + B_2 e_1.$$

Hence, from (2) and (3),

$$A = B + e_1 B_1 + B_2 e_2 + e_1 A e_1.$$

If B is not nilpotent, it contains an idempotent element e_2, such that $e_1 e_2 = 0 = e_2 e_1$, $e_1 + e_2$ is then also idempotent and may take the place of e_1 in the above discussion.

Again, if e_1 is not primitive, $e_1 A e_1$ can be broken up in the same manner as A, and so, by repeated application of this process, A can be expressed in the form

$$\begin{aligned} A &= B + eB_1 + B_2 e + eAe \\ &= B + \Sigma e_p B_1 + \Sigma B_2 e_p + \Sigma e_p A e_q, \end{aligned} \tag{5}$$

where

$$B^\beta = 0, \quad B_1 = B + eB_1, \quad B_2 = B + B_2 e, \quad e = \Sigma e_p, \quad e_p e_q = 0 \quad (p \neq q),$$

and e_p $(p = 1, 2, \ldots, r)$ are primitive idempotent elements of A. This form is due to Peirce.* e is called a *principal* idempotent of A. If A has a modulus, it is evidently the only principal idempotent element. Hence two principal idempotent elements differ only by an element of the maximal invariant nilpotent sub-algebra.

* Peirce (11), p. 109.

If A has a modulus e', B_1 and B_2 are zero, and $e = e'$. For

$$(e'-e)^2 = e'-e \quad \text{and} \quad (e'-e)\,e = 0 = e\,(e'-e).$$

Hence $e'-e < B$, and is therefore zero.

In (5), $B_1 B_2$ is nilpotent. For, from (5),

$$B_1 A = B_1 B_2 = A B_2,$$

and $$B_2 B_1 \leqslant B, \quad B^a = 0,$$

hence $$(B_1 A)^{a+1} \leqslant B_1 B^a B_2 = 0.$$

But $$A B_1 A = A \,.\, A B_2 \leqslant A B_2 \leqslant B_1 A,$$

$$B_1 A \,.\, A \leqslant B_1 A.$$

Hence $B_1 A = B_1 B_2$ is a nilpotent invariant sub-algebra of A. If $B_1 B_2 = 0$, then

$$(B_1+B_2)^2 = B_2 B_1 \leqslant B,$$

$$A\,(B_1+B_2) \leqslant A B_1 \leqslant B_1+B_2.$$

$$(B_1+B_2)\,A \leqslant B_2 A \leqslant B_1+B_2 \quad \text{and} \quad B_1+B_2 \neq 0,$$

unless A has a modulus. Hence, if an algebra has no modulus, it has a nilpotent invariant sub-algebra.

COROLLARY 1.—B_1 and B_2 are also nilpotent. For suppose $y^2 = y$, $y < B_1$. y can be expressed in the form $y = y_1 + y_2$, where $y_1 < B$, $y_2 < eB_1$, and therefore $y_2^2 = y_1 y_2 = 0$. It follows, then, that

$$y_1^2 = (y-y_2)^2 = y - yy_2 - y_2 y + y_2^2$$
$$= y_1 + y_2 - y_2 y_1.$$

But $eB_1 B \leqslant eB_1$ and $B^2 \leqslant B$; hence we must have

$$y_1^2 = y_1, \qquad y_2 = y_2 y_1,$$

which is impossible, since B, and therefore y_1, is nilpotent. Hence B_1 and B_2 are nilpotent.

COROLLARY 2.—Unless $eB_1 B_2 e = 0$, it is a nilpotent invariant sub-algebra of eAe.

COROLLARY 3.—If the index of A is 1, then $B = B_2 e B_1$, and conversely. For from $A^2 = A$ we deduce

$$B = B^2 + B_2 e B_1 = B^2 + C \text{ (say).}$$

If $B^a = 0$, then $$B = B^{a-1} C + B^{a-2} C + \ldots + C.$$

But $BC \leqslant C$; hence $B = C$, and

$$A = B_2 e B_1 + e B_1 + B_2 e + e A e.$$

If A has no modulus, it is always possible to add one to the algebra. Let e' be the added modulus and let $e_0 = e' - e$; then

$$A = e' A e' = e_0 B e_0 + e B_1 e_0 + e_0 B_2 e + e A e.$$

This form will be of use later.

Algebras which have no nilpotent invariant sub-algebra form a very important class. Such algebras are called *semi-simple.** A semi-simple algebra always has a modulus.

THEOREM 17.—*A semi-simple algebra, which is not simple, is reducible.*

Let A be the algebra and B an invariant sub-algebra. A, having no nilpotent invariant sub-algebra, has a modulus. Hence $AB = B = BA$. If B has no modulus, it has a nilpotent invariant sub-algebra N. BNB is a nilpotent invariant sub-algebra of A and is therefore zero, seeing that A is semi-simple. Also ANA is an invariant sub-algebra of A which is contained in B, and, since A has a modulus, it is not zero unless N is zero. Now, since $ANA \leqslant B$, we have

$$(ANA)^3 = ANA \,.\, N \,.\, ANA \leqslant BNB = 0.$$

Hence $N = 0$ and B has a modulus, and, by Theorem 13, A is reducible. It follows immediately that A can be expressed in the form

$$A = A_1 + A_2 + \ldots + A_n,$$

where $$A_p A_q = 0 = A_q A_p \quad (p \neq q)$$

and $$A_p \quad (p = 1, 2, \ldots, n)$$

are simple. A is therefore the direct sum of $A_1, A_2, \ldots, A_n$.

THEOREM 18.—*If e is an idempotent element of a semi-simple algebra A, then eAe is semi-simple.*

If eAe is not semi-simple, it must necessarily have a nilpotent sub-algebra N. Then ANA is an invariant sub-algebra of A which is not zero. Also $ANA \neq A$, since

$$eANAe = eAeNeAe = N < eAe.$$

Hence, if A is simple the theorem is proved. The main theorem can now be made to depend on this particular case, since any semi-simple algebra

* Cartan (1), p. 57.

can be expressed as the direct sum of simple algebras. The following proof is more direct and also more comprehensive. Let e' be the modulus of A. If, then, $e_1 = e' - e$, we have $ee_1 = 0 = e_1 e$; and therefore

$$e_1 N = 0 = N e_1. \tag{1}$$

We have also $$A = eAe + e_1 Ae + eAe_1 + e_1 Ae. \tag{2}$$

From (1) and (2), it follows that

$$ANA = eAeNeAe + eAeNeAe_1 + e_1 AeNeAe + e_1 AeNeAe_1$$

$$= N + NAe_1 + e_1 AN + e_1 ANAe_1,$$

and $$(ANA)^2 = ANANA = A(N^2 + N^2 Ae)$$

$$= N^2 + e_1 AN^2 + N^2 Ae_1 + e_1 AN^2 Ae_1 = AN^2 A.$$

Similarly $$(ANA)^3 = AN^3 A,$$

and so on. Hence ANA is nilpotent and therefore $N = 0$, since A is semi-simple.

COROLLARY.—If in the above theorem e is primitive, eAe is also primitive.

6. *The Classification of Potent Algebras.*

This section is chiefly concerned with the classification of semi-simple algebras. The result is, however, incomplete in so far as the classification is given in terms of primitive algebras which have themselves not yet been classified. At the same time, a considerable step is made towards the classification of non-nilpotent algebras in general.

Let e_p $(p = 1, 2, \ldots, n)$ be a set of primitive idempotent elements of A, which are so chosen that $e = \sum_{p=1}^{n} e_p$ is a principal idempotent element of A, and $e_p e_q = 0$ $(p \neq q)$. This was shown to be possible in the proof of Theorem 16, where it was also shown that A can be expressed in the form

$$A = B + eB_1 + B_2 e + eAe, \qquad eAe = \sum_{p,q} e_p A e_q.$$

The algebras $e_p A e_q$ occur so frequently in the sequel that the following notation is convenient, viz.,

$$e_p A e_q = A_{pq}, \qquad (e_p + e_q) A (e_r + e_s) = A_{p+q,\, r+s},$$

and so on. It is also convenient to denote elements of A_{pq} by $x_{pq}, y_{pq}, \ldots$.

THEOREM 19.—*If A is simple, $A_{pq} \neq 0$ for any p and q; and if semi-simple, but not simple, then $A_{pq} = 0$ entails $A_{qp} = 0$.*

Suppose that $A_{pq} = 0$, then

$$A_{p+q,\, p+q}\, A_{qp} = (A_{pp} + A_{qp} + A_{qq})\, A_{qp} \leqslant A_{qp},$$

$$A_{qp} A_{p+q,\, p+q} \leqslant A_{qp}.$$

Hence A_{qp} is a nilpotent invariant sub-algebra of $A_{p+q,\, p+q}$, and is therefore zero by Theorem 18. This proves the second part of the theorem. To prove the first part, we observe that, if $e' = e_p + e_q$, A_{pp} is an invariant sub-algebra of $A_{p+q,\, p+q} = e'Ae'$ when $A_{pq} = 0 = A_{qp}$. But $AA_{pp}A \neq A$, since*

$$e'AA_{pp}Ae' = e'Ae'A_{pp}e'Ae' \leqslant A_{pp} < A_{p+q,\, p+q};$$

and therefore $AA_{pp}A$ is an invariant sub-algebra of A. Hence we cannot have $A_{pq} = 0$, if A is simple.

THEOREM 20.—*If A is simple, then $A_{pq} A_{qr} = A_{pr}$, and the order of A_{pq} is the same for all values of p and q.*†

Let $$A' = A_{pq} A_{qp}.$$

From the definition of A_{pp}, we have

$$A' = e_p A' e_p \leqslant A_{pp}.$$

But $$A' A_{pp} \leqslant A' \quad \text{and} \quad A_{pp} A' \leqslant A'.$$

Therefore, either A' is identical with A_{pp} or it is zero. If it is zero, then also $A_{qp} A_{pq} = 0$. For, were $A_{qp} A_{pq} = A_{qq}$, we should have

$$A_{qq}^2 = A_{qp} . A_{pq} A_{qp} . A_{pq} = 0,$$

which is impossible, since A_{qq} is primitive. If $A' = 0$, then

$$A_{p+q,\, p+q}\, A_{pq} = (A_{pp} + A_{pq} + A_{qp} + A_{qq})\, A_{pq} \leqslant A_{pq},$$

$$A_{pq} A_{p+q,\, p+q} \leqslant A_{pq},$$

which is impossible by Theorem 18, since A is simple and A_{pq} is nilpotent.

Hence $$A_{pq} A_{qp} = A_{pp}. \tag{1}$$

Again, since

$$(A_{pp} + A_{pq} + A_{qp} + A_{qq})^2 = A_{p+q,\, p+q}^2 = A_{p+q,\, p+q} = A_{pp} + A_{pq} + A_{qp} + A_{qq},$$

* Cf. the proof of Theorem 18.

† Cartan (1), p. 50.

on multiplying on the left by e_p and on the right by e_q, we get

$$A_{pp}A_{pq}+A_{pq}A_{qq}=A_{pq}.$$

But $$A_{pp}A_{pq}=A_{pq}A_{qp}A_{pq}=A_{pq}A_{qq}$$

by (1); hence $$A_{pp}A_{pq}=A_{pq}=A_{pq}A_{qq}, \tag{2}$$

and, finally, from (1) and (2),

$$A_{pq}A_{qr}=A_{pr}A_{rq}A_{qr}=A_{pr}A_{rr}=A_{pr}.$$

It will now be shown that, if x_{pq} and x_{qr} are any elements, not zero, of A_{pq} and A_{qr} respectively, then $x_{pq}x_{qr}' \neq 0$.

If $x_{pq}x_{qr}=0$, then $x_{pq}x_{qr}A_{rq}=0$. But $x_{qr}A_{rq} \leqslant A_{qq}$, which is primitive; and therefore for any* y_{rq} such that $x_{qr}y_{rq}\neq 0$, there is an x_{qq}, such that $x_{qr}y_{rq}x_{qq}=e_q$. Hence, as $x_{pq}\neq 0$, $x_{pq}x_{qr}A_{rq}=0$ entails $x_{qr}A_{rq}=0$. It follows for any x_{rq} that $x_{qr}x_{rq}=0$; therefore, as above, $x_{rq}A_{qr}=0$; and, as this is true for any x_{rq}, we must have $A_{rq}A_{qr}=0$ in contradiction to the first part of the theorem. Hence $x_{pq}x_{qr}\neq 0$ for any x_{pq} and x_{qr}, and, since $x_{pq}A_{qr}\leqslant A_{pr}$ and $x_{qp}A_{pr}\leqslant A_{qr}$, we have evidently $x_{pq}A_{qr}=A_{pr}$, from which the second part of the theorem follows immediately.

COROLLARY.—For any $x_{pq}\neq 0$, there is an x_{qp} such that $x_{pq}x_{qp}=e_p$. This is evident from the relation $x_{pq}A_{qp}=A_{pp}$.

THEOREM 21.—*If A is simple, it is possible to find a set of n^2 elements e_{pq} $(p, q=1, 2, \ldots, n)$ such that $e_{pq}e_{qr}=e_{pr}$ and $e_{pq}e_{rs}=0$ $(q\neq r)$; and $e=\Sigma e_{rr}$ is the modulus of A.*†

Let $e_{pp}=e_p$ $(p=1, 2, \ldots, n)$. By the corollary to the previous theorem, we can find for any $x_{pq}\neq 0$ an x_{qp} such that $x_{pq}x_{qp}=e_{pp}$. Forming the square of $x_{qp}x_{pq}$, we get

$$x_{qp}x_{pq}x_{qp}x_{pq}=x_{qp}e_px_{pq}=x_{qp}x_{pq};$$

therefore, since e_q is primitive,

$$x_{qp}x_{pq}=e_q=e_{qq}.$$

It is therefore possible to find an algebra of order 4 which has the required laws of combination. Suppose that m^2 elements e_{pq} $(p, q=1, 2, \ldots, m)$

* As previously stated, x_{pq}, y_{pq}, ... will be used to denote elements of A_{pq}.

† Molien (10), p. 124; Cartan (1), p. 46; Frobenius (6), p. 527; Shaw (14), p. 275.

have been found which satisfy these laws, and let $e_{1,m+1}$ be any element of $A_{1,m+1}$. There is then an element $e_{m+1,1}$ of $A_{m+1,1}$ such that

$$e_{1,m+1}\, e_{m+1,1} = e_{11} = e_1.$$

Let

$$\left.\begin{aligned} e_{p1}\, e_{1,m+1} &= e_{p,m+1} \\ e_{m+1,1}\, e_{1p} &= e_{m+1,p} \end{aligned}\right\} \quad (p = 1, 2, \ldots, m).$$

Together with the previous m^2 elements and $e_{m+1,m+1}$, these form an algebra of $(m+1)^2$ elements satisfying the given laws; for

$$e_{pq}\, e_{q,m+1} = e_{pq}\, e_{q1}\, e_{1,m+1} = e_{p1}\, e_{1,m+1} = e_{p,m+1},$$

and similarly

$$e_{p,m+1}\, e_{m+1,r} = e_{pr}.$$

By induction it is therefore possible to find n^2 such elements.

This form of algebra we shall call a *simple* or *quadrate matric* algebra of order n^2.* When a semi-simple algebra is expressed as the sum of simple matric algebras, it is said to be a *matric* algebra.

In accordance with the corollary of Theorem 20, we have

$$A_{pp} = A_{p1}\, e_{1p} = e_{p1}\, A_{11}\, e_{1p}.$$

This gives a 1, 1-correspondence between the elements of the algebras A_{pp} and A_{11}, which is obviously preserved under the operations of addition and multiplication—*i.e.*, the two algebras are simply isomorphic. More generally,

$$A_{pq} = e_{p1}\, A_{11}\, e_{1q},$$

which establishes a 1, 1-relation between the elements of A_{pq} and A_{11}. Let x_{11} be any element of A_{11}, and let the element x_{pq} of A_{pq}, which is associated with it by the above relation, be denoted by

$$x_{pq} = \{x_{11}, e_{pq}\}.$$

Then

$$x_{pq} = \{x_{11}, e_{pq}\} = e_{p1}\, x_{11}\, e_{1q}.$$

Similarly, if $y_{rs} < A_{rs}$, we may write

$$y_{rs} = \{y_{11}, e_{rs}\} = e_{r1}\, y_{11}\, e_{1s},$$

if y_{11} corresponds to y_{rs}. This form of relation is preserved under addition and multiplication, since

$$x_{pq} + y_{pq} = e_{p1}\,(x_{11} + y_{11})\, e_{1q} = \{(x_{pq} + y_{pq}), e_{pq}\},$$

$$\begin{aligned} x_{pq}\, y_{rs} &= e_{p1}\, x_{11}\, e_{1q}\, e_{r1}\, y_{11}\, e_{1s} \\ &= \begin{cases} 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & (q \neq r), \\ e_{p1}\, x_{11}\, y_{11}\, e_{1s} = \{x_{11} y_{11}, e_{pq} e_{qr}\} = \{x_{11} y_{11}, e_{pr}\} & (q = r). \end{cases} \end{aligned}$$

* The algebra is also said to be of degree n. Cartan calls this type of algebra a quaternion.

This result can be expressed as follows. If C is an algebra simply isomorphic with A_{11}, and D is a simple matric algebra of order n^2; and if every element of C is commutative with every element of D; then $A = CD$. In general, if C and D are any algebras such that every element of the one is commutative with every element of the other, and if the order of the complex $A = CD$ is the product of the orders of C and D, then A is an algebra which is called the *direct product** of C and D. The final result can therefore be stated as follows.

THEOREM 22.—*Any simple algebra can be expressed as the direct product of a primitive algebra and a simple matric algebra.*†

Since semi-simple algebras can be reduced to the direct sum of several simple algebras, Theorem 22 amounts to a determination of the form of all semi-simple algebras.

THEOREM 23.—*The direct product A of a primitive algebra B and a quadrate matric algebra C is simple; and any element which is commutative with every other element of A is an element of B.*

Let the basis of C be e_{pq} $(p, q = 1, 2, \ldots, n)$, $e_p = e_{pp}$ $(p = 1, 2, \ldots, n)$ being a primitive set of idempotent elements. If D is any invariant sub-algebra, then $e_{pp} D e_{qq} \leqslant D$, and is not zero for some value of p and q unless $D = 0$. But every element of $e_{pp} D e_{qq}$ is the product of e_{pq} and an element of B; and if $x < B$, then $Bx = B = xB$. Hence $Be_{pq} \leqslant D$.

We have, however,

$$Be_{pq} e_{qr} = Be_{pr}, \qquad e_{sp} Be_{pr} \equiv e_{sp} e_{pr} B = e_{sr} B,$$

for every value of s and r. This gives $A = D$, which proves the first part of the theorem.

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* Scheffers used the term "product" in this sense. As this term is used in this paper in a different sense, I employ the term "direct product," which is used in the theory of groups in a similar sense. Cf. § 11.

† Cartan (1), p. 67, gives this form of a simple algebra in the field of all real numbers, apparently without observing that his result is capable of this simple description.

The theorem may also be proved as follows. If $x < A$, then

$$x = \Sigma x_{pq} \equiv \Sigma e_p x e_q,$$

$$x_{pq} = e_{pq} \sum_r e_{rp} x e_{qr} = \sum_r e_{rp} x e_{qr} \cdot e_{pq},$$

since

$$e_{rp} x e_{qr} = e_{rp} x_{pq} e_{qr}.$$

This method is fully developed in (9), where it is shown that, if B is any matric sub-algebra of A, which has the same modulus as A, then A can be expressed as the direct product of B and some other algebra C.

Again, if x is any element which is commutative with every element of A, then $x = \sum_{r,s} x_{rs} e_{rs}$, where $x_{rs} < B$. But $e_{pq}x = xe_{pq}$; hence

$$\sum_r x_{rp} e_{rq} = xe_{pq} = e_{pq}x = \sum_s x_{qs} e_{ps};$$

therefore $x_{rs} = 0$ $(r \neq s)$ and $x_{pp} = x_{qq}$, *i.e.*, x is an element of B.

This theorem is the converse of the preceding one.

COROLLARY.—The only element of a quadrate matric algebra which is commutative with every other element is the modulus.

THEOREM 24.—*If N is a maximal nilpotent invariant sub-algebra of an algebra A which possesses a modulus, and if $(A-N)$ is simple, then A can be expressed as the direct product of a simple matric algebra and an algebra which contains only one idempotent element.*

From Theorem 22, we have

$$A_{pq} A_{qp} = A_{pp} \pmod{N}.$$

Now $$A_{pp} A_{pq} A_{qp} \leqslant A_{pq} A_{qp}, \qquad A_{pq} A_{qp} A_{pp} \leqslant A_{pq} A_{qp}.$$

Hence, as any invariant sub-algebra of A_{pp} is necessarily nilpotent, we must have $A_{pq} A_{qp} = A_{pp}$. In particular, $A_{pp}^2 = A_{pp}$, and since, when $p = q$, the proof does not assume that e_p is primitive, we also have

$$A^2_{p+q,\,p+q} = A_{p+q,\,p+q}.$$

It may now be proved, as in Theorem 20, that $A_{pq} A_{qr} = A_{pr}$. If x_{pq} is an element of A_{pq} which is not contained in N_{pq}, then $x_{pq} A_{qr} = A_{pr}$. The proof of this is almost exactly as it is given in the proof of Theorem 20, and it is therefore only necessary to give it very briefly. If $x_{pq} A_{qr} < A_{pr}$, there must be some x_{qr} such that $x_{pq} x_{qr} = 0$. But, by Theorem 20, there is an x_{qp} such that $x_{qq} = x_{qp} x_{pq}$ is not zero, and therefore has an inverse, y_{qq}, with respect to e_q. Hence

$$x_{qr} = e_q x_{qr} = y_{qq} x_{qq} x_{qr} = y_{qq} x_{qp} x_{pq} x_{qr} = 0;$$

and therefore $x_{pq} A_{qr} = A_{pr}$. An important consequence of this is that, for any x_{pq} which is not contained in N_{pq}, there is an x_{qp} such that

$$x_{pq} x_{qp} = e_p.$$

It can now be proved, exactly as in Theorems 21 and 22, that A contains a simple matric sub-algebra, and that it can be expressed as the direct product of this matric algebra and an algebra containing only one idempotent element.

It is possible at this point to state Cartan's main theorem regarding

the classification of algebras in the field of ordinary complex or real numbers, if use is made of the fact that, in the latter field, quaternions is the only primitive algebra; and in the former the algebra of one idempotent unit. The result for an arbitrary field seems much more difficult to obtain, the difficulties centring round the proof of the theorem that an algebra with only one idempotent element can be expressed as the sum of a primitive and a nilpotent algebra; a theorem which is obvious in the above two special cases. The proof given in the next section is rather long, but much additional information is obtained in the course of the work.

7. *The Identical Equation.*

This section is not intended as a development of the theory of the identical equation, and so only those points are dealt with which are of importance from our present point of view.

If x is any element of an algebra A, which has a finite basis, the algebra generated by x, being a sub-algebra of A, must itself have a finite basis. x therefore satisfies a relation of the form

$$x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n = 0, \tag{1}$$

where $a_1, a_2, \ldots, a_n$ are marks of the given field, and a_n is to be taken as zero, if the algebra has no modulus, and otherwise as the product of the modulus and a mark of the field. If $x_1, x_2, \ldots, x_n$ is a basis of A and $x = \Sigma\xi_r x_r$, the r-th power of x can be expressed in the form

$$x^r = \Sigma\xi_s^{(r)}x_s,$$

where $\xi_s^{(r)}$ is a rational integral function of the ξ's; hence not more than a powers of x can be independent, and x satisfies an equation of the form (1), where $a_1, a_2, \ldots, a_n$ are now rational integral functions of the ξ's. This equation being an identity in the ξ's, there must be an equation of this form of lowest degree which is satisfied by x whatever values are assigned to the ξ's. This equation is called the *identical* or *characteristic* equation of the algebra. For particular values of the ξ's, x may satisfy an equation of lower degree; but there is evidently at least one x which satisfies no equation of lower degree. The equation of lowest degree satisfied by a particular x has been called by Frobenius the *reduced* equation of that element.

The characteristic of the identical equation will be denoted by $f(x)$, or by $f_x(x)$ where it is desirable to emphasise the fact that the coefficients are functions of x.

If N is the maximal nilpotent invariant sub-algebra of A, a being its index, and if $g(x) = 0$ is the identical equation of $(A-N)$, then $g(x) < N$, if $x < A$, and hence

$$\{g(x)\}^a = 0.$$

$\{g(x)\}^a$ is therefore divisible by $f(x)$. It may, of course, happen that $g(x) = f(x)$, as in the algebra

	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	0
e_4	e_4	0	0	0

where $$x^2 - (\xi_1 + \xi_2)\, x + \xi_1 \xi_2 = 0,$$

if $$x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4.$$

In a primitive algebra, $f(x)$ is irreducible; for otherwise the product of two rational elements would be zero. An immediate consequence of this is that, if the given field is so extended that every equation is soluble, the only primitive algebra in the extended field is the algebra of one unit, $e = e^2$.

THEOREM 25.—*If A is an algebra which is semi-simple in a given field F, and if F' is another field containing F, then A is also semi-simple in F'.**

Since a semi-simple algebra is the direct sum of a number of simple algebras and a simple algebra can be expressed as the direct product of a matric and a primitive algebra, it is sufficient to consider the latter type of algebra.

Let the identical equation of the primitive algebra A be

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_n = 0. \qquad (1)$$

If A has a nilpotent invariant sub-algebra N in the extended field, the identical equation of $(A-N)$ is also $f(x) = 0$, since the latter has no multiple roots. Hence, if z is any element of N and x any element of A, x and $x+z$ have the same identical equation, since they are equal modulo N.

* It is here assumed that rational elements which are independent in F are also independent in F'.

Let $z = \sum_1^a \zeta_r x_r$ be any element of N, the x's forming a rational basis for A. Then

$$z' \equiv x_s z - z x_s = \Sigma \zeta_r (x_s x_r - x_r x_s) \equiv \Sigma \zeta_r x'_r,$$

where x'_r $(r = 1, 2, \ldots, a)$ are rational and $x'_s = 0$. Similarly

$$z'' \equiv x'_t z' - z' x' = \Sigma \zeta_r (x' x'_r - x'_r x'_t) \equiv \Sigma \zeta_r x''_r,$$

where there are now at most $a-2$ terms under the summation sign. This process may be continued till each of the terms $x_r^{(p)}$ under the summation sign after the p-th operation is commutative with $z^{(p)}$, *i.e.*, $z^{(p+1)} = 0$. $z^{(p)}$, being commutative with each of $x_r^{(p)}$ $(r = 1, 2, \ldots, a)$, is also commutative with every element of the algebra generated by them. Let this algebra be denoted by B and its identical equation by $f(x) = 0$. Since $x_1^{(p)}, x_2^{(p)}, \ldots$ are rational, B has a rational basis and is therefore primitive in F. There is then a rational element x whose identical equation, with regard to B, is also its reduced equation, and a non-zero element z of B, which is also an element of N, such that $xz = zx$. Since z is nilpotent, we can obviously assume $z^2 = 0$. As before, $f(x+z) = 0$; hence, on expanding, we get

$$0 = f(x+z) = f(x) + f'(x) z = f'(x) z.$$

But, seeing that B is primitive, $f'(x)$, being of lower degree than $f(x)$, has an inverse; hence $z = 0$, *i.e.*, A has no nilpotent invariant subalgebra and is therefore semi-simple in F'.

THEOREM 26.—*If an algebra is rational in a field F and F' is any field containing F; and if B is the algebra composed of all elements of A which are, in F', commutative with every element of a sub-complex C of A; then, if a rational basis can be chosen for C every element of which possesses an inverse, B is also rational in F.*

Let $x_1, x_2, \ldots, x_a$ be a rational basis of A, then an arbitrary element y of B can be expressed in the form $y = \Sigma \xi_r x_r$, where ξ_r $(r = 1, 2, \ldots, a)$ are marks of F'. If b is the order of B, at least b of the ξ's are linearly independent in F. We may therefore suppose that the first n $(n \geqslant b)$ of the ξ's are linearly independent in F and that the remainder are zero.

Let x be any rational element of C which has an inverse. $xx_1, xx_2, \ldots, xx_a$ are then linearly independent and so also are $x_1 x, x_2 x, \ldots, x_a x$; hence

$$x_r x = \sum_1^a \eta_{rs} x x_s \quad (r = 1, 2, \ldots, a),$$

the η's being rational. Since $xy = yx$, we must have

$$0 = xy - yx = \sum_1^a (\xi_r - \sum_1^a \eta_{sr} \xi_s) x x_r ;$$

hence

$$\xi_r - \sum_{s=1}^a \eta_{sr} \xi_s = 0 \quad (r = 1, 2, \ldots, n).$$

But the ξ's are linearly independent and therefore these equations must reduce to identities. Hence

$$x x_r = x_r x \quad (r = 1, 2, \ldots, n). \tag{1}$$

Now a rational basis can be chosen for C in which every element has an inverse, so that (1) is true for every $x < C$. Hence it is possible to choose a rational basis for B, viz., $x_1, x_2, \ldots, x_n$.

THEOREM 27.—*If F' is a field, containing the given field F, in which every equation is soluble, and if a primitive algebra A is expressed in F' as the direct sum of r simple algebras $A_1, A_2, \ldots, A_r$, these algebras are simply isomorphic with each other and, in F', A can be expressed as the direct product of a commutative algebra, which is rational in F, and an algebra isomorphic with $A_1, A_2, \ldots, A_r$.*

Let $e_1, e_2, \ldots, e_r$ be the moduli of $A_1, A_2, \ldots, A_r$ respectively. Then every element of the algebra $B = e_1, e_2, \ldots, e_r$ is commutative with every element of A, and, conversely, every such element is, by Theorem 23, contained in B. Hence, by the previous theorem, a basis can be found for B which is rational in F. It is easily shown (as in the theory of finite groups) that we can find $a/b = c$ rational elements $x_1, x_2, \ldots, x_c$ such that any element of A can be expressed uniquely in the form

$$x = \sum_1^c y_r x_r,$$

y_r $(r = 1, 2, \ldots, c)$ being elements of B. Hence we have a primitive algebra C of c units in the field F'' obtained by adjoining B to F, and in this algebra, scalar multiples of the modulus are the only elements commutative with every element of C. In F'', C can therefore be expressed as a simple matric algebra $C = (e_{pq})$ of degree $n = \sqrt{c}$.* It follows that A can be expressed as the direct product of C and B.

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* This gives a proof of a theorem by Allan to the effect that the order of a primitive algebra is of the form bn^2. I have only seen an abstract of this paper. See *Amer. Math. Soc. Bull.*, Vol. XI. (1905), p. 351.

THEOREM 28.—*If A is an algebra in which every element, which has no inverse, is nilpotent, it can be expressed in the form $A = B+N$, where B is a primitive algebra and N is the maximal nilpotent invariant sub-algebra.*

We shall first show that the theorem is true in the case where $(A-N)$ is commutative. To do this it is only necessary to show that there is a sub-algebra of A which has the same identical equation, $f(x) = 0$, as $(A-N)$. Let x be an element of A which corresponds to an element of $(A-N)$ whose identical equation is also its reduced equation. If $f(x) = 0$, the theorem is proved. We therefore set $f(x) = z \neq 0$, z being then an element of N which is commutative with x. Let us first suppose that $N^2 = 0$. Then, putting $x - z/f'(x)$ for x in $f(x)$, we get

$$f[x - z/f'(x)] = f(x) - z = 0.$$

The theorem is therefore true in this case and so is also true of $(A-N^2)$ when $N^2 \neq 0$. Hence we can so choose B' in $A = B'+N$ that $B'^2 = B'$ (mod N^2), and therefore $B'+N^2$ is an algebra which can be treated as before. The theorem then follows for commutative algebras by induction. If the given field is a Galois field, it can be shown * that there is no non-commutative primitive algebra. In this case, therefore, the proof of the theorem is complete at this point.

Let us now consider the case where $(A-N)$ is not commutative. Suppose, first, that $(A-N)$ is not simple when the given field is sufficiently extended. There is then a commutative sub-algebra whose elements are commutative with every element of $(A-N)$. To this algebra there corresponds a sub-algebra of A, in which the primitive part B' can be separated from the nilpotent part as above. Hence,† by adjoining B' to the given field as in Theorem 27, we obtain an algebra A' such that $(A'-N)$ remains simple when the given field is extended. It is, therefore, sufficient to confine our attention to such algebras. We shall therefore suppose that, in the extended field F', A can be expressed as the direct product of a simple matric algebra B and an algebra M', which consists of the modulus and a nilpotent algebra M, of index a. Since $M^a = 0$ and every element of M is commutative with every element of B, it follows that every element of M^{a-1} is commutative with every element of A, and therefore, by Theorem 26, we can choose a basis for M^{a-1} which is rational in F. Similarly there is a rational sub-algebra of $(A-N^{a-1})$ corresponding to $(M^{a-2}-M^{a-1})$. This means that we can

* Maclagan Wedderburn (8). † See p. 117.

choose a basis for M^{a-2} such that each element consists of a rational element and an element of M^{a-1} which is not necessarily rational. But, since M^{a-2} contains M^{a-1}, which has a rational basis, we may neglect the non-rational parts, *i.e.*, we can choose a rational basis for M^{a-2}, and hence, by induction, for M. The problem can now be still further simplified by showing that the general case can be made to depend on the case where M consists of a single unit. Let y be any element of M which is not an element of M^2; then, as in Theorem 12, we can express M in the form $M = y+M_1$, where AM_1 is an invariant sub-algebra of A and $N = Ay+AM_1$ (since $N{=}BM{=}AM$). The algebra of $(A-AM_1)$ which corresponds to M then consists of a single unit. If, now, the theorem is true in this particular case, $(A-AM_1)$ can be expressed as the sum of a primitive and a nilpotent algebra, and hence A can be expressed in the form $A = B_1+N$, where $B_1^2 = B_1 \pmod{AM_1}$. Hence B_1+AM_1 is an algebra which can be treated as before, and so on till all the elements of M are exhausted. We shall, therefore, now suppose that the basis of M consists of a single element of y. N^2 is then zero.

For the remainder of the proof we require certain identities * which can be derived from the identical equation as follows:—

If in the identical equation $f_x(x) = 0$ we substitute $x+\xi y$ for x, ξ being a scalar, and expand as a polynomial in ξ, we have a relation which is true for any value of ξ, and hence the coefficients of the various powers of ξ vanish. The following notations are of value in expressing these identities. Let the coefficient of ξ^r in the expansion of $(x+\xi y)^n$ be denoted by $\begin{pmatrix} x & y \\ n-r & r \end{pmatrix}$ and, similarly, the coefficient of $\xi_1^{r_1}\xi_2^{r_2}\ldots\xi_s^{r_s}$ in $(\xi_1 x^{(1)}+\xi_2 x^{(2)}+\ldots+\xi_s x^{(s)})^n$ by $\begin{pmatrix} x^{(1)} & x^{(2)} & \ldots & x^{(s)} \\ r_1 & r_2 & \ldots & r_s \end{pmatrix}$. Thus

$$\begin{pmatrix} x & y \\ n & 1 \end{pmatrix} = x^n y + x^{n-1} y x + \ldots + y x^n.$$

Also let the coefficient of x^{n-r} in $f_x(x)$ be denoted by $\begin{bmatrix} x \\ r \end{bmatrix}$. $\begin{bmatrix} x \\ r \end{bmatrix}$ is of degree r in the coefficients of x. Finally, let $\begin{bmatrix} x^{(1)} & x^{(2)} & \ldots & x^{(s)} \\ r_1 & r_2 & \ldots & r_s \end{bmatrix}$ denote the coefficient of $\xi_1^{r_1}\xi_2^{r_2}\ldots\xi_s^{r_s}$ in the expansion of $\begin{bmatrix} \xi_1 x^{(1)}+\xi_2 x^{(2)}+\ldots+\xi_s x^{(s)} \\ r \end{bmatrix}$, where $r = r_1+r_2+\ldots+r_s$. We may here observe that

$$x\begin{pmatrix} x & y \\ r & s \end{pmatrix} - \begin{pmatrix} x & y \\ r & s \end{pmatrix}x \equiv \begin{pmatrix} x & y \\ r+1 & s-1 \end{pmatrix}y - y\begin{pmatrix} x & y \\ r+1 & s-1 \end{pmatrix}.$$

* Sylvester (15); Shaw (14), p. 284.

With this notation the above mentioned identities can be expressed as follows :—

$$f_x(x) \equiv \sum_{r=0}^{n} \begin{bmatrix} x \\ n-r \end{bmatrix} x^r = 0, \tag{0}$$

$$\sum_{r=0}^{n-1} \begin{bmatrix} x \\ r \end{bmatrix} \begin{pmatrix} x & y \\ n-r-1 & 1 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{bmatrix} x & y \\ r & 1 \end{bmatrix} x^{n-r-1} = 0, \tag{1}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\sum_{s=0}^{r} \sum_{t=0}^{n-r} \begin{bmatrix} x & y \\ t & s \end{bmatrix} \begin{pmatrix} x & y \\ n-r-t & r-s \end{pmatrix} = 0, \tag{r}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$f_y(y) \equiv \sum_{r=0}^{n} \begin{bmatrix} y \\ n-r \end{bmatrix} y^r = 0. \tag{n}$$

Similar identities can easily be obtained by the same method for three or more elements.

In the algebra we are considering, the primitive algebra $(A-N)$ is, in F', equivalent to a matric algebra e_{pq} $(p,\ q = 1, 2, \ldots, n)$, which, by Theorem 24, is a sub-algebra of A in the extended field F'. Hence, if $x_1', x_2', \ldots, x_m'$ are elements of $(A-N)$ corresponding to the rational elements $x_1, x_2, \ldots, x_m$ of A, we must have a relation of the form

$$x_r' = \Sigma \eta_{pqr} e_{pq}. \tag{1}$$

Consider these relations now as defining $x_1', x_2', \ldots, x_m'$ as elements of A and so giving a primitive algebra, isomorphic with $(A-N)$, but not necessarily rational in F. We have, however, $x_r = x_r' \pmod{N}$ or, say,

$$x_r' = x_r - x_r'' y,$$

where it is immaterial whether x_r'' is expressed in terms of $x_1', x_2', \ldots$ or $x_1, x_2, \ldots$, since these differ only by elements of N and $N^2 = 0$. We can choose one of the elements, say $x_1' \neq e$, so that $x_1' = x_1$. For this it is sufficient to choose x_1 so that $f_{x_1}(x_1) = 0$ and then to choose $e_{11}, e_{22}, \ldots, e_{nn}$ so that the primitive idempotent elements of the algebra generated by x_1 are linearly dependent on $e_{11}, e_{22}, \ldots, e_{nn}$. Further, if x_p' $(p \neq 1)$ is irrational, we may suppose $x_p'' = \Sigma \xi_{ps} x_s^{(3)}$, where $x_s^{(3)}$ $(s = 1, 2, \ldots)$ are rational and ξ_{ps} are irrational scalars which are linearly independent* in F. Let us now consider the r-th of the series of invariant relations

* In general we have $x_p'' = \Sigma \xi_{ps} x_s + x_{p_0}$, where x_{p_0} is rational. We may, however, suppose that the rational element $x_{p_0} y$ is included in x_p.

connecting x_1 and x'_p as elements of $(A-N)$, viz.,

$$\sum_s \sum_t \begin{bmatrix} x_1 & x'_p \\ t & s \end{bmatrix} \begin{pmatrix} x_1 & x'_p \\ n-r-t & r-s \end{pmatrix} = 0.$$

Putting $x'_p + x''_p y$ for x'_p in the left-hand side, we get

$$\sum_s \sum_t \begin{bmatrix} x_1 & x_p \\ t & s \end{bmatrix} \begin{pmatrix} x_1 & x'_p & x''_p \\ n-r-t & r-s-1 & 1 \end{pmatrix} y = z, \qquad \text{(i.)}$$

where z is rational, since $x_p = x'_p + x''_p y$ is rational. Also

$$y \begin{pmatrix} x_1 & x'_p + x''_p y & x''_p \\ n-r-t & r-s-1 & 1 \end{pmatrix}$$

$$= y \left\{ \begin{pmatrix} x_1 & x'_p & x''_p \\ n-r-t & r-s-1 & 1 \end{pmatrix} + y \begin{pmatrix} x_1 & x'_p & x''_p & x''_p \\ n-r-t & r-s-2 & 1 & 1 \end{pmatrix} \right\}$$

$$= y \begin{pmatrix} x_1 & x'_p & x''_p \\ n-r-t & r-s-1 & 1 \end{pmatrix},$$

since $y^2 = 0$. Hence we may put x_p for x'_p in (i.). The left-hand side of (i.) then becomes a linear and homogeneous expression in ξ_{ps} $(s = 1, 2, \ldots)$ with rational coefficients and, as the ξ's are linearly independent in F, it cannot equal a rational quantity. Hence it must vanish identically, *i.e.*, $z = 0$. Hence $f(\xi_1 x_1 + \xi_p x_p) = 0$ for all values of ξ_1 and ξ_p and for $p = 1, 2, \ldots, n$. By a repetition of this argument, $\xi_1 x_1 + \xi_p x_p$ taking the place of x_1, we can show that $f(\Sigma \xi_s x_s) = 0$. Furthermore, in the above process x_1 may be replaced by a rational integral function of it, say $h(x_1)$, and, since

$$h(x_1) x_p = h(x_1) x'_p + h(x_1) x''_p y,$$

which is linear in x''_p, x'_p may be replaced by $h(x_1) x'_p$. Hence

$$f(h_1(x_1) + h_2(x_1) x_p h_3(x_1)) = 0,$$

where $h_1(x_1)$, $h_2(x_1)$, and $h_3(x_1)$ are rational integral functions of x_1. Again,

$$x_p^2 = x'^2_p + (x_p x''_p + x''_p x_p) y = x'_{pp} + x''_{pp} y,$$

where
$$x''_{pp} = \Sigma \xi_{ps} (x_p x^{(3)}_{ps} + x^{(3)}_{ps} x_p),$$

the ξ's remaining linearly independent. Hence x_p^2, or any rational integral function of x_p, may take the place of x_p. Combining these results, we find that, if x is any element of the algebra C generated by x_1 and x_p, then $f_x(x) = 0$. This algebra cannot be identical with A. For it would then contain the element y which is commutative with every

other element. Hence, since $f_s(x) = 0$ is the identical equation both of $(A-N)$ and of $C = A$, therefore $f_s(x+y) = 0$. But

$$f_s(x+y) = f_s(x)+f_s'(x)\,y = f_s'(x)\,y \neq 0.$$

Let the theorem be now assumed to hold for algebras of order less than the order of A. C then has a rational primitive sub-algebra C_1, which contains elements congruent to x_1 and x_p modulo N, and is therefore of higher order than the algebra generated by x_1. Let D be any rational primitive sub-algebra of A of order r. Since in the extended field it is equivalent to a matric algebra, we may suppose e_{pq} $(p, q = 1, 2, \ldots, n)$ so chosen that $x_1', x_2', \ldots, x_r'$ form a rational basis of D, and hence $x_1'' = \ldots = x_r'' = 0$. But the algebra generated by D and x_p $(p > r)$ has, as we have shown, $f_s(x) = 0$ as its identical equation. As before, it cannot be equal to A; hence it has a rational primitive sub-algebra which is greater than D, since $x_p' \nless D$. Hence, by a repetition of this process, A can be expressed as the sum of a primitive and a nilpotent algebra. Now the theorem is obviously true of algebras of one unit. Hence, by induction, it is true for algebras of any order.

8. *The Classification of Potent Algebras (continued).*

The results of the preceding sections may be summarised as follows :—

(i.) An algebra can be expressed uniquely as the direct sum of two algebras, one of which has a modulus, and the other no modulus and no integral sub-algebra which has a modulus. (Theorem 10.)

(ii.) An algebra, which has a modulus, can be expressed uniquely as the direct sum of a number of irreducible algebras. (Theorem 10.)

(iii.) Any algebra can be expressed as the sum of a nilpotent algebra and a semi-simple algebra. The latter algebra is not unique, but any two determinations of it are simply isomorphic. (Theorems 24 and 28.)

(iv.) A semi-simple algebra can be expressed uniquely as the direct sum of a number of simple algebras. (Theorems 10 and 17.)

(v.) A simple algebra can be expressed as the direct product of a primitive and a simple quadrate algebra. (Theorems 22 and 23.)

(vi.) A simple quadrate algebra can be expressed as a matric algebra. (Theorem 22.)

The classification of algebras cannot be carried much further than this till a classification of nilpotent algebras has been found which is much more complete than any that has as yet been found.

9. *Non-associative Algebras.*

Many of the results of the previous sections are true of a much larger class of number-systems than the linear associative algebras. In this section I discuss the extension of some of these results to non-associative algebras.

A non-associative algebra differs from an associative one only in that, for some elements, the associative law does not hold true. Throughout this section the term "algebra" will be used to include non-associative algebras as well as associative ones, the appropriate adjective being affixed when it is necessary to distinguish between them.

The calculus of complexes is the same as in § 1, except that $A.BC$ is not necessarily the same as $AB.C$. Hence, any of the previous theorems which do not involve, directly or indirectly, products of more than two members, hold unaltered for non-associative algebras. Thus an invariant sub-complex of an algebra is itself an algebra, and so on, the terms "simple" and "invariant" being defined as in § 2. Hence also, if B_1 and B_2 are invariant sub-algebras of A, B_1+B_2 is also an invariant sub-algebra; and, if B_1 and B_2 are maximal, $A = B_1+B_2$, when $B_1 \neq B_2$.

If B is any sub-algebra of A and $A = B+C$, the elements of C define a new algebra if elements, which differ only by elements of B, are regarded as equal. This algebra, which may be said to be complementary to B, is not, however, unique, since C can be chosen in a variety of ways. But, if B is invariant, it is easily seen that the algebra is unique; it can therefore in this case be denoted by $(A-B)$. The proofs of Theorems 4–6 are therefore applicable word for word to non-associative algebras, the final result being that any two difference series of an algebra with a finite basis differ from one another merely in the order of their terms.

We may notice here a peculiar difference between associative and non-associative algebras, namely, that in the latter an algebra may have all its elements nilpotent and yet be simple. Consider the non-associative algebra A with three units whose multiplication table is

	e_1	e_2	e_3
e_1	0	e_1	e_2
e_2	e_1	0	e_3
e_3	e_2	e_3	0

the given field being $GF[2]$. Here

$$x^2 = (\xi_1 e_1+\xi_2 e_2+\xi_3 e_3)^2 = \xi_1^2 e_1^2+\ldots+\xi_1\xi_2(e_1e_2+e_2e_1)+\ldots = 0,$$

since $$e_1^2 = 0, \quad e_1 e_2 + e_2 e_1 = e_1 + e_1 = 0.$$

Also
$$e_1 x = \xi_2 e_1 + \xi_3 e_2 = x e_1,$$
$$e_2 x = \xi_1 e_1 + \xi_3 e_3 = x e_2,$$
$$e_3 x = \xi_1 e_2 + \xi_2 e_3 = x e_3.$$

At least two of these are independent, say $e_1 x$ and $e_2 x$. Then, if $B = e_1 x, e_2 x$, $AB = A$, this being also true if any other two be taken to be independent. A is therefore simple.

We may also observe that $A^2 = A$, although A has no idempotent element. This marks another difference between the two classes. Another interesting example of this is the algebra

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 + e_2 & e_2 \\ e_2 & e_2 & e_1 \end{array} \tag{2}$$

the field being the same as before. It is easily verified that, in this algebra, the equation $xy = z$ has, for given values of y and z, not both zero, a unique solution x. The algebra has therefore many of the properties of a primitive algebra, although it has no modulus.

The formation of powers in a non-associative algebra is rather complex. Thus $x . x^2$ is not necessarily the same as $x^2 . x$, nor $A . A^2$ the same as $A^2 . A$. We shall use the following notation :—

$$A(A(A \dots (A) \dots)) = A^n,$$
$$(A^n . A^m) A^p = A^{(n+m)+p},$$

and so on, the index indicating the manner in which the terms are grouped. All powers for which the sum of the indices is r, are said to be of the r-th degree.

If all the n-th powers of an algebra are zero, it is said to be a *nilpotent* algebra of *index n*. If A is nilpotent, the sum of the r-th powers is less than the sum of the $(r-1)$-th powers. To show this, let $A^{[s]}$ be the sum of the s-th powers, and suppose that the theorem holds for $s < r$. Then

$$A^{[r]} = A . A^{[r-1]} + A^{[r-1]} . A \leqslant A . A^{[r-2]} + A^{[r-2]} . A \leqslant A^{[r-1]}.$$

But $A^{[r]} \neq A^{[r-1]}$, and $A^{[2]} = A^2 < A$; hence the theorem follows by induction. Now $A^{[r]} A^{[s]} \leqslant A^{[r+s]}$, and $A^{[r]s} \leqslant A^{[r]}$. Hence, as in Theorem 7, we may express A in the form

$$A = B_1 + B_2 + \dots + B_{n-1},$$

where
$$B_p B_q \leqslant B_{p+q} + B_{p+q+1} + \dots .$$

Every element of a nilpotent algebra is nilpotent in the sense that, for some n, all its n-th powers are zero. This condition is, however, not sufficient to render the algebra nilpotent, as may be seen from the first of the examples given on p. 110. A sufficient condition is, however, not difficult to find. If n is the index of a nilpotent algebra A, then $A^{[n]} = 0$, and in particular, if x and y are any two elements,

$$y\,(y\,(\,\dots\,y\,(yx)\,\dots\,)\,) = 0.$$

Now the proof of Theorem 14 holds for non-associative algebras step for step, except that we cannot deduce from $A'x = A'$ that A has an idempotent element. There is, however, an element y such that $yx = x$, from which it follows that

$$y\,(y\,(\dots\,y\,(yx)\,\dots\,)\,) \neq 0,$$

and A is therefore not nilpotent. Hence a necessary and sufficient condition that A is nilpotent is that it contains no pair of elements y and x such that $yx = x$ $(x \neq 0)$. y, of course, need not be distinct from x.

Of the remaining theorems of Section 5, Theorems 9 and 13 hold also for non-associative algebras. The others deal chiefly with idempotent elements and do not seem to have any direct analogue in the general theory.

A rough classification of non-associative algebras may, however, be obtained as follows.

In an algebra A there will, in general, be a sub-algebra M_1 composed of all elements z, such that $z.xy = zx.y$ for any elements x and y of A. The modulus, if the algebra has one, will be contained in it. For this reason I shall call it the *modular sub-algebra of the first kind.* Similarly, the elements z such that $x.zy = xz.y$ form an associative algebra M_2 which may be called the *modular sub-algebra of the second kind;* and elements such that $x.yz = xy.z$ form an associative algebra M_3 called the *modular sub-algebra of the third kind.* The elements common to all three will be called the *principal modular sub-algebra* of A. For example, in the algebra

	e_1	e_2	e_3	e_4
e_1	e_1	0	0	0
e_2	0	e_2	e_3	e_4
e_3	e_3	0	0	e_4
e_4	e_4	0	e_4	0

we have $$M_1 = M_2 = M_3 = M = e_1,\ e_2\,;$$

and in

	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_2	e_3
e_3	e_2	e_3-e_2	e_2

$$M_1 = e_1,\ e_2,\ M_2 = M_3 = M = (e_1+e_2).$$

If $e_1, e_2, \ldots, e_m$ is a primitive set of idempotent elements of M, we have

$$A = \sum_{p,q} A_{pq}, \quad A_{pq}A_{rs} = 0 \ (q \neq r), \quad A_{pq}A_{qr} \leqslant A_{pr}.$$

This is analogous to Pierce's form for a linear associative algebra, and a partial classification of non-associative algebras can obviously be based upon it.

10. *Semi-invariant Sub-algebras.**

A sub-algebra B of A is said to be semi-invariant if either $AB \leqslant B$ or $BA \leqslant B$. We shall assume throughout this section that $AB \leqslant B$.

If B_1 and B_2 are two different maximal semi-invariant sub-algebras of A, then evidently $B_1+B_2 = A$, since $A\,(B_1+B_2) \leqslant B_1+B_2$. Further, if $B = B_1 \frown B_2$, it may be shown that the difference algebras complementary to B_1, B_2 and B, may be so chosen† that

$$(A-B_1) \sim (B_2-B), \qquad (A-B_2) \sim (B_1-B).$$

It then follows, as in Theorem 6, that, if

$$A,\ B_1,\ B_2,\ \ldots;\quad A,\ B_1',\ B_2',\ \ldots$$

are two series of algebras such that each of them is a maximal semi-invariant sub-algebra of the preceding term, then the corresponding series of difference algebras can be so chosen that they differ merely in regard to the order in which their terms occur.

In a potent associative algebra A, a maximal nilpotent semi-

* The proofs of the theorems of this section are merely repetitions of what has already been done and are, therefore, for the most part omitted.

† Since $(A-B_1)$, ... are not uniquely determined, these symbols have no meaning unless it is shown how these algebras are to be determined, *e.g.*, in this case by setting

$$B_1 = C_1+B,\quad C_1 \frown B = 0;\qquad B_2 = C_2+B,\quad C_2 \frown B = 0;$$
$$A = C_1+C_2+B.$$

$(A-B)$ is of course not necessarily simple when B is maximal.

invariant sub-algebra is invariant, and is therefore unique. For

$$AN \leqslant N, \quad NA.A \leqslant NA, \quad A.NA \leqslant NA,$$
$$(NA)^2 = NANA \leqslant N^2A, \quad (NA)^a \leqslant N^aA = 0,$$

if $N^a = 0$. Hence NA is a nilpotent invariant sub-algebra of A, and therefore either $NA \leqslant N$ or $A = NA+N$. In the latter case,

$$A^a \leqslant N^aA+N^a;$$

and therefore A is nilpotent contrary to our assumption. Hence we must have $NA \leqslant N$, which proves the theorem.

Suppose now that both A and B have a modulus, the moduli being respectively e and e_1. Then, if $e_2 = e-e_1$,

$$A = Ae_1+e_1Ae_2+e_2Ae_2 = B+C+D,$$

where $\qquad B = Ae_1, \quad C = e_1Ae_2, \quad D = e_2Ae_2,$

and $\qquad B \frown (C+D) = 0 \quad \text{and} \quad C \frown D = 0.$

Since $A^2 = A$, we have

$$A = Ae_1Ae_1+e_1Ae_2Ae_1+e_2Ae_2Ae_1+Ae_1Ae_2+e_1Ae_2Ae_2+e_2Ae_2Ae_2.$$

Therefore $D^2 = D$, and the multiplication table of A has the form

	B	C	D
B	B	C	0
C	0	0	C
D	0	0	D

C is a nilpotent invariant sub-algebra of A whose complementary algebra is reducible. Hence no semi-invariant sub-algebra of a semi-simple algebra has a modulus. We may also notice that D is a left-hand semi-invariant sub-algebra, and that $B+C$ and $D+C$ are invariant sub-algebras of A.

A primitive algebra is the only type of algebra which has no semi-invariant sub-algebra. For, if A has no semi-invariant sub-algebra, it must have a modulus, and if x is any element of A which has no inverse, Ax is a semi-invariant sub-algebra of A.

11. *The Direct Product.*

Let $A = x_1, x_2, \ldots, x_a$, $B = y_1, y_2, \ldots, y_b$ be two complexes of order a and b respectively, such that every element of A is commutative with

every element of B; and further, let all the elements

$$x_r y_s \ (r = 1, 2, \ldots, a; \ s = 1, 2, \ldots, b)$$

be linearly independent; then the complex

$$C = x_1 y_1, x_1 y_2, \ldots, x_r y_s, \ldots$$

is called the *direct product* of A and B.

The following is an alternative definition. Consider all pairs of elements of the form (x, y) where $x < A$ and $y < B$. Let

$$(x+x', y+y') = (x, y)+(x', y')+(x, y')+(x', y)$$

and

$$(x, y)(x', y') = (xx', yy').$$

The elements (x, y) generate an algebra of which they themselves form a complex of order ab which is said to be the direct product of A and B and is denoted by $A \times B$. $A \times B$ is of course the same as $B \times A$.

We shall generally take A and B to be algebras, in which case $A \times B$ is an algebra.

The following relations follow immediately from the definition of $A \times B$.

$$A \times (B \times C) = (A \times B) \times C,$$
$$A \times (B + C) = A \times B + A \times C,$$
$$A \times (B \frown C) = A \times B \frown A \times C.$$

If $A = B \times C$ has a modulus, B and C must each have a modulus and conversely. In this case there is also a sub-complex of A isomorphic with B, namely, the direct product of B and the modulus of C. Also, if B' and C' are the sub-complexes of A which correspond to B and C, then

$$A = C'B' = B'C'.$$

If B has an invariant sub-algebra B_1, $B_1 \times C$ is evidently an invariant sub-algebra of A; hence, if A is simple, B and C are also simple. The converse of this is, however, not always true. For instance, let

e_1	e_2
e_2	$-e_1$

be the table of B, and let $C = B$; then the table of A is

e_1	e_2	e_3	e_4
e_2	$-e_1$	e_4	$-e_3$
e_3	e_4	$-e_1$	$-e_2$
e_4	$-e_3$	$-e_2$	e_1

where $\qquad e_1 = (e_1, e_1), \quad e_2 = (e_1, e_2), \quad e_3 = (e_2, e_1),$

and $\qquad e_4 = (e_2, e_2).$

If we put $\qquad e_1' = \frac{1}{2}(e_1+e_4), \quad e_2' = \frac{1}{2}(e_2-e_3),$

$$e_3' = \frac{1}{2}(e_1-e_4), \quad e_4' = \frac{1}{2}(e_2+e_3),$$

the table becomes

	e_1'	e_2'	e_3'	e_4'
e_1'	e_1'	e_2'	0	0
e_2'	e_2'	$-e_1'$	0	0
e_3'	0	0	e_3'	e_4'
e_4'	0	0	e_4'	$-e_3'$

Hence $B \times C$ is reducible. If, however, the given field is such that every simple algebra is matric, the converse does hold; therefore, in any field, the product of two simple algebras is simple or semi-simple.

It is interesting to note that the algebra given above can also be expressed as the direct product of B and the algebra C_1 whose table is

	e_1	e_2
e_1	e_1	0
e_2	0	e_2

Hence, from $A = B \times C = B \times C_1$, it does not necessarily follow that $C \sim C_1$. This is, however, probably true if the field is sufficiently extended.

12. *Conclusion.*

It is remarkable that the properties of a field with regard to division are not used in many of the theorems of the preceding sections. The first place, where it is used, is where it is assumed that, if $A^2 < A$, the order of A^2 is less than the order of A. Thus, if the table of an algebra is

	e_1	e_2
e_1	$2e_1$	$2e_2$
e_2	$2e_2$	$2e_1$

and the set of positive and negative integers takes the place of the given field, then $A^2 = 2e_1, 2e_2$, which is not equivalent to A, but is still contained in A. In other words, if $B < A$ and $A = B + C$, then, for every such C, B is contained in C.

If we now call B a proper sub-complex of A when we can find C such that $A = B+C$, $B \frown C = 0$, and, in Theorem 2, substitute "proper invariant sub-complex" for "invariant sub-complex" throughout, we find that all the theorems of the section hold without further modification. Most of the theorems of the other sections can be modified in a similar fashion. Thus, Theorem 15, when modified, would read:—*If A is an algebra with not more than one idempotent element, and x is any element such that Ax is a proper sub-complex of A, then x is nilpotent.*

I have not carried out this process in detail, as the results obtained do not seem to be of sufficient importance.

[*Added February 1st*, 1908.—Since the above paper was in print I have noticed a mistake in the proof of Theorem 28; this mistake is, however, easily remedied. The notation used below is that of page 105.

It is there assumed that the algebra B' is commutative with every element of A. Suppose that this is not the case, and let M be the maximum sub-algebra of N which is composed of elements commutative with every element of B'. As on page 105, we may assume $N^2 = 0$. Let x, y, and z be elements of A, B', and M respectively. From the definition of B', we have $xy - yx < N$, and therefore, since $N^2 = 0$ and $M \leqslant N$, $xzy = xyz = yxz$. Hence $xz < M$, *i.e.*, M is invariant. Now, if we prove the theorem for $(A-M)$, it follows for A as in the text; for if the theorem is true for $(A-M)$, then A can be expressed in the form $A_1 + N_1$ where N_1 is nilpotent and A_1 is an algebra, containing B', of which M is the maximal invariant nilpotent sub-algebra; B' is then commutative with every element of A_1 and the proof proceeds as on page 105. We may therefore suppose that there are no elements of N commutative with every element of B', *i.e.*, $M = 0$.

If the given field is sufficiently extended, it follows from Theorems 22 and 27 that A contains a simple matric algebra A' such that $(A-N)$ is the direct product of A' and B'; and, since $M = 0$, evidently the elements of $A'B'$ are the only elements of A which are commutative with every element of B'. But B' is rational; hence, by Theorem 26, $A'B'$ is also rational if B' is of order greater than 1, *i.e.*, the theorem is true in this case. We may therefore assume that every element of B' is commutative with every element of A, as we have shown that the theorem follows if this is not the case.]

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3. L. E. Dickson.—"On Hypercomplex Number Systems," *Amer. Math. Soc. Trans.*, Vol. VI. (1905), p. 344.
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5. S. Epsteen and J. Wedderburn.—"On the Structure of Hypercomplex Number Systems," *Amer. Math. Soc. Trans.*, Vol. VI. (1905), pp. 172–178.
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7. H. E. Hawkes.—"On Hypercomplex Number Systems," *Amer. Math. Soc. Trans.*, Vol. III. (1902), pp. 312–330.
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11. B. Peirce.—"Linear Associative Algebra" (1870), *Amer. Journ. of Math.*, Vol. IV. (1881), p. 97.
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CONTENTS.

ADDENDUM TO A PAPER "ON THE INVERSION OF A REPEATED INFINITE INTEGRAL"*

By T. J. I'A. BROMWICH.

[Received June 29th, 1907.—Read November 14th, 1907.]

MR. G. H. HARDY has pointed out to me that the conditions (i.)–(v.) of § 5 of this paper are satisfied by the integral (*loc. cit.*, p. 192)

$$(1)\qquad \int_0^\infty x^{p-1}e^{-x}dx \int_0^\infty (e^{-y}-e^{-xy})\,\frac{dy}{y},$$

provided that q (the real part of p) is *positive;* so that it is unnecessary to suppose $q>1$ as I did in my original investigation. The following method is substantially the same as Mr. Hardy's.

It will be seen from my work on p. 193 that the condition $q>1$ is only introduced in proving that the integral

$$(2)\qquad \int_0^\xi x^{p-1}e^{-x}dx \int_\eta^\infty (e^{-xy}-e^{-y})\,\frac{dy}{y}\quad (0<\xi\leqslant 1)$$

tends to zero as η tends to infinity.

Now the absolute value of (2) is less than

$$(3)\qquad \int_0^\xi x^{q-1}dx \int_\eta^\infty (e^{-xy}-e^{-y})\,\frac{dy}{y}$$

and $$\int_\eta^\infty (e^{-xy}-e^{-y})\,\frac{dy}{y} = \int_{x\eta}^\infty e^{-u}\,\frac{du}{u} - \int_\eta^\infty e^{-u}\,\frac{du}{u} = \int_{x\eta}^\eta e^{-u}\,\frac{du}{u}.$$

Thus $$\int_\eta^\infty (e^{-xy}-e^{-y})\,\frac{dy}{y} < e^{-x\eta}\int_{x\eta}^\eta \frac{du}{u} = e^{-x\eta}\log\left(\frac{1}{x}\right),$$

and so (3) is less than †

$$(4)\qquad \int_0^\xi e^{-x\eta}\log\left(\frac{1}{x}\right)x^{q-1}dx \leqslant \int_0^1 e^{-x\eta}\log\left(\frac{1}{x}\right)x^{q-1}dx.$$

Now the last integral in (4) converges (at $x=0$) *uniformly* for all

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 176.

† This inequality constitutes the essential improvement introduced by Mr. Hardy; the method of my paper used $1/x$ instead of $\log(1/x)$.

positive values of η, since the integrand is less than that of the convergent integral

$$\int_0^1 \log\left(\frac{1}{x}\right) x^{q-1}\, dx \quad (\text{if } q > 0).$$

Thus, by a familiar result* relating to uniformly convergent integrals, we find

$$(5) \qquad \lim_{\eta\to\infty} \int_0^1 e^{-x\eta} \log\left(\frac{1}{x}\right) x^{q-1}\, dx = 0$$

because

$$\lim_{\eta\to\infty} e^{-x\eta} = 0 \quad (\text{if } x > 0).$$

From the results (3)–(5) we see that the integral (2) tends to zero, provided that q is *positive*: and so the conditions of my paper are then satisfied by the integral (1).

Now the condition $q > 0$ is certainly necessary† as well as sufficient for the inversion of the order of integration to be permissible in the integral (1). It is therefore suggested that the range of my conditions is not so limited as seemed probable from the original discussion of this example (see p. 194, top).

As a matter of fact, the conditions given in § 5 of my paper will be *necessary* as well as sufficient whenever the equation

$$(6) \qquad \int_\xi^\infty dx \int_\eta^\infty f(x, y)\, dy = \int_\eta^\infty dy \int_\xi^\infty f(x, y)\, dx$$

is true for *all* values of ξ, η. And this will be true in the ordinary applications of the conditions, although it would seem to be possible to build up examples in which (6) might hold for certain values of ξ, η, but not for others. Thus it appears that my conditions are necessary as well as sufficient whenever the difficulty is due *solely* to the presence of infinity in the upper limits of integration.

Dr. Hobson has remarked‡ that the condition (ii.) given on p. 185 of my paper really contains *both* conditions (i.) and (ii.) as given on p. 184; and therefore condition (i.) is rendered superfluous when the second form of condition (ii.) is used. This fact follows at once from the inequality (4) of p. 185: since, if ν is there allowed to tend to infinity, we find that

$$(7) \qquad |A - y_\mu| \leqslant \epsilon, \qquad |a - y_\mu| \leqslant \epsilon \quad (\text{if } \mu \geqslant m_0),$$

* See, for instance, Art. 172 of the Appendix to my book on *Infinite Series*.

† In fact, the integral (1) is not convergent unless $q > 0$.

‡ Hobson, *Theory of Functions of a Real Variable*, p. 446.

where A, a are the maximum and minimum limits of z_ν. Thus from (7) we have

$$A - a \leqslant 2\epsilon \quad (\text{if } \mu \geqslant m_0).$$

But A and a do not depend on the variable μ, and so we must have

$$A = a. \tag{8}$$

Thus from (7) and (8) we see that y_μ has a limit a whenever the inequality (4) of p. 185 is satisfied, and that z_ν has then the same limit a. Thus the sufficiency of condition (ii.) on p. 185 is completely established, and its necessity was proved by my former investigation; * the condition is therefore both necessary and sufficient.

On the other hand, if we apply the method given here to the inequality (3) on p. 184, the existence of $a = \lim z_\nu$ can be at once deduced; but we can only prove that a is *one* of the limits of y_μ, and accordingly the existence of $\lim y_\mu$ is here an additional necessary condition.

* My proof shows that, if $\lim y_\mu$ and $\lim z_\nu$ exist and are equal, then the inequality (4) of p. 185 can be inferred. To prove the sufficiency, I contented myself with the remark that the inequality (4) is more stringent than (3); the reason for this additional stringency is now evident, because the second inequality includes the first and *also* the condition (i.) of p. 184.

ON THE INVARIANTS OF A BINARY QUINTIC AND THE REALITY OF ITS ROOTS

By H. F. BAKER.

[Received October 19th, 1907.—Read November 14th, 1907.]

THE present note consists of three parts. In § 1 it is shewn that the known rational relation connecting the four fundamental invariants of a binary quintic is resoluble; namely, that three rational invariants, each rational in the four fundamental invariants, can be taken, in terms of which each of the four fundamental invariants can be rationally expressed; thus all the rational invariants of the quintic are rationally expressible in terms of three, these being two absolute invariants and one of effective degree 2, the quotient of an integral invariant of degree 18 by one of degree 16. In § 2 a discrimination of the roots of the quintic, *when its coefficients are real,* is given in terms of the two absolute invariants spoken of. In § 3, a proof of the results of § 2 is given, by means of Sylvester's canonical form. The roots of the quintic were first discriminated in terms of the invariants by Hermite (*Camb.* and *Dublin Math. Jour.*, Vol. IX.); much simpler results were obtained by Sylvester (*Phil. Trans.*, Vol. CLIV., 1864; *Compt. Rend.*, Vol. LIX., 1864), and these have been partly reconsidered, and the results, as stated by Sylvester in the first of these papers, corrected by Salmon, *Higher Algebra,* and Cayley, *Coll. Papers,* Vol. VI. In general the method followed here, in § 3, is put together from Sylvester's memoir; but for the case when the quintic has a repeated root, which is not considered by Salmon or Cayley, I have made an independent examination.

1.

Let I, J, K, L be the invariants, respectively, of degrees 18, 4, 8, 12, belonging to a binary quintic form, given for example in Salmon's *Higher Algebra*, 1885, Lesson XVIII.; they are connected by the equation

$$16I^2 = J(K^2-JL)^2+8K^3L-72JKL^2-432L^3.$$

Put $$\xi = \frac{L^4}{(K^2-JL)^3},\quad \eta = \frac{KL^2}{(K^2-JL)^2},\quad \zeta = \frac{I}{K^2-JL},$$

so that ξ, η, ζ are invariants respectively of degrees 0, 0 and 2; also let ϖ denote the absolute invariant, rational in ξ and η, given by

$$\varpi = -\xi+\eta^2+72\xi\eta-432\xi^2-64\eta^3;$$

then we have

$$\begin{aligned}\varpi(K^2-JL)^6 &= -L^4(K^2-JL)^3+K^2L^4(K^2-JL)^2+72KL^6(K^2-JL)\\ &\qquad -432L^8-64K^3L^6\\ &= L^5\{J(K^2-JL)^2+8K^3L-72JKL^2-432L^3\}\\ &= 16I^2L^5,\end{aligned}$$

so that $$\varpi = \frac{16I^2L^5}{(K^2-JL)^6};$$

hence $$\frac{2^{16}\xi^5\zeta^9}{\varpi^4} = \frac{2^{16}L^{20}I^9}{(K^2-JL)^{24}}\,\frac{(K^2-JL)^{24}}{2^{16}I^8L^{20}} = I,$$

$$\frac{2^4(\eta^2-\xi)\zeta^2}{\varpi} = \frac{2^4[K^2L^4-L^4(K^2-JL)]I^2}{(K^2-JL)^6}\,\frac{(K^2-JL)^6}{2^4I^2L^5} = J,$$

$$\frac{2^8\xi^2\eta\zeta^4}{\varpi^2} = \frac{2^8L^8KL^2I^4}{(K^2-JL)^{12}}\,\frac{(K^2-JL)^{12}}{2^8I^4L^{10}} = K,$$

$$\frac{2^{12}\xi^4\zeta^6}{\varpi^3} = \frac{2^{12}L^{16}I^6}{(K^2-JL)^{18}}\,\frac{(K^2-JL)^{18}}{2^{12}I^6L^{15}} = L,$$

that is $$I = \frac{2^{16}\xi^5\zeta^9}{\varpi^4},\quad J = \frac{2^4(\eta^2-\xi)\zeta^2}{\varpi},\quad K = \frac{2^8\xi^2\eta\zeta^4}{\varpi^2},\quad L = \frac{2^{12}\xi^4\zeta^6}{\varpi^3},$$

whereby the four invariants I, J, K, L are rationally expressed in terms of the three invariants ξ, η, ζ, which are themselves rational in I, J, K, L.

2.

We put $$X = 12^3.\xi,\qquad Y = 48.\eta,$$

and regard X, Y as rectangular Cartesian coordinates. The discriminant D, of the quintic, known to be equal to J^2-2^7K, is given by

$$\begin{aligned}D &= \frac{2^8\zeta^4}{\varpi^2}[(\eta^2-\xi)^2-2^7\xi^2\eta]\\ &= \frac{\zeta^4}{2^4.3^6\varpi^2}[(\tfrac{3}{4}Y^2-X)^2-\tfrac{8}{3}X^2Y].\end{aligned}$$

The curve $$(\tfrac{3}{4}Y^2 - X)^2 = \tfrac{8}{3}X^2Y$$

has a cusp at the origin, one cuspidal branch lying below the asymptote $Y = \frac{3}{8}$, the other cutting it, either of these being continuous at infinity with a branch lying above the asymptote for every point of which $X \leqslant -1$; this is the curve against which the letter D is placed in the diagram.

The curve $\varpi = 0$, easily seen to be the same as

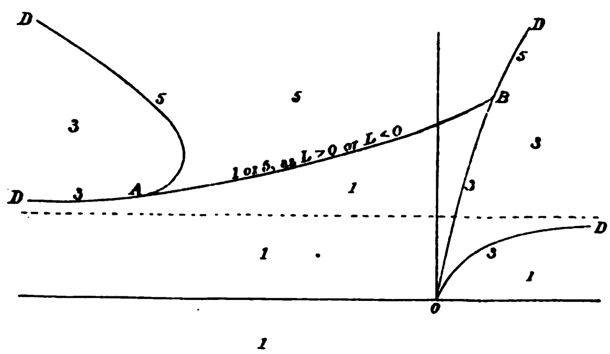
$$(X - 3Y + 2)^2 + 4\,(Y-1)^3 = 0,$$

touches the curve D at $[-2(\frac{6}{7})^3, \frac{24}{49}]$, and cuts it at $(2^8/11^3, 3.2^5/11^2)$, and for the present need only be drawn between these points.

Thereby the finite portion of the plane may be said to be divided into four regions. The quintic being supposed to have real coefficients, each of these regions corresponds to a definite number of real roots of the quintic; these numbers are placed in the diagram.

Further, the curve D is, by the arc of $\varpi = 0$ which is drawn, divided, we may say, into four arcs; each of these corresponds to a definite number of real roots of the quintic, shewn by the attached numbers; and the points of the arc of $\varpi = 0$ which is drawn correspond to one or other of two cases, in regard to the number of real roots, according as $L \gtrless 0$.

Finally, the origin, and the two points spoken of, where $\varpi = 0$ meets the curve D, are associated with certain numbers of real roots of the quintic. These are given in association with the diagram.



At A, if $L > 0$, there is 1 real root and two complex roots both repeated; if $L < 0$, there is 1 real root and two real roots both repeated, 5 in all.

At B, if $L > 0$, there are 3 real roots, one a double root; if $L < 0$, there are 5 real roots, one a double root.

At 0, if $D > 0$, there is 1 real root;
if $D \lessdot 0$, there are 3 real roots.

Analytically,

$D < 0$, 3 real roots.

$D > 0$, $\varpi < 0$, $Y > 24/49$, 5 real roots.

$D > 0$, $\varpi > 0$, or $Y < 24/49$, or both, 1 real root.

$D = 0$, $\varpi < 0$, $Y > 24/49$, 5 real roots.

$D = 0$, $\varpi > 0$, or $Y < 24/49$, or both, 3 real roots.

$D = 0$, $\varpi = 0$, $Y = 24/49$, $L > 0$, 1 real root; $L < 0$, 5 real roots.

$D = 0$, $\varpi = 0$, $Y = 3.2^5/11^2$, $L > 0$, 3 real roots; $L < 0$, 5 real roots.

$\left.\begin{array}{l} D > 0,\ \varpi = 0,\ Y = 0, \text{ 1 real root} \\ D \leqslant 0,\ \varpi = 0,\ Y = 0, \text{ 3 real roots} \end{array}\right\}$ included in a former statement.

3.

We consider the quintic with real coefficients

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

excluding from consideration the case when three or more roots are equal.

We assume, then, that the quintic can be brought to the form

$$ru^5 + sv^5 + tw^5,$$

where u, v, w are linear functions of x and y, chosen so that their sum is zero, and r, s, t are constants, provided that the invariant L does not vanish; and that, if L vanishes, and no other restrictive condition is satisfied, the quintic can be brought to the form

$$Au^5 + 5Euv^4 + Fv^5,$$

where u, v are real linear forms, and A, E, F are real constants, while if $L = 0$, $J = 0$, this form reduces to

$$Au^5 + 5Euv^4,$$

and that, in all other cases in which $L = 0$, save those in which the quintic has three or more equal roots, the first form

$$ru^5 + sv^5 + tw^5$$

is valid (Sylvester, *loc. cit.*). The invariants J, K of the form

$$Au^5 + 5Euv^4 + Fv^5$$

are $J = A^2F^2$, $K = -2A^3E^5$ (Salmon, *Higher Algebra*).

We put $$\xi = \frac{L^4}{(K^2-JL)^3}, \qquad \eta = \frac{KL^2}{(K^2-JL)^2},$$
and interpret $$X = 12^3\xi, \qquad Y = 48\eta$$
as rectangular Cartesian coordinates. The cases $L = 0$, or $L = 0$, $J = 0$, lead then to the single point $X = 0$, $Y = 0$, which we consider, briefly, later; for all other cases we may deal with the quintic under the form $ru^5+sv^5+tw^5$.

Calculated for this form the invariants of the quintic are

$$I = r^5s^5t^5(s-t)(t-r)(r-s), \quad K = r^2s^2t^2(st+tr+rs), \quad L = r^4s^4t^4,$$
$$J = (st+tr+rs)^2-4rst(r+s+t);$$

these give, if $\mu = 4(r+s+t)$,

$$\frac{1}{\mu^2}(st+tr+rs = \eta, \quad \frac{1}{\mu^4}rst(r+t+s) = \tfrac{1}{4}\xi, \quad \frac{1}{\mu^6}r^2s^2t^2 = \xi^2,$$

so that the three quantities $\mu^{-2}st$, $\mu^{-2}tr$, $\mu^{-2}rs$ are the roots of the cubic equation in ψ,
$$\psi^3-\eta\psi^2+\tfrac{1}{4}\xi\psi-\xi^2 = 0;$$
the discriminant of this is at once found to be
$$-\xi^2(-\xi+\eta^2+72\xi\eta-432\xi^2-64\eta^3),$$
or $$-\xi^2\varpi,$$
so that the roots of the cubic are real when ϖ is positive, and two of them conjugate imaginaries when ϖ is negative. We have found
$$\varpi/L = 16I^2L^4/(K^2-JL)^6,$$
so that the signs of L and ϖ are the same; thus, as L is a negative multiple of the discriminant of the canonizant
$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix},$$
of which u, v, w are the linear factors,

(1) When ϖ is positive, in the canonical form
$$ru^5+sv^5+tw^5 = 0,$$
u, v, w, r, s, t are real;

(2) When ϖ is negative, two of the linear forms u, v, w, which we take to be u and v, are conjugate imaginaries, and w is real, while, correspondingly, r and s are conjugate imaginaries and t is real. Taking account

of the fact that the original quintic is real, these results are easily found to hold also when $t = 0$, in which case both L and the canonizant vanish identically.

Conversely, when ξ and η are assigned, the above cubic determines ratios for the coefficients r, s, t, and, when ϖ, which is a definite function of ξ, η, is positive, we may take real linear functions u, v, w, while, when ϖ is negative, we may take u, v conjugate complex linear functions; thus, save when $\varpi = 0$, to each value of ξ and η belongs a class of quintic forms. When $\varpi = 0$, and $L \neq 0$, we may take $r = s$, both these being real since they are roots of a cubic equation with real coefficients, but the linear forms u, v may be either real or conjugate complexes.

Consider now the discriminant, D, of the quintic; it is known that, when it is negative, the quintic has three real roots, and, when it is positive, the quintic has either one or five real roots; its value is

$$\begin{aligned} D &= J^2 - 2^7 K \\ &= \frac{2^8 \zeta^4}{\varpi^2}\left[(\eta^2-\xi)^2 - 2^7\xi^2\eta\right], \end{aligned}$$

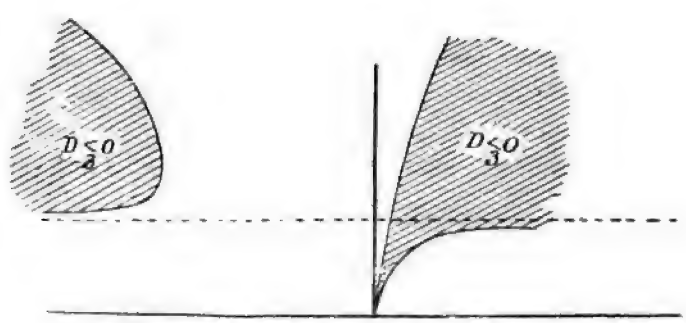
where, as before, $$\zeta = \frac{I}{K^2 - JL};$$

thus D is positive or negative according as $(\eta^2-\xi)^2 \gtrless 2^7\xi^2\eta$, or, substituting $\xi = 12^{-3}X$, $\eta = 48^{-1}Y$, according as

$$(\tfrac{3}{4}Y^2 - X)^2 \gtrless \tfrac{8}{3}X^2 Y,$$

namely, $$(1-\tfrac{8}{3}Y)\left\{X - \frac{\frac{3}{4}Y^2}{1+\left(\frac{8Y}{3}\right)^{\frac{1}{2}}}\right\}\left\{X - \frac{\frac{3}{4}Y^2}{1-\left(\frac{8Y}{3}\right)^{\frac{1}{2}}}\right\} \gtrless 0.$$

The curve $D = 0$ is thus as in the figure



having a cusp at the origin, the line $Y = \frac{3}{8}$ for an asymptote, cutting the

asymptote at $(3^3.2^{-9}, 3.2^{-3})$, or $(\cdot 05, 3.2^{-3})$, and having a vertical tangent at $(-1, \frac{2}{3})$. The portion of the plane for which D is negative, and the quintic has three real roots, is that shaded.

Consider next the function ϖ; putting

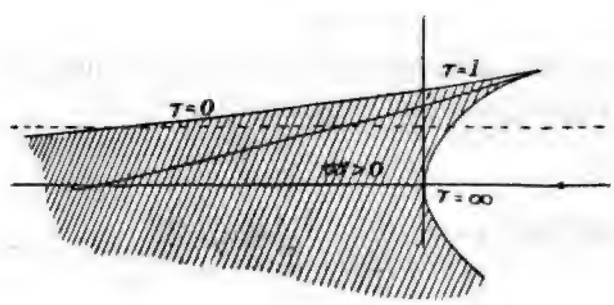
$$\xi = 12^{-3}X, \qquad \eta = 48^{-1}Y,$$

we have
$$\varpi = -\xi+\eta^2+72\xi\eta-432\xi^2-64\eta^3$$
$$= -\frac{1}{3^3.4^4}[4X-3Y^2-6XY+X^2+4Y^3]$$
$$= -\frac{1}{3^3.4^4}[(X-3Y+2)^2+4(Y-1)^3].$$

Thus the curve $\varpi = 0$, like $D = 0$, is unicursal, satisfied, for instance, by taking

$$\tau = \frac{4}{5}\,\frac{31.Y-7.X-24}{X-Y}, \quad X = 3^3.2^8\,\frac{5\tau-4}{(5\tau+28)^3}, \quad Y = 3.2^6\,\frac{5\tau+4}{(5\tau+28)^2},$$

where the parameter τ is chosen for its simplicity in a subsequent application. The curve is as in the figure



having a cusp at $X = 1$, $Y = 1$, whose tangent passes through $(-2, 0)$, one side of the curve cutting $Y = \frac{3}{8}$ in $(-1\cdot 86, 3.2^{-3})$ and $Y = 0$ in $(-4, 0)$, while the other side touches $X = 0$ at the origin and cuts $Y = \frac{3}{8}$ in $(\cdot 11, 3.2^{-3})$, that is, to the right of the point in which $D = 0$ cuts $Y = \frac{3}{8}$; in fact, this side of $\varpi = 0$ lies, so long as Y is positive, inside the shaded portion of $D = 0$ for which X is positive. Putting $\tau = 1$ in the formulæ above, we obtain

$$X = 2^8/11^3, \quad Y = 3.2^6/11^2,$$

which is a point also on the curve $D = 0$, obtained by taking, in the formulæ

$$X = 3^3.2^{-8}.\mu^4/(\mu+1), \quad Y = 3.2^{-3}.\mu^2,$$

which give a parametric representation of $D = 0$, the value $\mu = 2^4/11$;

putting, in the above formulæ for $\varpi = 0$, the parameter $\tau = 0$, we obtain

$$X = -2\left(\frac{6}{7}\right)^3, \quad Y = \frac{24}{49}, \quad \frac{dY}{dX} = \frac{7}{36},$$

which are also obtained by taking $\mu = -\frac{8}{7}$ in the formulæ just put down for $D = 0$; the curve $\varpi = 0$ thus touches $D = 0$ at this point. Finally, ϖ is positive in the shaded region of the diagram between the two sides of the curve.

There is another curve which it is useful to trace for the sake of comparison with Sylvester's results, though it is not necessary for our purpose. We have

$$2^{11}L - J^3 = -\frac{2^{12}\zeta^6}{12^9\varpi^3}\left[(\tfrac{3}{4}Y^2 - X)^3 - \frac{2^5}{3^3}X^4\right];$$

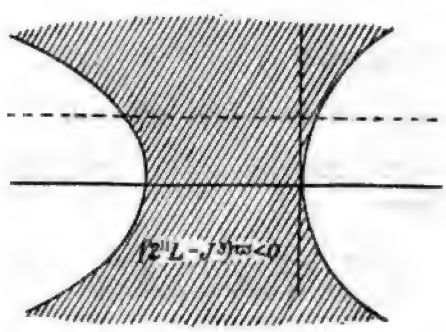
the curve $2^{11}L - J^3 = 0$ is thus given by

$$\tfrac{3}{4}Y^2 = X\left(1 + \frac{2^{\frac{1}{2}}}{3}X^{\frac{1}{2}}\right),$$

or by $$X = 3^3.2^{-5}(\sigma^2 - 1)^{-3}, \quad Y = 3.2^{-\frac{1}{2}}\sigma(\sigma^2 - 1)^{-2},$$

$$\sigma = 3^2.2^{-\frac{1}{2}}\frac{Y}{X}\left(\frac{3}{4}\frac{Y^2}{X} - 1\right);$$

the curve has therefore such a shape as given in the figure

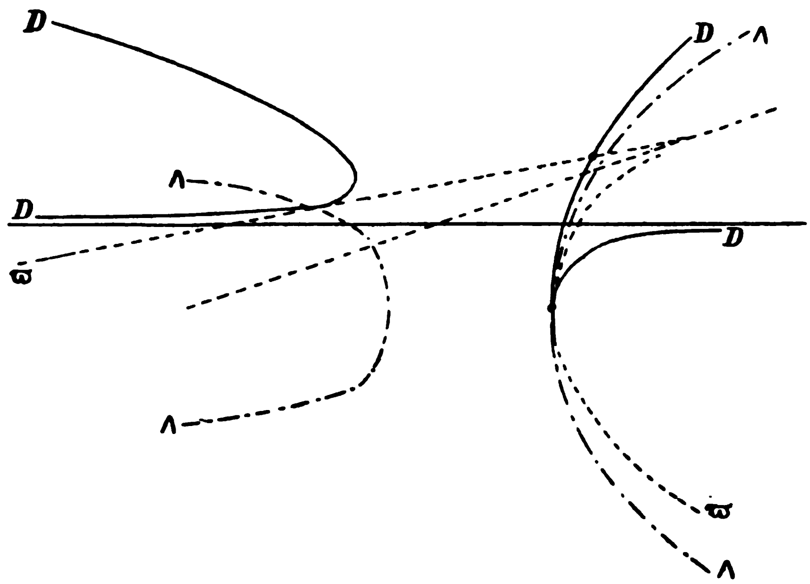


cutting $Y = 0$ in $(-\frac{27}{32}, 0)$ and $(0, 0)$, and cutting $Y = \frac{3}{8}$ in, approximately, $(-1, \frac{3}{8})$ and $(\cdot 065, \frac{3}{8})$, so that for positive X the right hand branch of the curve lies to the right of the branch $X = \frac{3}{4}Y^2/[1 + (\frac{8}{3}Y)^{\frac{1}{2}}]$ of $D = 0$, and to the left of the portion of $\varpi = 0$ which is on the right of the cuspidal tangent of this. It is important to notice that the curve passes through the point of contact of $D = 0$ and $\varpi = 0$, namely,

$$X = -2.6^3.7^{-3}, \quad Y = 24.7^{-2}.$$

The portion of the plane for which $(2^{11}L - J^3)\varpi < 0$ is that shaded in the figure; for part of this ϖ is positive and for part of it negative.

Denoting $2^{11}L-J^3$ by Λ, the relations of the three curves are shewn in the following diagram, which is drawn, however, with only approximate accuracy.



Consider now the equation

$$ru^5+sv^5-t\,(u+v)^5=0,$$

when $\varpi=0$, so that we may take $r=s$. Then, beside the root given by $u+v=0$, we have

$$\frac{u^4-u^3v+u^2v^2-uv^3+v^4}{u^4+4u^3v+6u^2v^2+4uv^3+v^4}=\frac{t}{r};$$

the left side is a function of $\theta=\dfrac{u^2+v^2}{uv}$,

and the equation gives, if $\tau=\frac{4}{5}\left(1+\dfrac{4t}{r}\right)$,

the values expressed by $\theta=\dfrac{-2\sqrt{\tau}}{\sqrt{\tau}\mp 2}$,

and thence, if $\epsilon^2=1$, $\zeta^2=1$, the four values for u/v expressed by

$$\frac{u}{v}=\frac{1+\zeta\,(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}}{1-\zeta\,(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}};$$

n terms of r, s, t, taking account of $r = s$, the invariants of the quintic are

$$J = (r^2+2tr)^2-4r^3t\,(2r+t), \quad K = r^4t^2\,(r^2+2rt), \quad L = r^8t^4,$$

$$K^2-JL = 4r^{10}t^5\,(2r+t),$$

from which X, Y are expressible in terms of τ; the result is that parametric expression for $\varpi = 0$ which was given above; we can thus obtain, as we know the values of this parameter for the various portions of this curve, the forms of the four values of u/v for the various portions.

Thus, for the portion of $\varpi = 0$ for which $X > 0$, $Y < 0$, and the portion for which $X < -2.6^8.7^{-3}$, through which the parameter

$$\tau = \tfrac{4}{5}\,\frac{31.Y-7.X-24}{X-Y}$$

remains negative, constantly increasing from $-\infty$ to 0, we have, if ω be a real positive quantity, and $\tan\theta = -\epsilon\omega$,

$$\frac{u}{v} = \frac{1+\zeta(\epsilon i\omega-1)^{\frac{1}{2}}}{1-\zeta(\epsilon i\omega-1)^{\frac{1}{2}}}$$

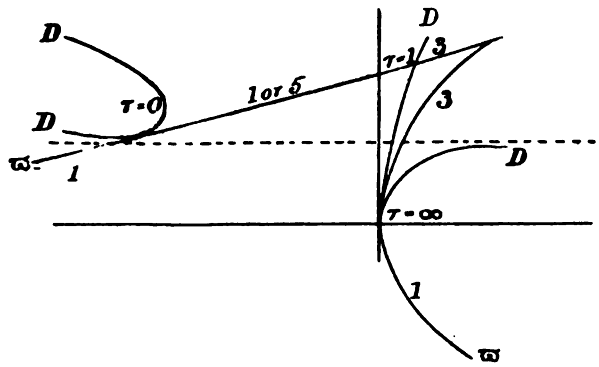
$$= \frac{1+\zeta(1+\omega^2)^{\frac{1}{4}}\cos\tfrac{1}{2}\theta+i\zeta(1+\omega^2)^{\frac{1}{4}}\sin\tfrac{1}{2}\theta}{1-\zeta(1+\omega^2)^{\frac{1}{4}}\cos\tfrac{1}{2}\theta-i\zeta(1+\omega^2)^{\frac{1}{4}}\sin\tfrac{1}{2}\theta},$$

and this is always a complex quantity of modulus other than unity (save for the final point $\cos\frac{1}{2}\theta = 0$, $\tau = 0$). Therefore, if L be positive, in which case u, v are real linear functions of the original variables x, y of the quintic equation, the quintic has only one real root, given by $u = -v$; while, if L be negative, in which case u, v are conjugate complex linear functions and u/v is of modulus unity, there is still only one real root, that given by $u = -v$. Next, for the portion of $\varpi = 0$ lying between the two branches of $D = 0$, for which X varies from $-2.6^8.7^{-3}$ to $2^8.11^{-3}$, and τ varies from 0 to 1, each of the four numbers expressed by

$$[1+\zeta(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}]/[1-\zeta(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}]$$

is complex, but of modulus unity; thus, when $L > 0$, the quintic has four complex roots corresponding to these, and one real root given by $u+v = 0$, while, when $L < 0$, the quintic has five real roots. Lastly, for the portion of $\varpi = 0$ lying between the cuspidal branches of $D = 0$ for which $X > 0$, $Y > 0$, the value of τ increases constantly from 1 to ∞; thus, for $\epsilon = 1$, the two corresponding numerical values of u/v are real, while those belonging to $\epsilon = -1$ are complex of modulus unity; wherefore, when $L > 0$, the quintic has three real roots, including that given

by $u+v=0$, and the same is true when $L<0$; this result only verifies a previously obtained result, for along this portion of the curve, as we have seen, we have $D<0$. These results are summarised in the diagram



We have already remarked that to any point (X, Y) not upon $\varpi = 0$, corresponds first a definite set of ratios for the coefficients r, s, t, and then, either three real linear forms u, v, w, or three of the form $-\frac{1}{2}(w+i\sigma)$, $-\frac{1}{2}(w-i\sigma)$, w, where w, σ are real, according as $\varpi > 0$ or $\varpi < 0$; and it is clear that all equations of the form

$$f(x, y) = r(\lambda x+\mu y)^5+s(\lambda' x+\mu' y)^5-t[(\lambda+\lambda')x+(\mu+\mu')y]^5 = 0,$$

in which r, s, t are given, have the same number of real roots whatever the values of the real coefficients λ, μ, λ', μ'; as also have all equations of the form

$$F(x, y) = r[\lambda x+\mu y+i(\lambda' x+\mu' y)]^5+s[\lambda x+\mu y-i(\lambda' x+\mu' y)]^5 \\ -t[2\lambda x+2\mu y]^5 = 0.$$

But further, still supposing $\varpi \neq 0$, and supposing also now that $D \neq 0$, this number will be unaffected by variation of X, Y, that is, of r, s, t, provided the variation be sufficiently limited in extent. For, putting $y=1$ and $t=1$, let x be a root of one of these equations, and $x+\delta x$ the root correspondingly belonging to the values $r+\delta r$, $s+\delta s$; we then have, in the two cases,

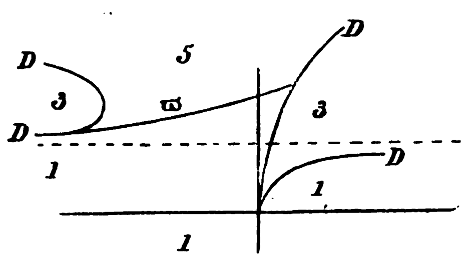
$$5(r\lambda u^4+s\lambda' v^4)\,\delta x+\delta r.u^5+\delta s.v^5 = 0,$$

or
$$5[r(\lambda+i\lambda')u^4+s(\lambda-i\lambda')v^4]\,\delta x+\delta r.u^5+\delta s.v^5 = 0,$$

shewing that δx is real when x is real, and complex (including real) when x is complex. The inference, however, holds equally well when r, s, t corre-

spond to a point upon $\varpi = 0$, provided that the variation of r, s, t is such as to preserve the forms of u, v, w, that is, such as to preserve the sign of L.

Now, for the portions of $\varpi = 0$ corresponding to $-\infty < \tau < 0$, we have seen that there is one real root, whether we have $L > 0$ or $L < 0$; there is thus one real root for the regions of the plane on either side of these portions of $\varpi = 0$, the limitation of these regions of the plane being only parts of the curves $D = 0$ or $\varpi = 0$. There are similarly three real roots on either side of the portion of the curve $\varpi = 0$ for which $1 < \tau < \infty$, a fact which we have proved independently. For the portion of $\varpi = 0$ for which $0 < \tau < 1$, we have proved that there is one or five real roots according as $L > 0$ or $L < 0$; when therefore we pass to the side of this portion of $\varpi = 0$ for which ϖ, $= 16I^2L^5/(K^2-JL)^6$, is positive, and therefore L positive, we shall move into a region for which there is one real root, and when we pass to the side of this portion of $\varpi = 0$ for which $\varpi < 0$ or $L < 0$, there will be five real roots; this is the region, in the figure, above $\varpi = 0$ and between the branches of $D = 0$. We therefore have the diagram



the only separations of regions being the curve $D = 0$ and the portion of $\varpi = 0$ for which $0 < \tau < 1$. Taking account of the behaviour of the curve $\Lambda = 0$, the results may be stated analytically by saying that if $D < 0$, there are three real roots, while if $D > 0$, there are five real roots when both $L < 0$ and $\Lambda > 0$, but only one real root if either or both of $L > 0$, $\Lambda < 0$. This is the result as stated by Salmon, correcting the form given by Sylvester in his paper in the *Phil. Trans.*, which, however, does not agree with the result of his own arguments. If an analytical statement is required, it is more natural from the present point of view to say that, when $D > 0$, there are five real roots, when L (or ϖ) < 0 and $Y > \frac{24}{49}$, but only one real root if $L > 0$, or $Y < \frac{24}{49}$, or both; the condition $Y > \frac{24}{49}$ is the same as $98KL^2 > (K^2-JL)^2$. Our diagrams show at once the reason for the indeterminate form which Sylvester was able to give to one of his criteria—the curve $\Lambda = 0$ only intervenes in the

analytic statement in order to divide the curve $\varpi = 0$ into portions.

A part verification of the preceding results is worth noticing: suppose $D > 0$, so that the number of real roots is 5 or 1; suppose further that $\varpi > 0$, so that in the form

$$ru^5+sv^5-t(u+v)^5 = 0,$$

all of r, s, t, u, v are real; if this equation had 5 real roots its derivative in regard to u, or

$$ru^4-t(u+v)^4 = 0,$$

would have 4; it has however at most 2, and this only when r and t are of the same sign. Thus for $D > 0$, $\varpi > 0$, the quintic has only one real root. We have seen that it has only one real root also in some cases in which these conditions are not satisfied.

We examine now equations for which $D = 0$. When the equation

$$ru^5+sv^5-t(u+v)^5 = 0$$

has the repeated root $u/v = \theta$, this is also a root of the two derivatives

$$ru^4-t(u+v)^4 = 0, \qquad sv^4-t(u+v)^4 = 0,$$

and the original equation may be written

$$\left(\frac{\theta+1}{\theta}\right)^4 u^5+(\theta+1)^4 v^5-(u+v)^5 = 0;$$

the left side of this, multiplied by θ, is

$$(u-v\theta)^2 (A\theta^{-3}u^3+B\theta^{-2}u^2v+C\theta^{-1}uv^2+Dv^3),$$

where

$$A = 4\theta^3+6\theta^2+4\theta+1,$$

$$B = 3\theta^3+12\theta^2+8\theta+2,$$

$$C = 2\theta^3+8\theta^2+12\theta+3,$$

$$D = \theta^3+4\theta^2+6\theta+4,$$

and the discriminant of the cubic factor of this, in regard to u/v, save for a positive numerical factor, is found to be

$$\left(\frac{\theta+1}{\theta}\right)^6 (\theta^2+1)(\theta^2+2\theta+2)(2\theta^2+2\theta+1).$$

Now, unless the original quintic have two pairs of equal roots which are

conjugate imaginaries, a case which will be proved to correspond only to the single point $X = -2.6^8.7^{-8}$, $Y = 24.7^{-2}$, the repeated root of the quintic will be real; thus when $L > 0$, and u, v are real, θ is real, and when $L < 0$, and u, v are conjugate imaginaries, θ is of the form $e^{i\beta}$. In the latter case, if $u = -\frac{1}{2}(w+i\sigma)$, $v = -\frac{1}{2}(w-i\sigma)$, the discriminant of the cubic in regard to the real quantities w, σ will differ from that just put down by a factor

$$\left[\frac{\partial(u, v)}{\partial(w, \sigma)}\right]^{\frac{1}{2}.3.4} = -\frac{1}{64},$$

and may be taken to be the negative of that; putting, when $\theta = e^{i\beta}$, $\cos\beta = t$, the discriminant of the cubic in regard to the real quantities w, σ may thus be found to be, save for a positive numerical factor,

$$-(1+t)^8\, t\,(5+12t+8t^2);$$

the sign of this is therefore the sign of $-t$, while the sign of the former discriminant, for θ real, is always positive.

Next, putting $r = (\theta+1)^4/\theta^4$, $s = (\theta+1)^4$, $t = 1$, and calculating the invariants J, K, L of the quintic, and hence the values of X and Y, we find

$$X = \frac{3^8\theta^8(\theta+1)^8}{[(\theta+1)^4+\theta^4+\theta^4(\theta+1)^4]^3}, \qquad Y = \frac{3\theta^4(\theta+1)^4[\theta^4+(\theta+1)^4+1]}{[(\theta+1)^4+\theta^4+\theta^4(\theta+1)^4]^2},$$

which, putting $$1-\frac{4X}{3Y^2} = \frac{\mu}{\mu+1} = \frac{4\theta^2(\theta+1)^2}{(\theta^2+\theta+1)^3},$$

lead to $$X = \frac{3^8.2^{-8}.\mu^4}{1+\mu}, \qquad Y = \tfrac{3}{8}\mu^2,$$

which is the previously obtained parametric expression of the curve $D = 0$; to each point of this curve there belong the six values of θ given by

$$\theta,\ \ \frac{1}{\theta},\ \ -(1+\theta),\ \ -\frac{1}{1+\theta},\ \ -\frac{\theta+1}{\theta},\ \ -\frac{\theta}{\theta+1},$$

which correspond to the six ways in which we may permute the coefficients r, s, t in the quintic. When $\theta = e^{i\beta}$, the expressions for X, Y in terms of t, $= \cos\beta$, are

$$X = \frac{3^8.2^4(t+1)^4}{(8t^3+12t^2-2t-7)^3(2t+1)^3}, \qquad Y = \frac{3.2^3(t+1)^2}{(8t^3+12t^2-2t-7)^2}.$$

Now, if θ is real and $0 < \theta < 1$, we have

$$-\infty < -\left(1+\frac{1}{\theta}\right) < -2 < -(1+\theta) < -1 < -\frac{1}{1+\theta}$$

$$< -\frac{1}{2} < -\frac{\theta}{1+\theta} < 0,$$

$$1 < \frac{1}{\theta} < \infty\,;$$

thus, in the equation $$\frac{4\theta^2(\theta+1)^2}{(\theta^2+\theta+1)^3} = \frac{\mu}{\mu+1},$$

to obtain the points of the curve $D = 0$ for which θ is real, it is sufficient to take $0 < \theta < 1$, and each point of the curve will only arise for one value of θ; then μ will vary from 0 to $2^4/11$ and (X, Y) from $(0, 0)$ to $(2^8/11^3,\ 3.2^5/11^2)$; we have in fact already seen that these are the extreme points of the arc of $D = 0$ which lies where $\varpi > 0$. On this arc the cubic obtained by dividing the quintic by $(u-v\theta)^2$ has a positive discriminant, and only one real root; the quintic has thus three real roots and two conjugate imaginary roots.

We have next, if $\theta = e^{i\beta}$, $\cos\beta = t$,

$$\frac{4\theta^2(\theta+1)^2}{(\theta^2+\theta+1)^3} = \frac{4\,(\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}})^2}{(\theta+\theta^{-1}+1)^3} = \frac{8\,(1+t)}{(2t+1)^3},$$

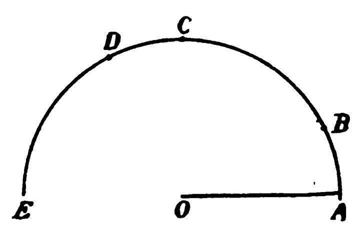
and the equation $$\frac{(2t+1)^3}{t+1} = \frac{(2t_0+1)^3}{t_0+1},$$

gives, beside $t = t_0$,

$$[2t+1+\tfrac{1}{2}\,(2t_0+1)]^2 = \tfrac{1}{4}\,(2t_0+1)^2\,\frac{t_0-1}{t_0+1},$$

of which, when $-1 < t_0 < 1$, the roots are imaginary; to a value t such that $-1 < t < 1$ correspond both $\theta = e^{i\beta}$ and $\theta = e^{-i\beta} = 1/\theta$. We shall thus obtain every point of the curve $D = 0$ for which $\theta = e^{i\beta}$ by supposing β to vary continuously from 0 to π.

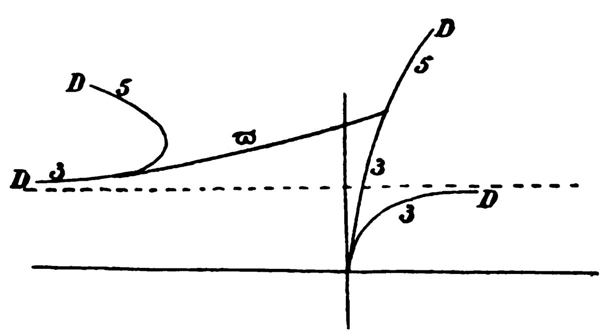
The whole path of θ, in its plane, corresponding to one description of the curve $D = 0$, is thus given by the open curve $OABCDE$



consisting of a radius OA and a semicircle; as θ varies from 0 to A, that is, from 0 to 1, the portion of $D = 0$ is described lying between the origin and the point $(2^8.11^{-3}, 3.2^5.11^{-2})$, where $D = 0$ is cut by $\varpi = 0$; along this arc the discriminant of the cubic factor is positive and the quintic has three real roots. As θ passes along the semicircle towards $\theta = i$, t, $= \cos\beta$, diminishes from 1 to a value (between ·7 and ·68) for which $(2t+1)^3 = 8(t+1)$, for which μ is infinite and $(X, Y) = (\infty, \infty)$; the corresponding value of β is denoted by the point B of the semicircle; the point (X, Y) of $D = 0$, corresponding to the arc AB of the semicircle, passes from the intersection of $D = 0$ and $\varpi = 0$, along the left hand cuspidal branch of $D = 0$ for which $X > 0$, to infinity; along this arc the discriminant

$$-(1+t)^3\, t\,(5+12t+8t^2)$$

is negative and the quintic has five real roots. As θ passes along the semicircle from B to C, the fraction $8(t+1)/(2t+1)^3$, or $\mu/(\mu+1)$, constantly increases from unity, being for $t = 0$ equal to 8, and μ is negative, passing from $-\infty$ to $-\frac{8}{7}$; thus (X, Y) describes the branch of $D = 0$ for which $X < 0$, from (∞, ∞) to the point $(-2.6^3.7^{-3}, 24.7^{-2})$ at which $D = 0$ is touched by $\varpi = 0$; along this portion the discriminant of the cubic is still negative and the quintic has five real roots. As θ passes along the semicircle from C to D, where $\beta = \frac{2}{3}\pi$, the fraction $8(t+1)/(2t+1)^3$ passes to infinity and μ passes to -1; the point (X, Y) thus passes to $(-\infty, \frac{3}{8})$; along this portion the discriminant of the cubic is positive and the quintic has three real roots. Finally, as β passes from $\frac{2}{3}\pi$ to π, the fraction $\mu/(\mu+1)$ becomes negative, and μ, which is negative, is numerically less than unity; thus X is positive, and (X, Y) passes from $(\infty, \frac{3}{8})$ to the origin, corresponding to $t = -1$, or the point E of the semicircle, along the portion of $D = 0$ which is below the asymptote; for this branch the discriminant of the cubic remains positive and the quintic has three real roots. The results are thus those represented in the diagram



and, analytically, upon $D=0$, there are five real roots for $\varpi<0$, $Y>\frac{24}{49}$, but three real roots in all other cases; or otherwise, introducing Sylvester's criterion $\Lambda=2^{11}L-J^3$, there are five real roots for $L<0$, $\Lambda>0$, and three in all other cases.

It remains now to examine the points of intersection of the curves $D=0$, $\varpi=0$.

At the point of contact $(-2.6^3.7^{-3}, 24.7^{-2})$, regarded as lying on $\varpi=0$, we have seen that the values of u/v are given, for $\tau=0$, by

$$\frac{u}{v}=-1,\quad \frac{u}{v}=\frac{1+\zeta(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}}{1-\zeta(\epsilon\sqrt{\tau}-1)^{\frac{1}{2}}},$$

that is, by

$$\frac{u}{v}=-1,\quad \frac{u}{v}=\frac{1+i}{1-i},\ \frac{1+i}{1-i},\ \frac{1-i}{1+i},\ \frac{1-i}{1+i},\ =i,\ i,\ -i,\ -i;$$

regarded as lying on $D=0$, we have found that the point corresponds to $\theta=i$, for which the discriminant

$$\theta^{-6}(\theta+1)^6(\theta^2+1)(\theta^2+2\theta+2)(2\theta^2+2\theta+1)$$

vanishes; the other vanishing factors $\theta+i$, $\theta^2+2\theta+2$, $2\theta^2+2\theta+1$ correspond in fact to the same point, arising by transforming the root $\theta=i$ by

$$\theta'=\theta^{-1},\quad \theta'=-(\theta+1),\quad \ldots.$$

Thus, if $L>0$, and u, v are real linear functions, there are two imaginary roots of the quintic, each repeated, and one real root given by $u+v=0$, while if $L<0$, and u, v are conjugate linear functions, there are two real roots, each repeated, beside the real root given by

$$u+v=0.$$

Conversely, if the original quintic have the form

$$10\,(cx_1+dy_1)\,x_1^2y_1^2,$$

proper to the case of two repeated roots, x_1 and y_1 being real or conjugate imaginary linear functions of the original variables x, y, the canonizant is

$$(cx_1+dy_1)(cx_1+i\,dy_1)(cx_1-i\,dy_1),$$

and, supposing neither c nor d to be zero, we have

$$10\,(cx_1+dy_1)\,x_1^2y_1^2$$
$$=\frac{1}{2c^2d^2}\left\{4\left[\tfrac{1}{2}(1-i)(cx_1+i\,dy_1)\right]^5+4\left[\tfrac{1}{2}(1+i)(cx_1-i\,dy_1)\right]^5-\left[-cx_1-dy_1\right]^5\right\},$$

from which we calculate, save for respective factors of the forms $\mu^{10}, \mu^{20}, \mu^{30}$,

$$J = (16-8)^2+4.16.7 = 2^9, \quad K = 16.16\,(16-8) = 2^{11}, \quad L = 2^{16},$$

and hence
$$X = -2\left(\tfrac{6}{7}\right)^3, \quad Y = \tfrac{24}{49}.$$

Next, at the point $(2^8.11^{-3}, 3.2^5.11^{-2})$, where $D = 0$, $\varpi = 0$ cut one another, given on $\varpi = 0$ by taking the parameter τ to be unity, and on $D = 0$ by taking the parameter μ to be $2^4/11$, we have

$$\frac{u}{v} = -1, \quad \frac{u}{v} = \frac{1+\zeta(\epsilon-1)^{\frac{1}{2}}}{1-\zeta(\epsilon-1)^{\frac{1}{2}}} = 1,\ 1,\ \frac{1+i\sqrt{2}}{1-i\sqrt{2}},\ \frac{1-i\sqrt{2}}{1+i\sqrt{2}};$$

thus, if $L > 0$, there are two conjugate complex roots of the original quintic, and three real roots, one of these repeated; and, if $L < 0$, and

$$u = -\tfrac{1}{2}(w+i\sigma), \quad v = -\tfrac{1}{2}(w-i\sigma),$$

there are five real roots given by $w = 0$, $\sigma = 0$ twice, and $\sigma/w = \pm\sqrt{2}$.

Finally, the origin corresponds to $L = 0$, and a canonical form

$$Au^5+5Euv^4+Fv^5,$$

where u, v are real linear forms, and A, E, F are real. This cannot have five real roots, since the derivative Au^4+Ev^4 can vanish only for two real values of u/v, and that only when E/A is negative. By examining the signs of $Au^5+5Euv^4+Fv^5$ for the values of u/v for which the derivative vanishes, we easily prove that the condition for three real roots is

$$AF^4+2^8E^5 \leqslant 0;$$

since
$$J = A^2F^2, \quad K = -2A^3E^5,$$

this is the same as
$$D = J^2-2^7K \leqslant 0;$$

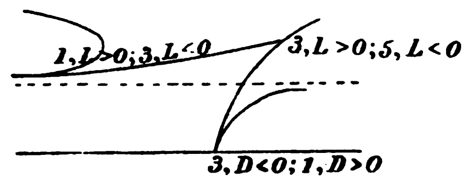
when $D > 0$ there is only one real root. In our figures, the curve we have denoted by $D = 0$ passes through the origin; but the discriminant D of the quintic does not necessarily vanish when $L = 0$, being in fact equal to

$$\frac{2^8\zeta^4}{\varpi^2}\left[(\eta^2-\xi)^2-2^7\xi^2\eta\right],$$

and the vanishing of ϖ allows D to be positive or negative. In fact

$L=0$, $K \neq 0$, gives $X=0$, $Y=0$ and $3Y^2/4X = K^2/(K^2-JL) = 1$. The case $L=0$, $K=0$ gives, on examination, $X=0$, $Y=$ finite.

The numbers of real roots of the quintic at the three points are then given by the diagram



A NEW DEVELOPMENT OF THE THEORY OF THE HYPERGEOMETRIC FUNCTIONS

By E. W. BARNES.

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1. The differential equation of the hypergeometric series may be written

$$\left[(\vartheta+a_1)(\vartheta+a_2)-\frac{1}{x}\,\vartheta(\vartheta+\rho-1)\right]y=0,$$

where $\vartheta = x\dfrac{d}{dx}$, or

$$\frac{d^2y}{dx^2}+\frac{\rho-(1+a_1+a_2)x}{x(1-x)}\,\frac{dy}{dx}-\frac{a_1a_2}{x(1-x)}\,y=0.$$

It is known to be satisfied by the hypergeometric series

$$F(a_1, a_2; \rho; x)=\frac{\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{n!\,\Gamma(\rho+n)}\,x^n,$$

which is convergent when $|x|<1$.

This series was first discussed in detail by Gauss[1] in 1812. Kummer[2], in 1836, obtained the twenty-four solutions of the hypergeometric equation usually given in the text-books by a process which can be traced back to Euler.[3] These twenty-four solutions are reducible to six sets of four, each four being identical functions differently expressed. The six sets can be divided into pairs Y_1, Y_2; Y_3, Y_4; Y_5, Y_6, each pair corresponding respectively to one of the three singularities 0, 1, ∞ of the differential equation.

Riemann,[4] in 1857, extended the theory by introducing his P-function, and discussed the general theory of transformation of the variable. Riemann did not connect his theory directly with that of Kummer, and it was reserved for Thomae,[5] in 1879, to work out in detail from the theory of linear differential equations the relations which connect any one of the twenty-four solutions of the hypergeometric equation with the two essentially different solutions which are valid in the neighbourhood of either of the singularities not associated with the particular solution chosen.

The differential equation for Riemann's P-function was first given by Papperitz,[6] in 1885, in the form

$$\frac{d^2y}{dx^2}+\left\{\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right\}\frac{dy}{dx}$$
$$+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right\}\frac{y}{(x-a)(x-b)(x-c)}=0,$$

wherein $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Denoting a linear relation between a, b, c by $(a, b, c) = 0$, these twelve relations will be

$(Y_1, Y_5, Y_6) = 0,$	I.;	$(Y_3, Y_4, Y_5) = 0,$	VII.;
$(Y_2, Y_5, Y_6) = 0,$	II.;	$(Y_3, Y_4, Y_6) = 0,$	VIII.;
$(Y_1, Y_2, Y_5) = 0,$	III.;	$(Y_1, Y_3, Y_4) = 0,$	IX.;
$(Y_1, Y_2, Y_6) = 0,$	IV.;	$(Y_2, Y_3, Y_4) = 0,$	X.;
$(Y_3, Y_5, Y_6) = 0,$	V.;	$(Y_1, Y_2, Y_3) = 0,$	XI.;
$(Y_4, Y_5, Y_6) = 0,$	VI.;	$(Y_1, Y_2, Y_4) = 0,$	XII.;

Now the advantage of the contour integrals which I have introduced is, that the contour integral I_R or the alternative contour integral I'_R gives at once, by an almost obvious transformation, the relation (R). The relations I., ..., XII. thus arise almost intuitively.

3. We can apply the same process to Riemann's P-function.

Since Papperitz's equation can be transformed into Kummer's, we can, from any contour integral which satisfies the latter equation, deduce a contour integral which satisfies Papperitz's equation for all values of $|x|$ on a suitably dissected plane.

A typical integral obtained in this way is

$$K_1 = -\frac{1}{2\pi i}\left(\frac{x-c}{x-a}\,\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(\alpha+\gamma-s)}{\Gamma(1-\alpha'-\gamma+s)}\,\Gamma(\beta+s)\,\Gamma(\beta'+s)\left(-\frac{x-a}{c-a}\,\frac{c-b}{x-b}\right)^{s}ds,$$

wherein
$$\left|\arg\left(-\frac{x-a}{c-a}\,\frac{c-b}{x-b}\right)\right| < \pi,$$

and the contour of the integral is parallel to the imaginary axis with loops if necessary to ensure that $\alpha+\gamma,\ \alpha+\gamma+1,\ \dots$ are to the right and $\begin{cases}-\beta,\ -\beta-1,\ \dots\\ -\beta',\ -\beta'-1,\ \dots\end{cases}$ are to the left of the contour.

This integral, by an obvious interchange of $\begin{cases}a,\ b,\ c,\\ \alpha,\ \beta,\ \gamma\\ \alpha',\ \beta',\ \gamma'\end{cases}$ can take twenty-four different forms. The twenty-four solutions of Papperitz's equation are thus in evidence. And the relations between them and Riemann's functions P_α, $P_{\alpha'}$; P_β, $P_{\beta'}$; P_γ, $P_{\gamma'}$, and Riemann's relations between the latter, arise with complete symmetry.*

4. The idea of taking contour integrals involving gamma functions in the subject of integration appears to be due to Pincherle,[10] who has been followed by Mellin,[11] though the type of contour and its use can be traced back to Riemann.[12] The author has made the method fundamental in several recent investigations.[13]

In conclusion, it may be observed that the contour integrals introduced in this paper are valid when any of the quantities a_1, a_2, ρ are integers or differ by integers, and in the case of Riemann's P-function, when $\alpha-\alpha'$, $\beta-\beta'$, or $\gamma-\gamma'$ are integers. The corresponding solutions, even when they involve logarithmic terms, are readily obtained. The results, in general, agree

* Some of the relations I.–XII. are given in Chapter vi. of Forsyth's *Treatise on Differential Equations* (Third Edition, London, 1903); but the forms there given are not in complete accord with the forms obtained in this paper. The twenty-four solutions of Papperitz's equation are given in Whittaker's *Course of Modern Analysis*, but the forms which he gives are not in complete accord with the forms given by Thomae (*loc. cit.*, p. 329). The fact that many-valued functions are involved in the expressions which Whittaker gives would be an obstacle in the way of determining Riemann's coefficients α_γ, $\alpha_{\gamma'}$ by the method which he suggests.

with those of Lindelöf.[14] As an example of this particularization, the differential equation of the quarter-periods of the Jacobian elliptic functions is discussed in Part III.

(1) Gauss, *Göttinger Commentationes Recentiores* (1812), T. II.; *Ges. Werke*, T. III., pp. 123–163.

(2) Kummer, *Crelle* (1836), T. XV., pp. 39-83 and 127–172.

(3) Euler, *Nova Acta Acad. Petropol*, T. XII. (1778), p. 58.

(4) Riemann, *Abh. d. Ges. d. Wiss. zu Göttingen*, T. VII. (1857); *Mathematische Werke* (2te *Auf.*), (1892), pp. 67 *et seq.*

(5) Thomae, *Crelle*, T. LXXXVII. (1879), pp. 222–349, especially pp. 306–333.

(6) Papperitz, *Mathematische Annalen*, T. XXV. (1885), p. 213.

(7) Riemann, *Mathematische Werke : Nachträge*, Edited by Noether and Wirtinger (1902), pp. 69-94.

(8) Klein, *Vorlesungen über die hypergeometrische Funktion* (lithographed), Göttingen (1894); *Mathematische Annalen* (1891), T. XXXVIII., pp. 144–152.

(9) Pochhammer, *Mathematische Annalen*, T. XXXV. (1890), pp. 470–494.

(10) Pincherle, *Atti d. R. Accademia dei Lincei*, Series IV., *Rendiconti*, Vol. IV., pp. 694–700 and 792–799.

(11) Mellin, *Acta Societatis Scientiarum Fennicae* (1895), T. XX., No. 12, p. 78.

(12) Riemann, *Mathematische Werke* (1892), pp. 145–147, or *Œuvres Mathématiques* (1898), pp. 166, 167.

(13) Barnes, *Proceedings of the London Mathematical Society* (1905), Ser. 2, Vol. 3, pp. 253–272; *Philosophical Transactions of the Royal Society*, (A) (1906), Vol. CCVI., pp. 249–297; *Quarterly Journal of Mathematics* (1907), Vol. XXXVIII., pp. 108–116 and pp. 116–140.

(14) Lindelöf, *Acta Societatis Scientiarum Fennicae* (1893), T. XIX., No. 1, pp. 1–31.

PART I.

The ordinary Hypergeometric Equation.

5. We take the differential equation of the ordinary hypergeometric functions in the form

$$\left[(\vartheta+a_1)(\vartheta+a_2)-\frac{1}{x}\,\vartheta(\vartheta+\rho-1)\right]y=0,$$

where $x\dfrac{d}{dx}\equiv\vartheta$. We shall refer to this as Kummer's equation. It may be written
$$\frac{d^2y}{dx^2}+\frac{\rho-(a_1+a_2+1)x}{x(1-x)}\frac{dy}{dx}-\frac{a_1a_2}{x(1-x)}\,y=0.$$

Suppose now that x has any value, real or complex, such that

$$|\arg(-x)|<\pi.$$

Then I say that the integrals

$$I_1=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)}{\Gamma(\rho+s)}\,\Gamma(-s)\,(-x)^s\,ds,$$

$$I_2=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}\,(-x)^s\,ds,$$

$$I_3=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-a_2-s)}\,(-x)^s\,ds,$$

$$I_4=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_2+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-a_1-s)}\,(-x)^s\,ds,$$

exist and are solutions of Kummer's equation. Each integral is taken along a contour which is parallel to the imaginary axis with loops if necessary to ensure that those sequences of poles of the respective subjects of integration which are ultimately positive are to the right of the contour, while those sequences which are ultimately negative lie to the left.

In the first place, the integrals exist. For, when s tends to infinity along a parallel to the imaginary axis in the finite part of the plane

$$|\Gamma(s+a)| \exp\left\{\left(\frac{\pi}{2}-\epsilon\right\}|v|\right.$$

tends uniformly to zero if $s = u+\iota v$, where u and v are real and $\epsilon > 0$. The integrals therefore exist if $|\arg(-x)| < \pi$.

In the second place, the integrals satisfy Kummer's equation. Take, for instance, the integral I_1. We may obviously differentiate it with regard to x by differentiating under the sign of integration.

Hence

$$(\vartheta+a_1)(\vartheta+a_2)I_1 = -\frac{1}{2\pi\iota}\int (s+a_1)(s+a_2)\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-x)^s ds$$

$$= -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(1-s)}{\Gamma(\rho+s-1)}(-x)^{s-1} ds$$

taken along a contour which is derived from the former by moving it through a distance unity in a positive direction parallel to the real axis.

The original contour was evidently so chosen that we may take the last integral along the original contour: we may therefore write it

$$-\frac{1}{2\pi\iota}\int \frac{s(\rho+s-1)}{x}\,\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-x)^s ds = \frac{d}{dx}(\vartheta+\rho-1)I_1.$$

Thus the integral satisfies Kummer's equation.

Evidently an almost identical proof will apply to the other integrals I_2, I_3, I_4.

6. Let us denote the hypergeometric series

$$1+\frac{a_1 a_2}{1.\rho}x+\frac{a_1(a_1+1)\,a_2(a_2+1)}{1.2.\rho.\rho+1}x^2+\ldots$$

by $F\{a_1, a_2; \rho; x\}$.

The series is convergent when $|x| < 1$. When $|x| > 1$,

$$F\{a_1, a_2; \rho; x\}$$

represents the continuation of the function represented by the series when $|x| < 1$.

We may now shew that, when $|x|<1$, $F\{a_1, a_2; \rho; x\}$ *is a solution of Kummer's equation and that, when* $|x|>1$, *the function* $F\{a_1, a_2; \rho; x\}$ *can be expressed in the form*

$$\frac{\Gamma(\rho)\,\Gamma(a_2-a_1)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,(-x)^{-a_1} F\{a_1,\, 1+a_1-\rho;\, 1+a_1-a_2;\, 1/x\}$$
$$+\frac{\Gamma(\rho)\,\Gamma(a_1-a_2)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,(-x)^{-a_2} F\{a_2,\, 1+a_2-\rho;\, 1+a_2-a_1;\, 1/x\}, \quad \text{(I.)}$$

provided $|\arg(-x)|<\pi$. This proviso uniquely prescribes $(-x)^{-a_1}$ and $(-x)^{-a_2}$, and indicates that $F\{a_1, a_2; \rho; x\}$ outside the circle $|x|=1$ needs a cross-cut from 1 to $+\infty$ along the real axis to make it one-valued.

Take the integral

$$I_1 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}\,(-x)^s\,ds,$$

wherein $|\arg(-x)|<\pi$, and suppose that $|x|<1$. Then we may bend the contour of the integral round until it embraces the positive half of the real axis and encloses the poles of $\Gamma(-s)$ but no other poles of the subject of integration. And by the asymptotic expansion of the gamma function this alteration of the contour will not affect the value of the integral, provided $|x|<1$.

But, by Cauchy's theorem, the value of the new integral is given by the sum of the residues within the contour. The residue of $\Gamma(-s)$ at $s=n$ is $(-)^{n-1}/n!$. We therefore have

$$I_1 = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{\Gamma(\rho+n)}\,\frac{(-)^n}{n!}\,(-x)^n,$$
$$= \frac{\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(\rho)}\,F\{a_1, a_2; \rho; x\}, \quad (\text{if } |x|<1).$$

Return to the original integral I_1 and suppose that $|x|>1$. Then we may bend the contour of the integral round till it becomes a contour which encloses the poles of $\Gamma(a_1+s)$ and $\Gamma(a_2+s)$, but not those of $\Gamma(-s)$.

By Cauchy's theory of residues we get, if $|x|>1$,

$$I_1 = \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\,\frac{\Gamma(a_2-a_1-n)}{\Gamma(\rho-a_1-n)}\,\Gamma(a_1+n)\,(-x)^{-a_1-n}$$

$$+ \text{ a similar series obtained by interchanging } a_1 \text{ and } a_2$$

$$= (-x)^{-a_1}\frac{\Gamma(a_2-a_1)\,\Gamma(a_1)}{\Gamma(\rho-a_1)}\,F\{a_1,\, 1+a_1-\rho;\, 1+a_1-a_2;\, 1/x\}$$
$$+(-x)^{-a_2}\frac{\Gamma(a_1-a_2)\,\Gamma(a_2)}{\Gamma(\rho-a_2)}\,F\{a_2,\, 1+a_2-\rho;\, 1+a_2-a_1;\, 1/x\}.$$

By equating the two values of I_1 we obtain the given theorem.

7. We may evidently apply the previous method to the integrals I_2, I_3, I_4 of § 5.

From I_2 we see that the series which represents

$$(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho;\ 2-\rho;\ x\}$$

is a solution of Kummer's equation, when $|x| < 1$, and that this function can, when $|x| > 1$ and $|\arg(-x)| < \pi$, be expressed in the form

$$\frac{\Gamma(a_2-a_1)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}(-x)^{-a_1} F\{a_1,\ 1+a_1-\rho;\ 1+a_1-a_2;\ 1/x\}$$
$$+\frac{\Gamma(a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}(-x)^{-a_2} F\{a_2,\ 1+a_2-\rho;\ 1+a_2-a_1;\ 1/x\}. \quad \text{(II.)}$$

From I_3 we see that the series which represents

$$(-x)^{-a_1} F\{a_1,\ 1+a_1-\rho;\ 1+a_1-a_2;\ 1/x\}$$

is a solution of Kummer's equation when $|x| > 1$, and that this function can, when $|x| < 1$ and $|\arg(-x)| < \pi$, be expressed in the form

$$\frac{\Gamma(1-\rho)\,\Gamma(1+a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)} F\{a_1,\ a_2;\ \rho;\ x\}$$
$$+\frac{\Gamma(\rho-1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho;\ 2-\rho;\ x\}. \quad \text{(III.)}$$

From I_4 we see that the series which represents

$$(-x)^{-a_2} F\{a_2,\ 1+a_2-\rho;\ 1+a_2-a_1;\ 1/x\}$$

is a solution of Kummer's equation when $|x| > 1$, and that this function can, when $|x| < 1$ and $|\arg(-x)| < \pi$, be expressed in the form

$$\frac{\Gamma(1-\rho)\,\Gamma(1+a_2-a_1)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)} F\{a_1,\ a_2;\ \rho;\ x\}$$
$$+\frac{\Gamma(\rho-1)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho;\ 2-\rho;\ x\}. \quad \text{(IV.)}$$

8. We may similarly shew that the integrals

$$I_5 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(1+a_1+a_2-\rho+s)}(x-1)^s\,ds,$$

$$I_6 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(\rho-a_1-a_2-s)}{\Gamma(1+s)}(x-1)^s\,ds,$$

$$I_7 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)\,\Gamma(-s)}{\Gamma(1-a_2-s)}(x-1)^s\,ds,$$

$$I_8 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(1-a_1-s)}(x-1)^s\,ds,$$

which exist when $|\arg(x-1)| < \pi$ for all values of $|x|$, are solutions of Kummer's equation. These integrals, when treated in the manner of § 6, lead to the relations

$$F\{a_1, a_2;\ 1+a_1+a_2-\rho;\ 1-x\}$$
$$= \frac{\Gamma(a_2-a_1)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_2)\,\Gamma(1+a_2-\rho)}(x-1)^{-a_1}F\left\{a_1, \rho-a_2;\ 1+a_1-a_2;\ \frac{1}{1-x}\right\}$$
$$+\frac{\Gamma(a_1-a_2)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_1)\,\Gamma(1+a_1-\rho)}(x-1)^{-a_2}F\left\{a_2, \rho-a_1;\ 1+a_2-a_1;\ \frac{1}{1-x}\right\}, \quad \text{(V.)}$$

$$(x-1)^{\rho-a_1-a_2}F\{\rho-a_1, \rho-a_2;\ 1+\rho-a_1-a_2;\ 1-x\}$$
$$= \frac{\Gamma(a_2-a_1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(\rho-a_1)}(x-1)^{-a_1}F\left\{a_1, \rho-a_2;\ 1+a_1-a_2;\ \frac{1}{1-x}\right\}$$
$$+\frac{\Gamma(a_1-a_2)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(\rho-a_2)}(x-1)^{-a_2}F\left\{a_2, \rho-a_1;\ 1+a_2-a_1;\ \frac{1}{1-x}\right\}, \quad \text{(VI.)}$$

$$(x-1)^{-a_1}F\left\{\rho-a_2, a_1;\ 1-a_2+a_1;\ \frac{1}{1-x}\right\}$$
$$= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_2+a_1)}{\Gamma(\rho-a_2)\,\Gamma(1-a_2)}F\{a_1, a_2;\ 1-\rho+a_1+a_2;\ 1-x\}$$
$$+\frac{\Gamma(\rho-a_1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(1+\rho-a_1-a_2)}(x-1)^{\rho-a_1-a_2}F\{\rho-a_1, \rho-a_2;\ 1+\rho-a_1-a_2;\ 1-x\}, \quad \text{(VII.)}$$

and

$$(x-1)^{-a_2}F\left\{\rho-a_1, a_2;\ 1-a_1+a_2;\ \frac{1}{1-x}\right\}$$
$$= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(\rho-a_1)\,\Gamma(1-a_1)}F\{a_1, a_2;\ 1-\rho+a_1+a_2;\ 1-x\}$$
$$+\frac{\Gamma(\rho-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(1+\rho-a_1-a_2)}(x-1)^{\rho-a_1-a_2}F\{\rho-a_1, \rho-a_2;\ 1+\rho-a_1-a_2;\ 1-x\}. \quad \text{(VIII.)}$$

Each of the series which represents any one of the four hypergeometric functions involved in these formulæ is, when it is convergent, a solution of Kummer's equation. The many-valued functions involved are limited by the cross-cut defined by $|\arg(x-1)| < \pi$.

9. In a similar manner we see that the integrals

$$I_9 = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(a_1+s)\,\Gamma(\rho-a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}\left(\frac{x}{1-x}\right)^s ds,$$
$$I_{10} = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(a_1+s)\,\Gamma(\rho-a_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I_{11} = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(a_1+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-\rho+a_2-s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I_{12} = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\rho-a_2+s)}{\Gamma(1-a_1-s)}\left(\frac{x}{1-x}\right)^s ds$$

are solutions of Kummer's equation which exist for all values of $|x|$ when $|\arg\{x/(1-x)\}| < \pi$. These integrals are therefore defined for the whole plane with a cross-cut along the entire real axis except between 0 and 1. These integrals, when treated in the manner of § 6, lead to the relations

$$\begin{aligned}(1-x)^{-a_1}&F\left\{a_1, \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\}\\ &= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,x^{-a_1}F\left\{a_1, 1+a_1-\rho;\ 1+a_1+a_2-\rho;\ \frac{x-1}{x}\right\}\\ &\quad+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}(1-x)^{\rho-a_1-a_2}x^{a_2-\rho}\\ &\qquad\times F\left\{1-a_2, \rho-a_2;\ 1+\rho-a_1-a_2;\ \frac{x-1}{x}\right\},\end{aligned} \quad \text{(IX.)}$$

$$\begin{aligned}(1-x)^{\rho-a_1-1}x^{1-\rho}&F\left\{1-a_2, 1+a_1-\rho;\ 2-\rho;\ \frac{x}{x-1}\right\}\\ &= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}\,x^{-a_1}F\left\{1-\rho+a_1, a_1;\ 1-\rho+a_1+a_2;\ \frac{x-1}{x}\right\}\\ &\quad+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(2-\rho)}{\Gamma(1+a_1-\rho)\,\Gamma(1+a_2-\rho)}\,x^{a_2-\rho}(1-x)^{\rho-a_1-a_2}\\ &\qquad\times F\left\{\rho-a_2, 1-a_2;\ 1+\rho-a_1-a_2;\ \frac{x-1}{x}\right\},\end{aligned} \quad \text{(X.)}$$

$$\begin{aligned}x^{-a_1}&F\left\{1+a_1-\rho, a_1;\ 1-\rho+a_1+a_2;\ \frac{x-1}{x}\right\}\\ &= \frac{\Gamma(1-\rho)\,\Gamma(1-\rho+a_1+a_2)}{\Gamma(1-\rho+a_1)\,\Gamma(1-\rho+a_2)}(1-x)^{-a_1}F\left\{a_1, \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\}\\ &\quad+\frac{\Gamma(1-\rho+a_1+a_2)\,\Gamma(\rho-1)}{\Gamma(a_1)\,\Gamma(a_2)}\,x^{1-\rho}(1-x)^{\rho-a_1-1}\\ &\qquad\times F\left\{1-\rho+a_1, 1-a_2;\ 2-\rho;\ \frac{x}{x-1}\right\},\end{aligned} \quad \text{(XI.)}$$

$$\begin{aligned}x^{a_2-\rho}(1-x)^{\rho-a_1-a_2}&F\left\{1-a_2, \rho-a_2;\ 1+\rho-a_1-a_2;\ \frac{x-1}{x}\right\}\\ &= \frac{\Gamma(1-\rho)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}(1-x)^{-a_1}F\left\{a_1, \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\}\\ &\quad+\frac{\Gamma(\rho-1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,x^{1-\rho}(1-x)^{\rho-a_1-1}\\ &\qquad\times F\left\{1-\rho+a_1, 1-a_2;\ 2-\rho;\ \frac{x}{x-1}\right\}.\end{aligned} \quad \text{(XII.)}$$

10. We have now given twelve integrals which are solutions of Kummer's equation.

If we consider the integrals I_1 and I_9, we see that they both give rise to hypergeometric series convergent near $x=0$ of the same (zero) exponent. By adjustment of the constant multiplier, we therefore see that

$$I_1\,\Gamma(\rho-a_2) = I_9\,\Gamma(a_2)$$

or
$$F\{a_1, a_2; \rho; x\} = (1-x)^{-a_1} F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\}. \qquad (1)$$

Similarly we obtain other relations set forth in § 13 *infra*.

11. All the twenty-four solutions of Kummer's equation can now be displayed as six sets of four equivalent solutions.

For, from the relation (1) of § 10, we have

$$(1-x)^{-a_1} F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} = F\{a_1, a_2; \rho; x\};$$

and therefore by symmetry each is equal to

$$(1-x)^{-a_2} F\left\{a_2, \rho-a_1; \rho; \frac{x}{x-1}\right\}.$$

Take now the equality

$$F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} = (1-x)^{a_1-a_2} F\left\{a_2, \rho-a_1; \rho; \frac{x}{x-1}\right\},$$

and in it put x for $x/(x-1)$, a_2 for $\rho-a_2$, and we get

$$F\{a_1, a_2; \rho; x\} = (1-x)^{\rho-a_1-a_2} F\{\rho-a_1, \rho-a_2; \rho; x\}.$$

We therefore have the first set of four equivalent solutions:

$$\begin{aligned} Y_1 = F\{a_1, a_2; \rho; x\} &= (1-x)^{\rho-a_1-a_2} F\{\rho-a_1, \rho-a_2; \rho; x\} \\ &= (1-x)^{-a_1} F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} \\ &= (1-x)^{-a_2} F\left\{a_2, \rho-a_1; \rho; \frac{x}{x-1}\right\}. \qquad \text{(A)} \end{aligned}$$

Change a_1 into $1+a_1-\rho$, a_2 into $1+a_2-\rho$; ρ into $2-\rho$, and we get the second set

$$\begin{aligned} Y_2 &= x^{1-\rho} F\{1+a_1-\rho, 1+a_2-\rho; 2-\rho; x\} \\ &= x^{1-\rho}(1-x)^{\rho-a_1-a_2} F\{1-a_1, 1-a_2; 2-\rho; x\} \\ &= x^{1-\rho}(1-x)^{\rho-a_1-1} F\left\{1+a_1-\rho, 1-a_2; 2-\rho; \frac{x}{x-1}\right\} \\ &= x^{1-\rho}(1-x)^{\rho-a_2-1} F\left\{1+a_2-\rho, 1-a_1; 2-\rho; \frac{x}{x-1}\right\}. \qquad \text{(B)} \end{aligned}$$

Thus Y_1 and Y_2 are the two linearly independent solutions of Kummer's equation valid near $x=0$.

If Y_3 and Y_4 are the two linearly independent solutions valid near $x=1$, we have from (A), by obvious transformations,

$$\begin{aligned} Y_3 &= F\{\alpha_1, \alpha_2;\, 1+\alpha_1+\alpha_2-\rho;\, 1-x\} \\ &= x^{1-\rho} F\{1+\alpha_1-\rho,\, 1+\alpha_2-\rho;\, 1+\alpha_1+\alpha_2-\rho;\, 1-x\} \\ &= x^{-\alpha_1} F\left\{\alpha_1,\, 1+\alpha_1-\rho;\, 1+\alpha_1+\alpha_2-\rho;\, \frac{x-1}{x}\right\} \\ &= x^{-\alpha_2} F\left\{\alpha_2,\, 1+\alpha_2-\rho;\, 1+\alpha_1+\alpha_2-\rho;\, \frac{x-1}{x}\right\}, \end{aligned} \tag{C}$$

$$\begin{aligned} Y_4 &= (1-x)^{\rho-\alpha_1-\alpha_2} F\{\rho-\alpha_1,\, \rho-\alpha_2;\, 1+\rho-\alpha_1-\alpha_2;\, 1-x\} \\ &= x^{1-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_1,\, 1-\alpha_2;\, 1+\rho-\alpha_1-\alpha_2;\, 1-x\} \\ &= x^{\alpha_2-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\left\{\rho-\alpha_2,\, 1-\alpha_2;\, 1+\rho-\alpha_1-\alpha_2;\, \frac{x-1}{x}\right\} \\ &= x^{\alpha_1-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\left\{\rho-\alpha_1,\, 1-\alpha_1;\, 1+\rho-\alpha_1-\alpha_2;\, \frac{x-1}{x}\right\}. \end{aligned} \tag{D}$$

And, finally, if Y_5 and Y_6 are the two linearly independent solutions valid near $x=\infty$, we have

$$\begin{aligned} Y_5 &= x^{-\alpha_1} F\{\alpha_1,\, 1+\alpha_1-\rho;\, 1+\alpha_1-\alpha_2;\, 1/x\} \\ &= x^{\alpha_2-\rho}(x-1)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_2,\, \rho-\alpha_2;\, 1+\alpha_1-\alpha_2;\, 1/x\} \\ &= (x-1)^{-\alpha_1} F\left\{\alpha_1,\, \rho-\alpha_2;\, 1+\alpha_1-\alpha_2;\, \frac{1}{1-x}\right\} \\ &= x^{1-\rho}(x-1)^{\rho-1-\alpha_1} F\left\{1-\alpha_2,\, 1+\alpha_1-\rho;\, 1+\alpha_1-\alpha_2;\, \frac{1}{1-x}\right\}, \end{aligned} \tag{E}$$

$$\begin{aligned} Y_6 &= x^{-\alpha_2} F\{\alpha_2,\, 1+\alpha_2-\rho;\, 1+\alpha_2-\alpha_1;\, 1/x\} \\ &= x^{\alpha_1-\rho}(x-1)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_1,\, \rho-\alpha_1;\, 1+\alpha_2-\alpha_1;\, 1/x\} \\ &= (x-1)^{-\alpha_2} F\left\{\alpha_2,\, \rho-\alpha_1;\, 1+\alpha_2-\alpha_1;\, \frac{1}{1-x}\right\} \\ &= x^{1-\rho}(x-1)^{\rho-1-\alpha_2} F\left\{1-\alpha_1,\, 1+\alpha_2-\rho;\, 1+\alpha_2-\alpha_1;\, \frac{1}{1-x}\right\}, \end{aligned} \tag{F}$$

We assume that in each case the principal values (whose argument lies

between $\pm n\pi$) of n-th powers of x and $(x-1)$ are taken, where n is any quantity.

12. An examination of the transformation formulæ I., ..., XII. shews us that these formulæ are precisely the relations between the solutions $Y_1, \ldots, Y_6$ set forth in § 2.

Thus the relation (I.) may be written

$$Y_1 = \frac{\Gamma(\rho)\,\Gamma(a_2-a_1)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,e^{\pm\pi\iota a_1}\,Y_5 + \frac{\Gamma(\rho)\,\Gamma(a_1-a_2)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,e^{\pm\pi\iota a_2}\,Y_6,$$

also

$$Y_2 e^{\mp\pi\iota(1-\rho)} = \frac{\Gamma(a_2-a_1)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\,e^{\pm\pi\iota a_1}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,e^{\pm\pi\iota a_2}\,Y_6, \qquad \text{II.}$$

$$Y_5 e^{\pm\pi\iota a_1} = \frac{\Gamma(1-\rho)\,\Gamma(1+a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,e^{\mp\pi\iota(1-\rho)}\,Y_2, \qquad \text{III.}$$

$$Y_6 e^{\pm\pi\iota a_2} = \frac{\Gamma(1-\rho)\,\Gamma(1+a_2-a_1)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,e^{\mp\pi\iota(1-\rho)}\,Y_2, \qquad \text{IV.}$$

$$Y_3 = \frac{\Gamma(a_2-a_1)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_2)\,\Gamma(1+a_2-\rho)}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_1)\,\Gamma(1+a_1-\rho)}\,Y_6, \qquad \text{V.}$$

$$Y_4 e^{\pm\pi\iota(\rho-a_1-a_2)} = \frac{\Gamma(a_2-a_1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(\rho-a_1)}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(\rho-a_2)}\,Y_6, \qquad \text{VI.}$$

$$Y_5 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_2+a_1)}{\Gamma(\rho-a_2)\,\Gamma(1-a_2)}\,Y_3 + \frac{\Gamma(\rho-a_1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(1+\rho-a_1-a_2)}\,e^{\pm\pi\iota(\rho-a_1-a_2)}\,Y_4, \qquad \text{VII.}$$

$$Y_6 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(\rho-a_1)\,\Gamma(1-a_1)}\,Y_3 + \frac{\Gamma(\rho-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(1+\rho-a_1-a_2)}\,e^{\pm\pi\iota(\rho-a_1-a_2)}\,Y_4, \qquad \text{VIII.}$$

$$Y_1 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,Y_3 + \frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}\,Y_4, \qquad \text{IX.}$$

$$Y_2 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}\,Y_3 + \frac{\Gamma(a_1+a_2-\rho)\,\Gamma(2-\rho)}{\Gamma(1+a_1-\rho)\,\Gamma(1+a_2-\rho)}\,Y_4, \qquad \text{X.}$$

$$Y_3 = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho+a_1+a_2)}{\Gamma(1-\rho+a_1)\,\Gamma(1-\rho+a_2)}\,Y_1 + \frac{\Gamma(1-\rho+a_1+a_2)\,\Gamma(\rho-1)}{\Gamma(a_1)\,\Gamma(a_2)}\,Y_2, \qquad \text{XI.}$$

$$Y_4 = \frac{\Gamma(1-\rho)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,Y_2. \qquad \text{XII.}$$

In each case the upper or lower sign is taken as $I(x)$ is positive or negative.

We have, however, hitherto only written down the integrals which lead directly to twelve of the twenty-four solutions of Kummer's equation.

We now proceed to give the other set of twelve, and we number them in such a way that I'_R, when treated in the manner of § 6, leads to the relation (R). The integrals are

$$I'_1 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(\rho+s)}(-x)^s\,ds,$$

$$I'_2 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(1-\rho-s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(1+s)}(-x)^s\,ds,$$

$$I'_3 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(1-\rho-s)\,\Gamma(-s)\,\Gamma(\rho-\alpha_2+s)}{\Gamma(1-\rho+\alpha_1-s)}(-x)^s\,ds,$$

$$I'_4 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\rho-\alpha_1+s)\,(1-\rho-s)}{\Gamma(1-\rho+\alpha_2-s)}(-x)^s\,ds,$$

$$I'_5 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(1-\rho+\alpha_2+s)}{\Gamma(1-\rho+\alpha_1+\alpha_2+s)}(x-1)^s\,ds,$$

$$I'_6 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(1-\rho+\alpha_2+s)}{\Gamma(1+s)}(x-1)^s\,ds,$$

$$I'_7 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)}{\Gamma(\rho-\alpha_2-s)}(x-1)^s\,ds,$$

$$I'_8 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_2+s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)}{\Gamma(\rho-\alpha_1-s)}(x-1)^s\,ds,$$

$$I'_9 = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(\rho+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{10} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{11} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\alpha_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1-\rho+\alpha_1-s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{12} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(\rho-\alpha_1+s)\,\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1-\alpha_2-s)}\left(\frac{x}{1-x}\right)^s ds.$$

13. By considering the hypergeometric series to which the integrals give rise, we obtain an extension of the equalities indicated in § 10.

Thus, if $J_1, \ldots, J_6$ be suitable multiples of $Y_1, \ldots, Y_6$ respectively, we have

$$\begin{aligned}
J_1 &= I_1\Gamma(\rho-a_1)\Gamma(\rho-a_2) = I_9\Gamma(\rho-a_1)\Gamma(a_2) = I_1'\Gamma(a_1)\Gamma(a_2) \\
&= I_9'\Gamma(a_1)\Gamma(\rho-a_2), \\
J_2 &= I_2\Gamma(1-a_1)\Gamma(1-a_2) = I_{10}e^{\pm\pi i(\rho-1)}\Gamma(1-a_1)\Gamma(1+a_2-\rho) \\
&= I_2'\Gamma(1+a_1-\rho)\Gamma(1+a_2-\rho) = I_{10}'e^{\pm\pi i(\rho-1)}\Gamma(1-a_2)\Gamma(1+a_1-\rho). \\
J_3 &= I_5\Gamma(1-\rho+a_1)\Gamma(1-\rho+a_2) = I_{11}\Gamma(a_2)\Gamma(1-\rho+a_2) \\
&= I_5'\Gamma(a_1)\Gamma(a_2) = I_{11}'\Gamma(a_1)\Gamma(1-\rho+a_1), \\
J_4 &= I_6\Gamma(1-a_1)\Gamma(1-a_2) = I_{12}e^{\pm\pi i(\rho-a_1-a_2)}\Gamma(\rho-a_1)\Gamma(1-a_1) \\
&= I_6'\Gamma(\rho-a_1)\Gamma(\rho-a_2) = I_{12}'e^{\pm\pi i(\rho-a_1-a_2)}\Gamma(\rho-a_2)\Gamma(1-a_2), \\
J_5 &= I_3\Gamma(1-a_2)\Gamma(\rho-a_2) = I_7e^{\pm\pi i a_1}\Gamma(1-a_2)\Gamma(1-\rho+a_1) \\
&= I_3'\Gamma(a_1)\Gamma(1-\rho+a_1) = I_7'e^{\pm\pi i a_1}\Gamma(a_1)\Gamma(\rho-a_2), \\
J_6 &= I_4\Gamma(1-a_1)\Gamma(\rho-a_1) = I_8e^{\pm\pi i a_2}\Gamma(1-a_1)\Gamma(1-\rho+a_2) \\
&= I_4'\Gamma(a_2)\Gamma(1-\rho+a_2) = I_8'e^{\pm\pi i a_2}\Gamma(a_2)\Gamma(\rho-a_1).
\end{aligned}$$

In each case the upper or lower sign is taken as $I(x)$ is $\pm$: when x is real we have a point on one of the various systems of cross-cuts by which the integrals $I_1, \ldots, I_{12}$; $I_1', \ldots, I_{12}'$, are limited, and therefore some of the formulæ are illusory.

14. We have now obtained the twenty-four solutions of Kummer's equation and we have found the relations between each of the six sets of four which are substantially the same, and also the relations connecting any three of the six fundamental solutions. Our transformations (I.), ..., (XII.) are all, however, transformations by which substantially the variable is changed into its reciprocal. We now proceed to shew that the contour integrals can be so modified as to give directly the change of x into $1-x$; and, in fact, any other of the six transformations

$$x, \quad 1/x, \quad 1-x, \quad 1/(1-x), \quad x/(x-1), \quad (x-1)/x.$$

And, further, we will obtain directly from the theory of the contour integrals the relations given in § 13.

For this purpose we need a lemma which proves to be of fundamental importance in the theory.

15. LEMMA.—*If α_1, α_2, β_1, β_2 be any complex quantities of finite modulus, certain special cases excepted, and if the contour of the integral be parallel to the imaginary axis with loops if necessary to ensure that*

positive sequences of poles of the subject of integration lie to the right of the contour and negative sequences to the left,

$$-\frac{1}{2\pi\iota}\int \Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(\beta_1-s)\,\Gamma(\beta_2-s)\,ds$$
$$=\frac{\Gamma(a_1+\beta_1)\,\Gamma(a_1+\beta_2)\,\Gamma(a_2+\beta_1)\,\Gamma(a_2+\beta_2)}{\Gamma(a_1+a_2+\beta_1+\beta_2)}.$$

It is evident from the asymptotic expansion of the gamma function that, unless a_1, a_2, β_1, β_2 have such relations that the contour cannot be drawn, the integral will exist and have a definite finite value.

When a_1, a_2, β_1, β_2 are such that $R(1-a_1-a_2-\beta_1-\beta_2)>0$, the contour of the integral can be bent round so as to include negative sequences of poles of the subject of integration, and by Cauchy's theorem it will be equal to

$$\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\,\Gamma(\beta_1+a_1+n)\,\Gamma(a_2-a_1-n)\,\Gamma(a_1+\beta_2+n)$$
$$+\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\,\Gamma(\beta_1+a_2+n)\,\Gamma(a_1-a_2-n)\,\Gamma(a_2+\beta_2+n)$$
$$=\Gamma(a_1+\beta_1)\,\Gamma(a_2-a_1)\,\Gamma(a_1+\beta_2)\,F\{a_1+\beta_1,\ a_1+\beta_2;\ 1+a_1-a_2;\ 1\}$$
$$+\Gamma(a_2+\beta_1)\,\Gamma(a_1-a_2)\,\Gamma(a_2+\beta_2)\,F\{a_2+\beta_1,\ a_2+\beta_2;\ 1+a_2-a_1;\ 1\}.$$

By Gauss's theorem this is equal to

$$\Gamma(a_1+\beta_1)\,\Gamma(a_2-a_1)\,\Gamma(a_1+\beta_2)\,\frac{\Gamma(1+a_1-a_2)\,\Gamma(1-a_1-a_2-\beta_1-\beta_2)}{\Gamma(1-a_2-\beta_1)\,\Gamma(1-a_2-\beta_2)}$$

$+$ a similar expression obtained by interchanging a_1 and a_2

$$=-\Gamma(1-a_1-a_2-\beta_1-\beta_2)\,\Gamma(a_1+\beta_1)\,\Gamma(a_1+\beta_2)\,\Gamma(a_2+\beta_1)\,\Gamma(a_2+\beta_2)$$
$$\times\frac{1}{\pi\sin\pi(a_1-a_2)}\{\sin\pi(a_2+\beta_1)\sin\pi(a_2+\beta_2)-\sin\pi(a_1+\beta_1)\sin\pi(a_1+\beta_2)\}$$
$$=\Gamma(1-a_1-a_2-\beta_1-\beta_2)\,\Gamma(a_1+\beta_1)\,\Gamma(a_1+\beta_2)\,\Gamma(a_2+\beta_1)\,\Gamma(a_2+\beta_2)$$
$$\times\frac{\sin\pi(a_1+a_2+\beta_1+\beta_2)}{\pi}$$
$$=\frac{\Gamma(a_1+\beta_1)\,\Gamma(a_1+\beta_2)\,\Gamma(a_2+\beta_1)\,\Gamma(a_2+\beta_2)}{\Gamma(a_1+a_2+\beta_1+\beta_2)}.$$

We have limited ourselves by the restriction that

$$R(1-a_1-a_2-\beta_1-\beta_2)>0.$$

But the original integral and the final expression are, except for isolated points, analytic functions of a_1, a_2, β_1, and β_2. The theorem is therefore true in general.

16. *We now proceed to shew that, if J_1 denote the integral*

$$-\frac{1}{2\pi\iota}\int_C \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)\,(1-x)^s\,ds,$$

wherein $|\arg(1-x)| < 2\pi$, *and the contour C is parallel to the imaginary axis and passes between the sequences of positive and negative poles of the subject of integration, then*

$$J_1 = I_1\,\Gamma(\rho-a_1)\,\Gamma(\rho-a_2),$$

where I_1 is the integral defined in § 5.

I have previously shewn* that, if D be a contour parallel to the imaginary axis with loops leaving the positive sequence of poles 0, 1, 2, ... on the right and the negative sequence s, $s-1$, $s-2$, ... on the left,

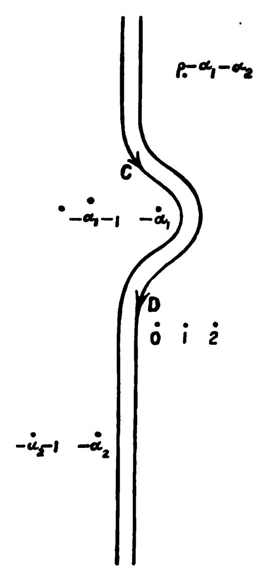
$$\Gamma(-s)\,(1-x)^s = -\frac{1}{2\pi\iota}\int_D \Gamma(\phi-s)\,\Gamma(-\phi)\,(-x)^\phi\,d\phi,$$

wherein $\quad |\arg(-x)| < \pi, \qquad |\arg(1-x)| < \pi.$

We therefore have

$$J_1 = -\frac{1}{2\pi\iota}\int_C\left(-\frac{1}{2\pi\iota}\right)\int_D \Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)$$
$$\times\,\Gamma(\phi-s)\,\Gamma(-\phi)\,(-x)^\phi\,d\phi.$$

Now we may invert the order of integration, for each integral is absolutely and uniformly convergent. We may conveniently take the contours as in the figure. Hence we have



$$\begin{aligned} J_1 = {} & -\frac{1}{2\pi\iota}\int_D \Gamma(-\phi)\,(-x)^\phi\,d\phi \\ & \times -\frac{1}{2\pi\iota}\int_C \Gamma(\rho-a_1-a_2-s)\,\Gamma(\phi-s) \\ & \qquad\qquad \times\,\Gamma(a_1+s)\,\Gamma(a_2+s)\,ds \\ = {} & -\frac{1}{2\pi\iota}\,\Gamma(\rho-a_1)\,\Gamma(\rho-a_2) \\ & \times\int_D \frac{\Gamma(-\phi)\,\Gamma(a_1+\phi)\,\Gamma(a_2+\phi)}{\Gamma(\rho+\phi)}\,(-x)^\phi\,d\phi \\ & \qquad\qquad \text{(by the lemma of § 15)} \\ = {} & \Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,I_1. \end{aligned}$$

In the investigation we have assumed that

$$|\arg(-x)| < \pi$$

which gives the cross-cut necessary to define I_1.

* *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 108–116.

This includes the condition $|\arg(1-x)| < \pi$. We notice that the integral J_1 represents the function over the extended range

$$|\arg(1-x)| < 2\pi.$$

17. Suppose now that $|1-x| < 1$. We may bend round the contour of the integral J_1 so as to include the positive sequences of poles of the subject of integration, and we therefore obtain, by Cauchy's theorem,

$$\frac{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}{\Gamma(\rho)} F\{a_1, a_2; \rho; x\}$$

$$= \Gamma(-a_1-a_2)\,\Gamma(a_1)\,\Gamma(a_2)\,F\{a_1, a_2; 1+a_1+a_2-\rho; 1-x\}$$
$$+\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,\Gamma(a_1+a_2-\rho)\,(1-x)^{\rho-a_1-a_2}$$
$$\times F\{\rho-a_1, \rho-a_2; 1+\rho-a_1-a_2; 1-x\}$$

or $F\{a_1, a_2; \rho; x\}$

$$= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)} F\{a_1, a_2; 1+a_1+a_2-\rho; 1-x\}$$
$$+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)} (1-x)^{\rho-a_1-a_2} F\{\rho-a_1, \rho-a_2; 1+\rho-a_1-a_2; 1-x\}.$$

This is a direct transformation from argument x to argument $(1-x)$. It is equivalent to our old relation (IX.) of § 12.

Suppose that similarly $|1-x| > 1$ and we bend the contour of the integral J_1 round to the left so as to include the negative sequences of poles of the subject of integration, we obtain

$F\{a_1, a_2; \rho; x\}$

$$= \frac{\Gamma(a_2-a_1)\,\Gamma(\rho)}{\Gamma(a_2)\,\Gamma(\rho-a_1)} (1-x)^{-a_1} F\{a_1, \rho-a_2; 1+a_1-a_2; 1/(1-x)\}$$
$$+\frac{\Gamma(a_1-a_2)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(\rho-a_2)} (1-x)^{-a_2} F\{a_2, \rho-a_1; 1+a_2-a_1; 1/(1-x)\}.$$

This is equivalent to our old relation (I.) of § 12.

The formula thus gives a direct transformation from x to $1/(1-x)$.

18. We can now shew that the integral J_1 gives rise directly to the set of equalities obtained indirectly in § 13 between I_1, I_9, I_1', I_9'. We have already seen that $J_1 = \Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,I_1$.

Again, if in the integral we write $s-a_1-a_2+\rho$ for s, it becomes

$$J_1 = (1-x)^{\rho-a_1-a_2}$$
$$\times\left(-\frac{1}{2\pi\iota}\right)\int \Gamma(-s)\,\Gamma(a_1+a_2-\rho-s)\,\Gamma(\rho-a_2+s)\,\Gamma(\rho-a_1+s)\,(1-x)^s ds.$$

This is the same integral as the former when a_1 is replaced by $\rho-a_1$ and

a_2 by $\rho-a_2$. The transformation of § 16 therefore gives us

$$J_1 = \Gamma(a_1)\,\Gamma(a_2)\,(1-x)^{\rho-a_1-a_2}$$
$$\times\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(\rho-a_1+\phi)\,\Gamma(\rho-a_2+\phi)\,\Gamma(-\phi)}{\Gamma(\rho+\phi)}\,(-x)^{\phi}\,d\phi$$
$$= \Gamma(a_1)\,\Gamma(a_2)\,I_1',$$

Similarly, by an obvious change of the variable, J_1 may be written

$$-\frac{1}{2\pi\iota}\,(1-x)^{-a_1}\int \Gamma(-s)\,\Gamma(a_2-a_1-s)\,\Gamma(a_1+s)\,\Gamma(\rho-a_2+s)\,(1-x)^{-s}ds$$
$$= \Gamma(a_2)\,\Gamma(\rho-a_1)$$
$$\times\left(-\frac{1}{2\pi\iota}\right)(1-x)^{-a_1}\int \frac{\Gamma(-\phi)\,\Gamma(a_1+\phi)\,\Gamma(\rho-a_2+\phi)}{\Gamma(\rho+\phi)}\left(\frac{x}{1-x}\right)^{\phi}d\phi$$
$$= \Gamma(a_2)\,\Gamma(\rho-a_1)\,I_9,$$

and, by symmetry, $\qquad J_1 = \Gamma(a_1)\,\Gamma(\rho-a_2)\,I_9'.$

We thus have by direct transformation the first set of equalities of § 13. or, if we prefer so to regard it, the equalities (A) of § 11.

19. We can similarly obtain by direct transformation the other five sets of equalities of § 13, and all possible transformations and relations between three integrals of the system by means of the analogous integrals

$$J_2 = -\frac{1}{2\pi\iota}\,(-x)^{1-\rho}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)$$
$$\times\Gamma(1+a_1-\rho+s)\,\Gamma(1+a_2-\rho+s)\,(1-x)^{s}\,ds.$$

$$J_3 = -\frac{1}{2\pi\iota}\int \Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)\,x^{s}\,ds,$$

$$J_4 = -\frac{1}{2\pi\iota}\,(x-1)^{\rho-a_1-a_2}\int \Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\rho-a_1+s)\,\Gamma(\rho-a_2+s)\,x^{s}\,ds,$$

$$J_5 = -\frac{1}{2\pi\iota}\,(-x)^{-a_1}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)$$
$$\times\Gamma(1+a_1-\rho+s)\left(\frac{x-1}{x}\right)^{s}ds,$$

$$J_6 = -\frac{1}{2\pi\iota}\,(-x)^{-a_2}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_2+s)$$
$$\times\Gamma(1+a_2-\rho+s)\left(\frac{x-1}{x}\right)^{s}ds.$$

The general theory of the hypergeometric solutions of Kummer's equation is evidently complete. When the quantities a_1, a_2, ρ or their differences are integers, the integrals are still valid representations of the solutions even though the hypergeometric series degenerate into different forms involving logarithmic terms. The case when $\rho = 1$, $a_1 = a_2 = \frac{1}{2}$ is discussed later: it gives rise to the quarter-periods of the Jacobian elliptic functions.

We now proceed to an analogous development of the Riemann P-functions, where the greater symmetry of the theory shews the elegance of Riemann's generalisation.

Part II.

The Riemann P-Functions.

20. The differential equation for Riemann's P-function is, in the form due to Papperitz,*

$$\frac{d^2y}{dx^2}+\left\{\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right\}\frac{dy}{dx}$$
$$+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right\}$$
$$\times\frac{y}{(x-a)(x-b)(x-c)}=0,$$

where $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

Put $$z=\frac{x-a}{x-b}\frac{c-b}{c-a} \quad\text{and}\quad y=z^{\alpha}(1-z)^{\gamma}w,$$

and we obtain

$$\frac{d^2w}{dz^2}+\left(\frac{1+\alpha-\alpha'}{z}+\frac{1+\gamma-\gamma'}{z-1}\right)\frac{dw}{dz}+\{\beta\beta'+(\alpha+\gamma)(1-\alpha'-\gamma')\}\frac{w}{z(z-1)}=0.$$

Take now
$$a_1=\alpha+\beta+\gamma,$$
$$a_2=\alpha+\beta'+\gamma,$$
$$\rho=1+\alpha-\alpha',$$

so that $$a_1a_2=\beta\beta'+(\alpha+\gamma)(1-\alpha'-\gamma'),$$

since $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1,$$

and we have $$\frac{d^2w}{dz^2}+\frac{\rho-(1+a_1+a_2)z}{z(1-z)}\frac{dw}{dz}-\frac{a_1a_2w}{z(1-z)}=0.$$

This is the ordinary equation of the hypergeometric function, and we have seen that a solution is

$$w=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)}{\Gamma(\rho+s)}\Gamma(-s)(-z)^s\,ds,$$

when $|\arg(-z)|<\pi$.

* Papperitz, *Mathematische Annalen*, T. xxv., p. 213.

Hence a solution of Papperitz's equation is

$$-\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(\alpha+\gamma-s)\,\Gamma(\beta+s)\,\Gamma(\beta'+s)}{\Gamma(1-\alpha'-\gamma+s)}\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds,$$

when $\left|\arg -\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}\right| < \pi$.

21. Leaving for the present the question of the precise determination of the functions $\left(\dfrac{x-c}{x-a}\dfrac{b-a}{b-c}\right)^{\gamma}$ and $\left(-\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}\right)^{s}$ we can, by making all possible symmetrical interchanges in this solution, obtain the twenty-four integrals which satisfy Papperitz's equation. These we now proceed to indicate, denoting them by K with such suffixes and accents as correspond in order to the twenty-four solutions (A), ..., (F) (§ 11) of Kummer's equation when $a=0$, $b=\infty$, $c=1$, and when a_1, a_2, ρ are connected with α, β, γ, α', β', γ' by the relations of the previous paragraph, coupled with $\alpha=0$, $\gamma=0$. Thus we have $\alpha=0$, $\alpha'=1-\rho$, $\beta=a_1$, $\beta'=a_2$, $\gamma=0$, $\gamma'=\rho-a_1-a_2$. We thus write down the integrals in groups of four, each of which will be subsequently proved to be substantially the same functions of x. And in brackets we indicate the corresponding solution I of the more special case.

$$\left.\begin{aligned}
K_1 &= -\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(\alpha+\gamma-s)}{\Gamma(1-\alpha'-\gamma+s)}\Gamma(\beta+s)\,\Gamma(\beta'+s)\\
&\qquad\times\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds\ (I_1)\\
K_2 &= -\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma'}\int\frac{\Gamma(\alpha+\gamma'-s)}{\Gamma(1-\alpha'-\gamma'+s)}\Gamma(\beta+s)\,\Gamma(\beta'+s)\\
&\qquad\times\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds\ (I_1')\\
K_3 &= -\frac{1}{2\pi\iota}\left(\frac{x-b}{x-a}\frac{c-a}{c-b}\right)^{\beta}\int\frac{\Gamma(\alpha+\beta-s)}{\Gamma(1-\alpha'-\beta+s)}\Gamma(\gamma+s)\,\Gamma(\gamma'+s)\\
&\qquad\times\left(-\frac{x-a}{b-a}\frac{b-c}{x-c}\right)^{s}ds\ (I_9)\\
K_4 &= -\frac{1}{2\pi\iota}\left(\frac{x-b}{x-a}\frac{c-a}{c-b}\right)^{\beta'}\int\frac{\Gamma(\alpha+\beta'-s)}{\Gamma(1-\alpha'-\beta'+s)}\Gamma(\gamma+s)\,\Gamma(\gamma'+s)\\
&\qquad\times\left(-\frac{x-a}{b-a}\frac{b-c}{x-c}\right)^{s}ds\ (I_9')
\end{aligned}\right\}. \quad (A)$$

If we interchange α and α' in this set we get in order K_5, K_6, K_7, K_8,

which correspond respectively to I_2, I_2', I_{10}, I_{10}' and form a set (B) of four integrals.

Interchange in (A) γ and α, γ' and α', b and c, and we get a set (C) of four integrals K_9, K_{10}, K_{11}, K_{12} which correspond to I_5, I_5', I_{11}, I_{11}' respectively.

Interchange γ and γ' in the set (C) and we get the set (D) of four integrals K_{13}, K_{14}, K_{15}, K_{16} which correspond to I_6, I_6', I_{12}, I_{12}'.

Interchange in (A) a and b, α and β, α' and β', and we get the set (E) of four integrals K_{17}, K_{18}, K_{19}, K_{20} which correspond to I_3, I_3', I_7, I_7'.

Finally, interchanging β and β' in (E) we get the set (F) of four integrals K_{21}, K_{22}, K_{23}, K_{24} which correspond to I_4, I_4', I_8, I_8'.

22. We have now to give an accurate definition of the many-valued functions which occur in the previous integrals $K_1, \ldots, K_{24}$.

Represent the three quantities a, b, c by points A, B, C by the usual Argand diagram, and, as in Riemann's memoir, let us assume that these points are so placed that, when we go round the circle through A, B, and C in a positive counter-clockwise direction, we pass from A to C to B.

We now define $\{(x-c)/(x-a)\}^\gamma$ by a cross-cut along the arc AC. (By this we mean the arc which excludes the point B: the other arc will be denoted by ABC.) When x lies within the circle, arg $\{(x-c)/(x-a)\}$ is taken to lie between $\pi+B$, its value just inside the arc AC, and B, its value on the arc ABC. When x lies outside the circle, arg $\{(x-c)/(x-a)\}$ is taken to lie between B, its value on the arc ABC, and $B-\pi$, its value just outside the arc AC.

The argument of $\dfrac{b-c}{b-a}$ is thus B. Hence $\dfrac{x-c}{b-c}\dfrac{b-a}{x-a}$ has a cross-cut along AC. Its argument ranges from π just within AC to zero on ABC, and to $-\pi$ just outside AC.

Similarly $\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}$ has a cross-cut along BA, and its argument ranges from π just within BA to zero on BCA and to $-\pi$ just outside BA. And $\dfrac{x-b}{a-b}\dfrac{a-c}{x-c}$ has a cross-cut along CB, and its argument ranges from π just within CB to zero on CAB and to $-\pi$ just outside CB.

Further, in the preceding integrals we always take such values of $\left(-\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}\right)^s$ and similar terms as have arguments less than π. Hence

$$\arg\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right) = \arg\frac{x-a}{c-a}\frac{c-b}{x-b} \begin{matrix} -\pi \\ +\pi \end{matrix} \text{ as } x \text{ lies } \begin{matrix}\text{within} \\ \text{without}\end{matrix} \text{ the circle.}$$

Hence $-\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}$ has a cross-cut along the arc ACB, and the value of its argument ranges from $-\pi$ just within ACB to zero on AB, and to π just outside ACB. The argument of the reciprocal of this expression is minus the argument of the expression. Similar definitions apply to the other similar terms which intervene in the integrals defined in the previous paragraph.

We see that, with such definitions,

$$\arg\frac{x-a}{x-b}+\arg\frac{x-b}{x-c}+\arg\frac{x-c}{x-a} = \left.\begin{matrix} 2\pi \text{ within} \\ 0 \text{ without}\end{matrix}\right\} \text{ the circle;}$$

and also $$\left(\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{a}\left(\frac{x-b}{a-b}\frac{a-c}{x-c}\right)^{a}\left(-\frac{x-c}{b-c}\frac{b-a}{x-a}\right)^{a}=1,$$

whether x be within or without the circle, for the three terms in order have cross-cuts along AB, BC, and ABC, and these cross-cuts neutralise one another.

For brevity we shall write

$$u=\frac{x-b}{a-b}\frac{a-c}{x-c},\quad v=\frac{x-c}{b-c}\frac{b-a}{x-a},\quad w=\frac{x-a}{c-a}\frac{c-b}{x-b}.$$

The Functions P_a, $P_{a'}$, ..., $P_{\gamma'}$.

23. By § 21 it is evident that there is a solution of Papperitz' equation which near $x = a$ admits an expansion of the form

$$w^{a}\{1+C_1w+C_2w^2+\ldots\}.$$

This solution when $|\arg w| < \pi$ we denote by P_a. Thus P_a is defined with respect to a cross-cut along the arc AB: on the inside of this arc $\arg w = \pi$, on the outside it is $-\pi$, and on the arc ACB it is zero.

From the theory of the differential equation we know that equally P_a must admit near $x = a$ an expansion in powers of

$$1/v=\frac{x-a}{b-a}\frac{b-c}{x-c},$$

with index a at $x = a$. Now v^a has a cross-cut along AC and $\arg v$ ranges from π just within AC to zero on ABC, and then to π just outside AC. Hence, if $|\arg(-1/v)| < \pi$, $(-1/v)^a$ has a cross-cut along ABC, and $\arg(-1/v)$ ranges from π just within ABC to zero on AC, and then to $-\pi$ just outside ABC. Hence the arguments of w^a and $(-1/v)^a$ have the

same range, and near $x = a$ they have the same cross-cut. Also

$$\frac{(-1/v)^{a}}{w^{a}} = \left(\frac{c-a}{c-x}\,\frac{x-b}{a-b}\right)^{a},$$

and this ratio is unity at $x = a$. Therefore P_a may be equally expressed in the form

$$(-1/v)^{a}\{1+D_1/v+D_2/v^2+\ldots\},$$

wherein $|\arg(-1/v)| < \pi$. By writing a' for a we derive $P_{a'}$ from P_a.

Similarly P_β near $x = b$ has a cross-cut along the arc BC and may be expressed in either of the forms

$$u^{\beta}\{1+E_1u+E_2u^2+\ldots\}, \text{ wherein } |\arg u| < \pi;$$

or $$(-1/w)^{\beta}\{1+F_1/w+F_2/w^2+\ldots\}, \text{ wherein } |\arg(-1/w)| < \pi.$$

And P_γ has a cross-cut near $x = c$ along the arc CA, and may be expressed in either of the forms

$$v^{\gamma}\{1+G_1v+G_2v^2+\ldots\}, \text{ wherein } |\arg v| < \pi;$$

or $$(-1/u)^{\gamma}\{1+H_1/u+H_2/u^2+\ldots\}, \text{ wherein } |\arg(-1/u)| < \pi.$$

24. We proceed now to shew that

$$P_a = \left(\frac{x-c}{a-c}\,\frac{a-b}{x-b}\right)^{\gamma}\left(\frac{x-a}{c-a}\,\frac{c-b}{x-b}\right)^{a}$$
$$\times F\left\{a+\beta+\gamma,\ a+\beta'+\gamma;\ 1-a'+a;\ \frac{x-a}{c-a}\,\frac{c-b}{x-b}\right\},$$

$$K_1 = e^{\mp\pi\iota a}\,\frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}\,P_a,$$

the upper or lower sign being taken as x lies within or without the circle. We have

$$K_1 = -\frac{1}{2\pi\iota}\,v^{\gamma}\int\frac{\Gamma(a+\gamma-s)}{\Gamma(1-a'-\gamma+s)}\,\Gamma(\beta+s)\,\Gamma(\beta'+s)(-w)^{s}\,ds,$$

with the previous specification of v^{γ} and $(-w)^{s}$.

When $|w| < 1$, we may bend round the contour so as to include the positive sequence of poles of the subject of integration, and we have, by Cauchy's theorem,

$$K_1 = v^{\gamma}(-w)^{a+\gamma}\,\frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$
$$\times F\{a+\beta+\gamma,\ a+\beta'+\gamma;\ 1-a'+a;\ w\}.$$

Now, by § 22, $v^{\gamma}(-w)^{\gamma} = (1/u)^{\gamma}$, and $(-w)^{a} = w^{a} e^{\mp\pi i a}$.

Hence

$$K_1 = (1/u)^{\gamma} w^{a} e^{\mp\pi i a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$
$$\times F\{a+\beta+\gamma,\ a+\beta'+\gamma;\ 1-a'+a;\ w\}.$$

Now, at $x = a$, $(1/u)^{\gamma}$ approaches the value unity.

Hence, by the definition of § 23,

$$K_1 = e^{\mp\pi i a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)} P_a.$$

We thus have the equalities given.

25. If we treat the integral K_2 in the same way, we get

$$K_2 = e^{\mp\pi i a} \frac{\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}{\Gamma(1-a'+a)} P_a,$$

$$P_a = (1/u)^{\gamma'} w^{a} F\{a+\beta+\gamma',\ a+\beta'+\gamma';\ 1-a'+a;\ w\}.$$

Similarly $$K_3 = \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma')}{\Gamma(1-a'+a)} P_a,$$

$$P_a = u^{\beta}(-1/v)^{a} F\{a+\beta+\gamma,\ a+\beta+\gamma';\ 1-a'+a;\ 1/v\}.$$

Finally, $$K_4 = \frac{\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma')}{\Gamma(1-a'+a)} P_a,$$

$$P_a = u^{\beta'}(-1/v)^{a} F\{a+\beta'+\gamma,\ a+\beta'+\gamma';\ 1-a'+a;\ 1/v\}.$$

We have thus deduced from the theory of the differential equation the equalities

$$\begin{aligned} K_1 e^{\pm\pi i a}\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma') &= K_2 e^{\pm\pi i a}\,\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma) \\ &= K_3 \Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma') \\ &= K_4 \Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma'). \end{aligned}$$

These equalities shew that the group of integrals (A) of § 21 are substantially equivalent to P_a. Similarly (B), (C), (D), (E), (F) respectively are equivalent to $P_{a'}$, P_{γ}, $P_{\gamma'}$, P_{β}, $P_{\beta'}$.

26. We can verify the relations just found by direct transformation. Apply the result of § 16 to the function

$$K_1 = (1/u)^{\gamma} w^{a} e^{\mp\pi i a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$

$$\times F\{a+\beta+\gamma,\ a+\beta'+\gamma;\ 1-a'+a\ ;\ w\}.$$

We find

$$K_1 (1/w)^{a} e^{\pm\pi i a}\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta+\gamma')$$

$$= -\frac{1}{2\pi i} u^{a} \int \Gamma(a+\gamma+s)\,\Gamma(a+\gamma'+s)\,\Gamma(\beta-s)\,\Gamma(\beta'-s)\, u^{s}\, ds,$$

the direction of integration of the contour being downwards.

Interchanging β and γ, β' and γ', b and c, we find

$$K_1 (1/w)^{a} e^{\pm\pi i a}\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta+\gamma')$$

$$= K_3 u^{a} (-v)^{a}\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta'+\gamma),$$

so that $$K_1 e^{\pm\pi i a}\,\Gamma(a+\beta+\gamma') = K_3\,\Gamma(a+\beta'+\gamma),$$

since $$w^{a} u^{a} (-v)^{a} = 1 \quad (\S\ 22).$$

The other relations can be obtained in like manner.

We note that incidentally we have shewn that

$$P_a \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}{\Gamma(1-a'+a)}$$

$$= -\frac{1}{2\pi i} w^{a} \int \Gamma(\gamma-s)\,\Gamma(\gamma'-s)\,\Gamma(a+\beta+s)\,\Gamma(a+\beta'+s)\, u^{-s}\, ds,$$

wherein $|\arg u| < \pi$. This and analogous expressions constitute the best definitions of the fundamental solutions P_a, $P_{a'}$, P_{β}, $P_{\beta'}$, P_{γ}, $P_{\gamma'}$.

27. We know from the theory of linear differential equations that a linear relation connects any three independent solutions. Adopting Riemann's notation we must therefore have relations of the form*

$$P_a = a_{\beta} P_{\beta} + a_{\beta'} P_{\beta'},$$

where a_{β} and $a_{\beta'}$ are constants.

* Such relations are of course part of Riemann's definition. He did not define his P-functions as solutions of Papperitz's equation.

We may easily establish this relation directly, and shew that

$$a_\beta = \frac{\Gamma(1-a'+a)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma')}\, e^{\pm\pi\iota a},$$

$$a_{\beta'} = \frac{\Gamma(1-a'+a)\,\Gamma(\beta-\beta')}{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma')}\, e^{\pm\pi\iota a},$$

the upper or lower sign being, as usual, taken as x lies within or without the circle.

We have

$$P_a e^{\mp\pi\iota a}\frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$

$$= K_1 = -\frac{1}{2\pi\iota} v^\gamma \int \frac{\Gamma(a+\gamma-s)}{\Gamma(1-a'-\gamma+s)}\,\Gamma(\beta+s)\,\Gamma(\beta'+s)(-w)^s\, ds.$$

If $|w| > 1$ we may bend round the contour so as to include the negative sequences of poles of the subject of integration, and we have by Cauchy's theorem

$$K_1 = v^\gamma(-w)^{-\beta}\frac{\Gamma(a+\beta+\gamma)\,\Gamma(\beta'-\beta)}{\Gamma(1-a'-\gamma-\beta)}$$

$$\times F\{a+\beta+\gamma,\ a'+\gamma+\beta;\ 1-\beta'+\beta;\ 1/w\}$$

+ a similar expression obtained by interchanging β and β'.

Hence, from § 23, since

$$a+a'+\beta+\beta'+\gamma+\gamma' = 1,$$

$$P_a e^{\mp\pi\iota a}\frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$

$$= \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma')} P_\beta + \frac{\Gamma(a+\beta'+\gamma)\,\Gamma(\beta-\beta')}{\Gamma(a+\beta+\gamma')} P_{\beta'}.$$

We thus have the given values of a_β and $a_{\beta'}$.

Similarly we have the relation

$$P_a = a_\gamma P_\gamma + a_{\gamma'} P_{\gamma'},$$

wherein

$$a_\gamma = \frac{\Gamma(1-a'+a)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}\, e^{\mp\pi\iota\gamma}$$

$$a_{\gamma'} = \frac{\Gamma(1-a'+a)\,\Gamma(\gamma-\gamma')}{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}\, e^{\mp\pi\iota\gamma}.$$

These values of a_γ, $a_{\gamma'}$ are not symmetrical with those just obtained for a_β, $a_{\beta'}$, and this may be expected *a priori*, since P_γ, $P_{\gamma'}$ have a cross-cut along CA, while P_β, $P_{\beta'}$ have a cross-cut along BC.

The new result is easily obtained. We have

$$P_a \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma')}{\Gamma(1-a'+a)}$$

$$= K_3 = -\frac{1}{2\pi\iota}\,w^{-\beta}\int \frac{\Gamma(a+\beta-s)}{\Gamma(1-a'-\beta+s)}\,\Gamma(\gamma+s)\,\Gamma(\gamma'+s)\,(-1/v)^s\,ds$$

$$= w^{-\beta}(-1/v)^{-\gamma}\,\frac{\Gamma(a+\beta+\gamma)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta'+\gamma')}$$

$$\times F\{a+\beta+\gamma,\ a'+\beta+\gamma;\ 1-\gamma'+\gamma;\ v\}$$

$+$a similar expression obtained by interchanging γ and γ'

$$= \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta'+\gamma')}\,P_\gamma e^{\mp\pi\iota\gamma} + \frac{\Gamma(a+\beta+\gamma')\,\Gamma(\gamma-\gamma')}{\Gamma(a+\beta'+\gamma)}\,e^{\mp\pi\iota\gamma'}P_{\gamma'}.$$

We thus have the given values of a_γ, $a_{\gamma'}$.

We can now write down by cyclical interchange the values of β_γ, $\beta_{\gamma'}$, ...

28. The preceding results verify Riemann's manuscript* relations between the ratios of the coefficients, and we can immediately obtain the relations given by him in his memoir.†

We have

$$\frac{a_\beta}{a'_\beta} = \frac{\Gamma(1-a'+a)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma')}\,e^{\pm\pi\iota a} \Big/ \frac{\Gamma(1-a+a')\,\Gamma(\beta'-\beta)}{\Gamma(a'+\beta'+\gamma)\,\Gamma(a'+\beta'+\gamma')}\,e^{\pm\pi\iota a'}.$$

Therefore $$\frac{a_\beta \sin\pi(a+\beta+\gamma')\,e^{\mp\pi\iota a}}{a'_\beta \sin\pi(a'+\beta+\gamma')\,e^{\mp\pi\iota a'}}$$

$$= \frac{\Gamma(1-a'+a)\,\Gamma(a'+\beta'+\gamma')\,\Gamma(a'+\beta+\gamma')}{\Gamma(1-a+a')\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')} = \frac{a_\gamma}{a'_\gamma}.$$

Hence $$\frac{a_\gamma}{a'_\gamma} = \frac{a_\beta \sin(a+\beta+\gamma')\pi\, e^{\mp a\pi\iota}}{a'_\beta \sin(a'+\beta+\gamma')\pi\, e^{\mp a'\pi\iota}} = \frac{a_{\beta'} \sin(a+\beta'+\gamma')\pi\, e^{\mp a\pi\iota}}{a'_{\beta'} \sin(a'+\beta'+\gamma')\pi\, e^{\mp a'\pi\iota}}.$$

* Note (1), p. 84, Riemann, *Œuvres Mathématiques* (Paris, 1898), or *Mathematische Werke* (1892), p. 86. These relations are found in several places among Riemann's manuscripts, but they are not in the first German edition of his collected works.

† *Loc. cit.*, p. 70.

This is equivalent to Riemann's result, for he takes the case when $a = 0$, $b = \infty$, $c = 1$, and $I(x)$ is positive, which corresponds to the case when x lies within the circle ABC in our more general investigation.

29. It is an obvious investigation to try to obtain for P_α an expansion near $x = a$ in ascending powers of $x-a$, our previous expansions having been in ascending powers of

$$w = \frac{x-a}{c-a}\frac{c-b}{x-b} \quad \text{or} \quad \frac{1}{v} = \frac{x-a}{b-a}\frac{b-c}{x-c}.$$

This investigation leads to hypergeometric functions of two variables of the type first introduced into analysis by Appell,* and is, in fact, equivalent to a theorem given by him.

We will briefly indicate a proof that when $|x-a| < |b-a|$ or $|c-a|$,

$$P_\alpha \left(\frac{a-b}{x-a}\frac{c-a}{c-b}\right)^\alpha \left(\frac{a-b}{x-b}\right)^\beta \left(\frac{a-c}{x-c}\right)^\gamma \frac{\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(m+\alpha+\beta'+\gamma)\,\Gamma(n+\alpha+\beta+\gamma')\,\Gamma(m+n+\alpha+\beta+\gamma)}{m!\, n!\, \Gamma(1-\alpha'+\alpha+m+n)}$$

$$\times \left(\frac{x-a}{c-a}\right)^m \left(\frac{x-a}{b-a}\right)^n,$$

We take for simplicity the case when x does not lie within or on the sides of the smaller angle whose vertex is A and whose sides are AB and AC produced indefinitely in the directions AB and AC respectively. In this case, as may be readily verified by reference to a figure,

$$u^{-s} = \left(\frac{x-c}{c-a}\frac{b-a}{x-b}\right)^s = \left(1-\frac{x-a}{c-a}\right)^{s-\gamma} \left(1-\frac{x-a}{b-a}\right)^{-s-\alpha-\beta} \left(\frac{x-c}{a-c}\right)^\gamma \left(\frac{x-b}{a-b}\right)^{\alpha+\beta},$$

where $\quad \left|\arg\frac{x-c}{a-c}\right| < \pi \quad$ and $\quad \left|\arg\frac{x-b}{a-b}\right| < \pi.$

Also $\quad \frac{x-c}{a-c} = 1+\frac{x-a}{a-c}, \qquad \frac{x-b}{a-b} = 1+\frac{x-a}{a-b},$

where $\quad \left|\arg\frac{x-a}{a-c}\right| < \pi, \quad$ and $\quad \left|\arg\frac{x-a}{a-b}\right| < \pi.$

* Appell, *Liouville* (1882), Sér. 3, T. VIII., pp. 173–216. Previous notes had appeared in the *Comptes Rendus*.

Therefore, by § 26 and the theorem quoted in § 16,

$$P_a \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}{\Gamma(1-a'+a)}\, w^{-a} \times \left(\frac{x-c}{a-c}\right)^{-\gamma}\left(\frac{x-b}{a-b}\right)^{-a-\beta}$$

$$= -\frac{1}{2\pi\iota}\int \Gamma(\gamma-s)\,\Gamma(\gamma'-s)\,\Gamma(a+\beta+s)\,\Gamma(a+\beta'+s) \times\left(1-\frac{x-a}{c-a}\right)^{s-\gamma}\left(1-\frac{x-a}{b-a}\right)^{-s-a-\beta} ds$$

$$= \left(-\frac{1}{2\pi\iota}\right)^3\int ds\int d\phi\int d\psi\,\Gamma(\gamma'-s)\,\Gamma(\gamma+\phi-s)\,\Gamma(-\phi)\,\Gamma(\psi+a+\beta+s) \times\Gamma(-\psi)\,\Gamma(a+\beta'+s)\left(\frac{x-a}{a-c}\right)^{\phi}\left(\frac{x-a}{a-b}\right)^{\psi}.$$

The order of integration may be inverted, and we get, by § 15,

$$\Gamma(a+\beta'+\gamma')\left(-\frac{1}{2\pi\iota}\right)^2\iint\Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(\psi+a+\beta+\gamma')\,\Gamma(\phi+a+\beta'+\gamma) \times\frac{\Gamma(\phi+\psi+a+\beta+\gamma)}{\Gamma(\phi+\psi+1-a'+a)}\left(\frac{x-a}{a-c}\right)^{\phi}\left(\frac{x-a}{a-b}\right)^{\psi} d\phi\, d\psi$$

$$= \Gamma(a+\beta'+\gamma')\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \times\frac{\Gamma(m+a+\beta'+\gamma)\,\Gamma(n+a+\beta+\gamma')\,\Gamma(m+n+a+\beta+\gamma)}{m!\; n!\;\Gamma(1-a'+a+m+n)}\left(\frac{x-a}{c-a}\right)^{m}\left(\frac{x-a}{b-a}\right)^{n}.$$

We thus have the given result if that value of $\left(\frac{a-b}{x-a}\frac{c-a}{c-b}\right)^a$ is taken which is equal to $w^{-a}\left(\frac{a-b}{x-b}\right)^a$, where w has a cross-cut along the arc AB, and $|\arg w| < \pi$. P_a is thus defined with respect to a cross-cut along the arc AB, and the result is therefore, by § 23, valid for all values of x for which the series is convergent.

Appell would denote the double series, with unity for its first term, by

$$F_1\left\{a+\beta+\gamma;\; a+\beta'+\gamma,\; a+\beta+\gamma';\; 1-a'+a;\; \frac{x-a}{c-a},\; \frac{x-a}{b-a}\right\}.$$

And the preceding result is equivalent to his theorem that

$$F\{a_1, a_2;\; \rho;\; \xi+\eta\} = (1-\eta)^{-a_1}F_1\left\{a_1;\; a_2,\; \rho-a_2;\; \rho;\; \frac{\xi}{1-\eta};\; \frac{-\eta}{1-\eta}\right\}.$$

The theory of the transformation of Appell's series can be developed entirely by the contour integrals introduced in the present paragraph.

PART III.

The Jacobian Elliptic Integrals K and K' as Functions of k^2.

30. It has been stated that the contour integrals introduced into the preceding theory are valid even when degenerate cases of the hypergeometric series arise which involve a logarithmic term. As an example we will take the important case when $a_1 = a_2 = \frac{1}{2}$, $\rho = 1$, when Kummer's equation becomes the differential equation for Jacobi's elliptic integrals K and K' considered as functions of $x = k^2$.

The equation has been often considered, among others by Fuchs* and Tannery.† But the theory is invariably complicated, and most writers have, explicitly or implicitly, confined themselves to the case when $I(x)$ is positive. Tannery's investigation, which is given in Forsyth's treatise,‡ is cumbrous, and the investigation given by Schlesinger§ is difficult to follow, and not altogether accurate. The fundamental relation

$$K(x) \mp \iota K'(x) = x^{-\frac{1}{2}} K(1/x),$$

is, in fact, difficult to obtain from the Jacobian elliptic integrals. There appears to be an error in Forsyth's formula,‖ and in his corresponding substitution for $\iota K'/K$ corresponding to $x = \infty$. The complete investigation, whether $I(x)$ is $\pm$, is given in § 36 of the present paper.

31. The differential equation of the quarter-periods of the Jacobian elliptic functions is

$$x(1-x)\frac{d^2y}{dx^2} + (1-2x)\frac{dy}{dx} - \tfrac{1}{4}y = 0.$$

It corresponds to Kummer's equation with $a_1 = a_2 = \frac{1}{2}$, $\rho = 1$.

From the general theory, or by direct substitution, we readily see that a solution is

$$-\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 (1-x)^s\, ds.$$

The integral is convergent if $|\arg(1-x)| < 2\pi$, if we assume as usual that the contour of integration is parallel to the imaginary axis with loops

* Fuchs, *Crelle*, T. LXXI. (1870), pp. 121–127.

† Tannery, *Annales de l'École Normale Supérieure*, Sér. 2, T. VIII. (1879), pp. 169–194.

‡ Forsyth, *Theory of Differential Equations*, Part III., Vol. IV. (1902), pp. 128–135.

§ Schlesinger, *Linearen Differential Gleichungen*, Bd. II., pp. 476–484.

‖ Forsyth, *Theory of Functions* (2nd edition, 1900), p. 731.

if necessary to ensure that the positive sequence of poles of the subject of integration lies to the right, and the negative sequence to the left of the contour. We uniquely prescribe the integral as a function of x by the condition $|\arg(1-x)| < \pi$, and then we say that it defines $2\pi K(x)$. *Thus $K(x)$ has a cross-cut along the real axis from $+1$ to $+\infty$.*

Since the equation is unaltered when we write $1-x$ for x, we see that similarly

$$-\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 x^s\,ds$$

is a solution. When $|\arg x| < \pi$, we denote this solution by $2\pi K'(x)$. *Thus $K'(x)$ has a cross-cut along the real axis from $-\infty$ to 0.*

32. These definitions at once lead to a number of relations between K and K' of arguments belonging to the set x, $1/x$, $1-x$, $1/(1-x)$, $x/(x-1)$, $(x-1)/x$.

We have at once

$$\left.\begin{aligned} K(x) &= K'(1-x), && \text{if } |\arg(1-x)| < \pi \\ K(1-x) &= K'(x), && \text{if } |\arg x| < \pi \end{aligned}\right\}. \tag{A}$$

Again, in the integral defining $K(x)$ write $-s-\frac{1}{2}$ for s. Then the direction of the contour as always being downwards, we have

$$2\pi K(x) = (1-x)^{-\frac{1}{2}}\left(-\frac{1}{2\pi\iota}\right)\int \{\Gamma(\tfrac{1}{2}+s)\,\Gamma(-s)\}^2\left(1-\frac{x}{x-1}\right)^s ds$$

or $\quad K(x) = (1-x)^{-\frac{1}{2}} K\left(\frac{x}{x-1}\right)$, $\quad$ if $|\arg(1-x)| < \pi$

therefore

$$K(1-x) = x^{-\frac{1}{2}} K\left(\frac{x-1}{x}\right), \qquad \text{if } |\arg x| < \pi \tag{B}$$

Applying the same process to the integral which defines $K'(x)$, we have

$$\left.\begin{aligned} K'(x) &= x^{-\frac{1}{2}} K'(1/x), && \text{if } |\arg x| < \pi \\ K'(1-x) &= (1-x)^{-\frac{1}{2}} K'\left(\frac{1}{1-x}\right), && \text{if } |\arg(1-x)| < \pi \end{aligned}\right\}. \tag{C}$$

From (A), we have

$$\left.\begin{aligned} K(1/x) &= K'\left(\frac{x-1}{x}\right), && \text{if } \left|\arg\frac{x-1}{x}\right| < \pi \\ K\left(\frac{x-1}{x}\right) &= K'(1/x), && \text{if } |\arg x| < \pi \end{aligned}\right\}. \tag{D}$$

Finally, from (B) and (C) by changing x into $1/x$,

$$\left.\begin{aligned} K(1/x) &= (1-1/x)^{-\frac{1}{2}} K\left(\frac{1}{1-x}\right), \quad \text{if} \quad |\arg(1-1/x)| < \pi \\ K'\left(\frac{x-1}{x}\right) &= (1-1/x)^{-\frac{1}{2}} K'\left(\frac{x}{x-1}\right), \quad \text{if} \quad |\arg(1-1/x)| < \pi \end{aligned}\right\}. \tag{E}$$

These relations may be summed up in the equalities

$$Y_1 = K(x) = K'(1-x) = (1-x)^{-\frac{1}{2}} K\left(\frac{x}{x-1}\right) = (1-x)^{-\frac{1}{2}} K'\left(\frac{1}{1-x}\right),$$

$$\begin{aligned} Y_3 = x^{-\frac{1}{2}} K(1/x) = x^{-\frac{1}{2}} K'\left(\frac{x-1}{x}\right) &= (x-1)^{-\frac{1}{2}} K\left(\frac{1}{1-x}\right) \\ &= (x-1)^{-\frac{1}{2}} K'\left(\frac{x}{x-1}\right), \end{aligned}$$

and $Y_2 = K(1-x) = K'(x) = x^{-\frac{1}{2}} K'\left(\frac{1}{x}\right) = x^{-\frac{1}{2}} K\left(\frac{x-1}{x}\right)$,

though when written in this form we have no indication of the system of cross-cuts by which our equalities are limited. Such limitations must be those which have just been specified.

33. The three functions Y_1, Y_2, Y_3 are evidently solutions of the differential equation. They include all functions that can be obtained from $K(x)$, $K'(x)$ by the fundamental homographic transformations. It remains to find the linear relation connecting these three quantities, and explicit expansions for $K(x)$, $K'(x)$, when $|x| < 1$.

In the first place, by § 16,

$$\begin{aligned} 2\pi K(x) &= -\frac{1}{2\pi\iota} \int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 (1-x)^s\, ds \\ &= -\frac{\pi}{2\pi\iota} \int \frac{\{\Gamma(\frac{1}{2}+s)\}^2\,\Gamma(-s)}{\Gamma(1+s)} (-x)^s\, ds, \end{aligned}$$

when $|\arg(-x)| < \pi$.

From the last integral we see, by applying Cauchy's theorem in the usual manner, that, when $|x| < 1$,

$$2K(x) = \sum_{n=0}^{\infty} \frac{\{\Gamma(\frac{1}{2}+n)\}^2}{\{n!\}^2} x^n$$

or

$$K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left\{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right\}^2 x^n.$$

This is the explicit expansion for $K(x)$.

34. Again,

$$2\pi K'(x) = -\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 x^s\, ds,$$

and when $|x| < 1$, we may take the contour to include the positive sequence of poles $0, 1, 2, \ldots, \infty$.

Now, when $s = n+\epsilon$, and ϵ is small,

$$\Gamma(-s) = \frac{\Gamma(1-\epsilon)}{(-\epsilon)(-\epsilon-1)\ldots(-\epsilon-n)}$$

$$= \frac{(-)^{n-1}}{n!\,\epsilon}\left\{1-\epsilon\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]+\ldots\right\}.$$

Hence the residue of $-\{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 x^s$ at $s = n+\epsilon$ is the coefficient of $1/\epsilon$ in the expansion in ascending powers of ϵ of

$$-\frac{1}{\epsilon^2(n!)^2}\left\{1-2\epsilon\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]+\ldots\right\}$$

$$\times\{\Gamma(\tfrac{1}{2}+n)\}^2\{1+2\epsilon\psi(\tfrac{1}{2}+n)+\ldots\}\, x^{n+\epsilon}$$

$$= -\left\{\frac{\Gamma(\tfrac{1}{2}+n)}{\Gamma(n+1)}\right\}^2 x^n\left\{\log x+2\psi(n+\tfrac{1}{2})-2\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]\right\}.$$

Now $$\psi(x+1) = \frac{1}{x}+\psi(x).$$

Also $$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\,\Gamma(x+\tfrac{1}{2}),$$

so that $$2\psi(2x) = 2\log 2+\psi(x)+\psi(x+\tfrac{1}{2});$$

and therefore $$\psi(1)-\psi(\tfrac{1}{2}) = 2\log 2.$$

Hence $$2\psi(n+\tfrac{1}{2})-2\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]$$

$$= 2\left[\psi(\tfrac{1}{2})-\psi(1)+\frac{2}{2n-1}+\frac{2}{2n-3}+\ldots+\frac{2}{1}-\frac{1}{1}-\frac{1}{2}-\ldots-\frac{1}{n}\right]$$

$$= 4\left[-\log 2+\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right].$$

Hence, when $|x| < 1$ and $|\arg x| < \pi$,

$$2\pi K'(x) = -\sum_{n=0}^{\infty}\left\{\frac{\Gamma(\tfrac{1}{2}+n)}{\Gamma(n+1)}\right\}^2 x^n\left\{\log x-4\log 2+4\left(\frac{1}{1}-\frac{1}{2}+\ldots-\frac{1}{2n}\right)\right\},$$

$$\text{or } K'(x) = -\frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right\}^2 x^n$$
$$\times\left[\log x - 4\log 2 + 4\left(\frac{1}{1}-\frac{1}{2}+\ldots-\frac{1}{2n}\right)\right].$$

This is the explicit expansion for $K'(x)$.

35. *We can now shew that**

$$Y_1 = Y_3 \pm \iota Y_2,$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

When $|x| > 1$, we may take the contour of the integral

$$2K(x) = -\frac{1}{2\pi\iota}\int\frac{\{\Gamma(\frac{1}{2}+s)\}^2\,\Gamma(-s)}{\Gamma(1+s)}(-x)^s\,ds$$

to include the sequence of negative poles of the subject of integration.

Now the residue of

$$\frac{\{\Gamma(\frac{1}{2}+s)\}^2\,\Gamma(-s)}{\Gamma(1+s)}(-x)^s$$

at $$s = -n-\tfrac{1}{2}+\epsilon$$

is the coefficient of $1/\epsilon$ in the expansion in ascending powers of ϵ of

$$\frac{\sin\pi(n+\frac{1}{2})\,\{\Gamma(n+\frac{1}{2})\}^2}{\pi\epsilon^2\,\{\Gamma(n+1)\}^2}\left[1-2\epsilon\,\{\psi(n+\tfrac{1}{2})-\psi(n+1)\}+\ldots\right](-x)^{-n-\frac{1}{2}+\epsilon};$$

and is therefore

$$\frac{(-)^n}{\pi}\left\{\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right\}^2(-x)^{-n-\frac{1}{2}}\left[\log(-x)+4\log 2-4\left(\frac{1}{1}-\frac{1}{2}+\ldots-\frac{1}{2n}\right)\right].$$

Therefore

$$2K(x) = (-x)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\left\{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right\}^2\frac{1}{x^n}$$
$$\times\left\{-\log\frac{1}{x}\mp\pi\iota+4\log 2-4\left(\frac{1}{1}-\frac{1}{2}-\ldots-\frac{1}{2n}\right)\right\}$$
$$= \mp\pi\iota(-x)^{-\frac{1}{2}}\frac{2}{\pi}K\left(\frac{1}{x}\right)+(-x)^{-\frac{1}{2}}\,2K'\left(\frac{1}{x}\right),$$

or $$K(x) = \mp\iota\,(-x)^{-\frac{1}{2}}K(1/x)+(-x)^{-\frac{1}{2}}K'(1/x),$$

when $|\arg(-x)| < \pi$, the upper or lower sign being taken as $I(x)$ is

* Cf. Tannery et Molk, *Fonctions Elliptiques*, T. III., p. 205.

positive or negative. Now

$$(-x)^{-\frac{1}{2}} = \pm \iota x^{-\frac{1}{2}}.$$

Hence
$$K(x) = x^{-\frac{1}{2}} K(1/x) \pm \iota x^{-\frac{1}{2}} K'(1/x),$$

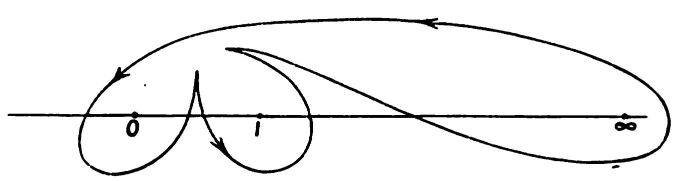
or
$$Y_1 = Y_3 \pm \iota Y_2.$$

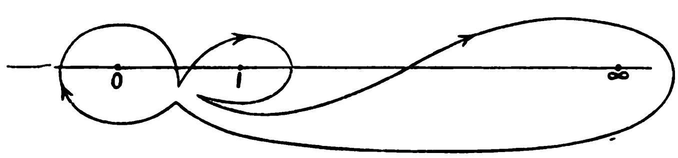
36. From the preceding we may at once obtain the fundamental substitutions which lead to the theory of elliptic modular functions.

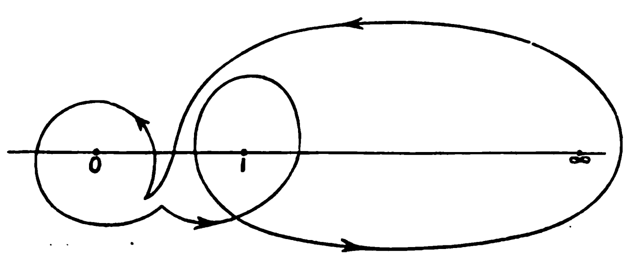
First we recall that near $x = 0$ $K(x)$ is uniform and $K'(x)$ has a cross-cut from $-\infty$ to 0; near $x = 1$ $K(x)$ has a cross-cut from $+1$ to $+\infty$, and $K'(x)$ is uniform, and near $x = \infty$ $K(x)$ has a cross-cut from $+\infty$ to $+1$, $K'(x)$ has a cross-cut from $-\infty$ to 0.

We wish to obtain for the three points such substitutions that the product of all three will be equivalent to a circuit round a point of no singularity—that is to say, to unity.

The first figure represents a possible combination of circuits when $I(x)$ is positive, the second figure when $I(x)$ is negative. The third figure shews that when $I(x)$ is negative, the positive circuits possible in the first case, when $I(x)$ is positive, are no longer available.







We therefore define a *possible* circuit as one which is positive or negative as $I(x)$ is positive or negative.

And now after a possible circuit round the origin

$$\begin{array}{lcl} K(x) & \text{becomes} & K(x), \\ K'(x) & ,, & K'(x) \mp 2\iota K(x), \end{array}$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

Putting $x = 1-\xi$, we see that a possible circuit round $x = 1$ is equivalent to the reverse description of a possible circuit round $\xi = 0$. Therefore after a possible circuit round $x = 1$,

$$\begin{array}{lcl} K(x) & \text{becomes} & K(x) \mp 2\iota K'(x), \\ K'(x) & ,, & K'(x). \end{array}$$

Putting $x = 1/t$, we see that a possible circuit round $x = \infty$ is equivalent to the reverse description of a possible circuit round $t = 0$. Hence, after a possible circuit round $x = \infty$,

$$K(x) = t^{\frac{1}{2}} K(t) \pm \iota t^{\frac{1}{2}} K'(t)$$

becomes $-t^{\frac{1}{2}} K(t) \mp \iota t^{\frac{1}{2}} \{K'(t) \mp 2\iota K(t)\}$

$$= -3t^{\frac{1}{2}} K(t) \mp \iota t^{\frac{1}{2}} K'(t) = -3K(x) \pm 2\iota K'(x),$$

and
$$K'(x) = t^{\frac{1}{2}} K'(t)$$

becomes
$$-t^{\frac{1}{2}} \{K'(t) \mp 2\iota K(t)\} = K'(x) \pm 2\iota K(x).$$

Hence, corresponding to possible circuits round 0, 1, ∞, the corresponding substitutions for $K(x)$, $\pm \iota K'(x)$ are

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix},$$

agreeing with Schlesinger.*

If $w = \pm \dfrac{\iota K'(x)}{K(x)}$ and $S_a(w)$ be the substitution corresponding to a possible circuit for w round the point a,

$$S_0(w) = w+2,$$

$$S_1(w) = \frac{w}{1-2w},$$

$$S_\infty(w) = \frac{w-2}{2w-3}.$$

---

* Schlesinger, *Linearen Differentialgleichungen*, Bd. II. (2), p. 46.

37. We may readily verify that the result of a possible circuit round 0, 1, ∞ in succession is a unit substitution.

For, if in $S_0 S_1$, S_0 operates after S_1 so that $S_0 S_1$ represents a circuit round 0 and 1 successively,

$$S_0 S_1 (w) = \frac{-3w+2}{1-2w},$$

$$S_0 S_1 S_\infty (w) = S_0 (w-2) = w.$$

The theory of elliptic modular functions can now be developed in the usual way.

ON THE APPLICATION OF QUATERNIONS TO THE PROBLEM OF THE INFINITESIMAL DEFORMATION OF A SURFACE

By J. E. CAMPBELL.

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VOLTERRA* has remarked that Weingarten's characteristic function in the deformation problem admits of a simple kinematical interpretation—viz., the normal component of the rotation which the element of the surface experiences in the deformation. This remark suggested the attempt to obtain the characteristic equation directly from the property, and thus I was led to extend the principle of moving axes dependent on two parameters† to that of moving axes dependent on three parameters.

Having obtained the equation I proceeded to apply the kinematical method to obtain the chief results in Bianchi, Kap. XI., and Darboux, Part IV., Ch. II. and III. The work occupied a considerable space, but I then saw that the application of the mere elements of quaternions would give what I wanted more directly, and add geometrical unity to the theory. This is the justification of the present paper, which does not pretend to add much to results already known, but aims rather at fuller kinematical illustration and greater simplicity of proof.

1. Consider a set of moving axes with a fixed origin whose motion is defined by the angular displacements

$$p'\,du+q'\,dv+r'\,dw, \qquad p''\,du+q''\,dv+r''\,dw, \qquad p'''\,du+q'''\,dv+r'''\,dw,$$

where the coefficients of the differential elements du, dv, dw are functions of the parameters u, v, w.

Let p, q, r respectively denote the vectors

$$p'i+p''j+p'''k, \qquad q'i+q''j+q'''k, \qquad r'i+r''j+r'''k,$$

* The article is "Sulla deformazione delle superficie flessibile ed inestendibili," *Rendiconti della Reale Accad. dei Lincei*, Sitzung von 6 April, 1884. I have not been able to consult the article itself and only quote from Lukat's translation of Bianchi, *Vorlesungen über Differential Geometrie*, p. 289.

† Darboux, *Théorie des Surfaces*, I., p. 49, and II., p. 348.

where i, j, k are unit vectors along the axes of coordinates and therefore mutually perpendicular.

Let $$z = z'i+z''j+z'''k,$$

where z', z'', z''' are the coordinates of a point with reference to the moving axes; and let $\bar{z}', \bar{z}'', \bar{z}'''$ be the coordinates of the same point with reference to a set of fixed axes, and let

$$\bar{z} = \bar{z}'i+\bar{z}''j+\bar{z}'''k.$$

Let the fixed axes be so chosen that the moving ones coincide with them when

$$u = u_0, \quad v = v_0, \quad w = w_0.$$

For these values of the parameters we know that

$$\frac{\partial \bar{z}'}{\partial u} = \frac{\partial z'}{\partial u} - z''p'''+z'''p'',$$

$$\frac{\partial \bar{z}''}{\partial u} = \frac{\partial z''}{\partial u} - z'''p' +z'p''',$$

$$\frac{\partial \bar{z}'''}{\partial u} = \frac{\partial z'''}{\partial u} - z'p'' +z''p'.$$

These three equations may be replaced by the single quaternion equation

$$\frac{\partial \bar{z}}{\partial u} = \frac{\partial z}{\partial u} + Vpz = z_1, \text{ say.}$$

Similarly we see that $$\frac{\partial \bar{z}}{\partial v} = \frac{\partial z}{\partial v} + Vqz = z_2,$$

$$\frac{\partial \bar{z}}{\partial w} = \frac{\partial z}{\partial w} + Vrz = z_3.$$

We also have $$\frac{\partial^2 \bar{z}}{\partial v\, \partial u} = \frac{\partial z_1}{\partial v} + Vqz_1 = z_{21},$$

$$\frac{\partial^2 \bar{z}}{\partial u\, \partial v} = \frac{\partial z_2}{\partial u} + Vpz_2 = z_{12}.$$

Since $z_{21} = z_{12}$, we have

$$\frac{\partial z_1}{\partial v} + Vqz_1 = \frac{\partial z_2}{\partial u} + Vpz_2;$$

and therefore $$\frac{\partial^2 z}{\partial v\,\partial u}+V\frac{\partial p}{\partial v}z+Vp\frac{\partial z}{\partial v}+Vq\frac{\partial z}{\partial u}+V(qVpz)$$

$$=\frac{\partial^2 z}{\partial u\,\partial v}+V\frac{\partial q}{\partial u}z+Vq\frac{\partial z}{\partial u}+Vp\frac{\partial z}{\partial v}+V(pVqz).$$

It follows that $$V\left(\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}\right)z+V(qVpz)-V(pVqz)=0;$$

but $$V(qVpz)-V(pVqz)=V(zVpq);$$

and therefore $$V\left(\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}-Vpq\right)z=0.$$

Since z may be *any* vector we must therefore have

$$\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}=Vpq.$$

Similarly we have

$$\frac{\partial q}{\partial w}-\frac{\partial r}{v}=Vqr,\qquad \frac{\partial r}{\partial u}-\frac{\partial p}{\partial w}=Vrp.$$

The vectors p, q, r which define the motion of the axes are thus connected by the above three equations.

One set of vectors satisfying these equations are p and q zero, and r a function of w only; that gives a motion of the axes depending on w only.

Another set would be obtained by taking $r=0$, and making p and q depend on the parameters u and v only in such a way that

$$\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}=Vpq.$$

This is the motion of the axes of which Darboux makes much use in considering the properties of a surface.

The first set of vectors would be sufficient for the investigations of this paper in connection with the deformation of a known surface. I have, however, preferred to keep to the most general motion defined by the three vectorial equations, as that will allow of the application of Codazzi's formulæ to the deformed surface, and may be of use in some further investigation.

2. If α is any vector by definition

$$\alpha_1=\frac{\partial\alpha}{\partial u}+Vp\alpha,\qquad \alpha_2=\frac{\partial\alpha}{\partial v}+Vq\alpha,$$

$$\alpha_{11}=\frac{\partial\alpha_1}{\partial u}+Vp\alpha_1,\qquad \alpha_{22}=\frac{\partial\alpha_2}{\partial v}+Vq\alpha_2,$$

$$\alpha_{12}=\alpha_{21}=\frac{\partial\alpha_2}{\partial u}+Vp\alpha_2=\frac{\partial\alpha_1}{\partial v}+Vq\alpha_1.$$

By aid of the formulæ

$$V(aV\beta\gamma)+V(\beta V\gamma a)+V(\gamma Va\beta)=0,$$

$$Sa\beta\gamma+S\beta a\gamma=0,$$

it easily follows that

$$\left.\begin{aligned}\frac{\partial}{\partial u}Va\beta+V(pVa\beta) &= Va_1\beta+Va\beta_1\\ \frac{\partial}{\partial u}Sa\beta &= Sa_1\beta+Sa\beta_1\end{aligned}\right\}. \tag{1}$$

If z is a vector which only depends on u and v we shall write

$$dz=z_1du+z_2dv,$$

$$\partial z=z_1\partial u+z_2\partial v,$$

and shall speak of dz and ∂z as elements of the z surface.

The square of the element of arc on the z surface is then

$$-(dz)^2=z_1^2du^2+2Sz_1z_2du\,dv+z_2^2dv^2.$$

The vector Vz_1z_2 is parallel to the normal. We do not, however, take this particular vector, but *any* parallel vector ρ in order to obtain the condition that two elements may be conjugate.

From the definition of conjugate elements the element ∂z is conjugate to dz if, and only if, ∂z is perpendicular to the normals at z and $z+dz$.

It follows that $V\rho\,d\rho$ is parallel to ∂z, and therefore

$$V(\rho\,d\rho)\,\partial z=\partial z\,V\rho\,d\rho.$$

From this, combined with $\quad\rho\,\partial z+\partial z\,\rho=0,$

we deduce $\quad\rho\,S\,d\rho\,\partial z=0;$

and therefore for conjugate elements

$$S\,d\rho\,\partial z=0.$$

This condition is, of course, equivalent with

$$S\,\partial\rho\,dz=0,$$

and if we take ρ to be of unit length, it expresses the known fact that any element on a surface is perpendicular to the element which corresponds in the spherical representation to the conjugate element.

For self conjugate elements, that is, for asymptotic lines,

$$S\,d\rho\,dz=0,$$

or, expanded,

$$S(\rho_1z_1)du^2+(S\rho_1z_2+S\rho_2z_1)du\,dv+S(\rho_2z_2)\,dv^2=0.$$

3. Let z be a vector which for a given value w_0 of the parameter w is known in terms of u and v, and let p and q also be known for the same value of w.

Let w_0 be changed into w_0+dw_0, and let

$$\zeta = (z_3)_{w=w_0} = \left(\frac{\partial z}{\partial w}\right)_{w=w_0} + Vrz_{w=w_0}.$$

The surface traced out by $z_{w=w_0}+\zeta\, dw_0$

will have for the square of its element of arc

$$-\left[\left(z_1+\frac{\partial z_1}{\partial w}\zeta\, dw_0\right) du + \left(z_2+\frac{\partial z_2}{\partial w}\zeta\, dw_0\right) dv\right]^2,$$

where w_0 is supposed to be substituted for w after differentiations have been carried out.

If, then,* $$\frac{\partial z_1}{\partial w} = \frac{\partial z_2}{\partial w} = 0,$$

the surface $z+\zeta dw_0$ will have the same element of arc as the surface traced out by z; that is, the surface will be infinitesimally deformed.

Now $$\frac{\partial z_1}{\partial w} = z_{31}+Vz_1 r = z_{13}+Vz_1 r = \zeta_1+Vz_1 r,$$

since $$z_3 = \zeta;$$

it follows that the equations which define the infinitesimal transformation are $$\zeta_1+Vz_1 r = 0, \quad \zeta_2+Vz_2 r = 0,$$

or $$d\zeta+Vdz\,r = 0.$$

In this equation z is a known function of u and v, and ζ, r are functions of u and v which depend on the solution of a partial differential equation of the second order whose form has now to be found.

4. Let $\bar{\zeta}$ bear the same relation to ζ that $\bar{z}$ bore to z in § 1, then, as we suppose the position of the moving axes known for all values of u and v when $w = w_0$, we know ζ in terms of u and v when we know $\bar{\zeta}$, and conversely.

Suppose now that r has been found, then since ζ_1 and ζ_2 are known in terms of u and v, we know $\partial\bar{\zeta}/\partial u$ and $\partial\bar{\zeta}/\partial v$ with reference to the fixed axes through which the moving ones are passing for any given values of u and v. We therefore know $\partial\bar{\zeta}/\partial u$ and $\partial\bar{\zeta}/\partial v$ with reference to any axes fixed once for all; and as $\partial\bar{\zeta}/\partial w = 0$, we can thus obtain $\bar{\zeta}$ by quadratures,

* It might seem as though these equations only define a particular class of deformations, but as the vector r is undetermined we can choose it so as to make one of these equations hold when the other must hold necessarily.

and therefore we can obtain ζ. The constant of integration which is thus introduced is immaterial to the real problem of deformation as it merely denotes an infinitesimal translation of the surface.

From the fundamental formulæ

$$\zeta_1 + Vz_1 r = 0, \qquad \zeta_2 + Vz_2 r = 0,$$

we deduce
$$\zeta_{12} + Vz_1 r_2 + Vz_{12} r = 0,$$

by aid of (1), and similarly, $\zeta_{21} + Vz_2 r_1 + Vz_{21} r$.

It follows that
$$Vz_1 r_2 = Vz_2 r_1 ;$$

and therefore, as the vectors z_1, z_2, r_1, r_2 are thus shown to be all parallel to the same plane, the surface traced out by the vector r—the r surface we shall call it—and the z surface have their normals parallel at corresponding points.

5. We have shown that the deformation is given when r is known, for then ζ is obtained by quadratures, and we shall now show how r depends on the solution of a partial differential equation of the second order.

Let λ be a unit vector drawn through the origin parallel to the normal (outwards) to the z surface. The λ surface will be the spherical representation of the z surface and also of the r surface.

Let
$$R + S\lambda r = 0,$$

so that R is the perpendicular from the origin on the r surface.

The z surface being known, λ is also known in terms of u and v, and, of course, λ_1 and λ_2 are known; we shall first express r, r_1 and r_2 in terms of R, λ, λ_1 and λ_2. From
$$R + S\lambda r = 0$$

we deduce, since
$$S\lambda r_1 = S\lambda r_2 = 0,$$

that
$$\frac{\partial R}{\partial u} + S\lambda_1 r = 0, \qquad \frac{\partial R}{\partial v} + S\lambda_2 r = 0.$$

It follows that

$$rS\lambda\lambda_1\lambda_2 + RV\lambda_1\lambda_2 + \frac{\partial R}{\partial u} V\lambda_2\lambda + \frac{\partial R}{\partial v} V\lambda\lambda_1 = 0.$$

Since
$$\lambda^2 = -1 \quad \text{and} \quad S\lambda\lambda_1 = S\lambda\lambda_2 = 0,$$

we have
$$\lambda S\lambda\lambda_1\lambda_2 + V\lambda_1\lambda_2 = 0;$$

and therefore
$$r = R\lambda + V\lambda\mu,$$

where $$\mu S\lambda\lambda_1\lambda_2 = \frac{\partial R}{\partial u}\lambda_2 - \frac{\partial R}{\partial v}\lambda_1;$$

thus r is expressed in terms of R and known vectors.

Again, $$\lambda\frac{\partial R}{\partial u} + V\lambda_1\mu = 0,$$

$$\lambda\frac{\partial R}{\partial v} + V\lambda_2\mu = 0;$$

and therefore, by (1), $$r_1 = R\lambda_1 + V\lambda\mu_1,$$

$$r_2 = R\lambda_2 + V\lambda\mu_2.$$

The function R is called the characteristic function because on it depends, as we have now proved, the infinitesimal deformation.

Before proceeding to obtain the differential equation which R satisfies, we shall deduce some formulæ required for that equation. From the expression for μ we deduce

$$S\lambda\lambda_1\mu_1 = S\lambda_{11}\lambda\mu + \frac{\partial^2 R}{\partial u^2},$$

$$S\lambda\lambda_2\mu_1 = S\lambda\lambda_1\mu_2 = S\lambda_{12}\lambda\mu + \frac{\partial^2 R}{\partial u\,\partial v},$$

$$S\lambda\lambda_2\mu_2 = S\lambda_{22}\lambda\mu + \frac{\partial^2 R}{\partial v^2};$$

and therefore $$S\lambda_1 r_1 = R\lambda_1^2 - \frac{\partial^2 R}{\partial u^2} - S\lambda_{11}\lambda\mu,$$

$$S\lambda_1 r_2 = S\lambda_2 r_1 = RS\lambda_1\lambda_2 - \frac{\partial^2 R}{\partial u\,\partial v} - S\lambda_{12}\lambda\mu,$$

$$S\lambda_2 r_2 = R\lambda_2^2 - \frac{\partial^2 R}{\partial v^2} - S\lambda_{22}\lambda\mu.$$

Similarly, if $$Z + S\lambda z = 0,$$

so that Z is the perpendicular on the z surface, and if

$$\nu S\lambda\lambda_1\lambda_2 = \frac{\partial Z}{\partial u}\lambda_2 - \frac{\partial Z}{\partial v}\lambda_1,$$

then $$z = Z\lambda + V\lambda\nu,$$

$$z_1 = Z\lambda_1 + V\lambda\nu_1,$$

$$z_2 = Z\lambda_2 + V\lambda\nu_2,$$

and
$$S\lambda_1 z_1 = Z\lambda_1^2 - \frac{\partial^2 Z}{\partial u^2} - S\lambda_{11}\lambda\nu,$$
$$S\lambda_1 z_2 = S\lambda_2 z_1 = ZS\lambda_1\lambda_2 - \frac{\partial^2 Z}{\partial u\,\partial v} - S\lambda_{12}\lambda\nu,$$
$$S\lambda_2 z_2 = Z\lambda_2^2 - \frac{\partial^2 Z}{\partial v^2} - S\lambda_{22}\lambda\nu.$$

We can now obtain the characteristic equation satisfied by R.

We saw that
$$Vr_1 z_2 = Vr_2 z_1;$$
and therefore
$$SV\lambda_1\lambda_2 . Vr_1 z_2 = SV\lambda_1\lambda_2 . Vr_2 z_1.$$
Applying the formula
$$SV\alpha\beta . V\gamma\delta = S\alpha\delta . S\beta\gamma - S\alpha\gamma . S\beta\delta,$$
and remembering that
$$S\lambda_1 z_2 = S\lambda_2 z_1 \quad \text{and} \quad S\lambda_1 r_2 = S\lambda_2 r_1,$$
we obtain the equation
$$S\lambda_1 z_1 . S\lambda_2 r_2 + S\lambda_2 z_2 . S\lambda_1 r_1 = 2S\lambda_1 z_2 . S\lambda_2 r_1.$$
On substituting for $S\lambda_1 z_1$, $S\lambda_1 r_1$, ..., the expressions just found, we have a differential equation of the second order in R which involves Z—a known function—symmetrically with R and also λ and its derivatives.

6. Darboux has shown that there are twelve surfaces, such that when the deformation problem is solved for one of the surfaces it is solved for all the others, and he has pointed out many interesting relations between these surfaces.

To express these surfaces in vector notation we consider the vector defined by
$$\rho = -\lambda/R,$$
and can now express them all in terms of
$$r, \quad \rho, \quad z, \quad \text{and} \quad \zeta.$$
From the definition of R we see that
$$Sr\rho = 1.$$
The vectors of the twelve surfaces are
$$z, \quad \rho, \quad r/Sr\zeta, \quad \zeta/Sz\zeta, \quad (z+V\zeta\rho)/Sz\rho, \quad \zeta+Vzr,$$
which we shall denote respectively by
$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6,$$

where the suffixes now have no longer any meaning of differentiation, but are merely a convenient notation ; and

$$\zeta, \quad r, \quad \rho/Sz\rho, \quad z/Sz\rho, \quad (\zeta+Vzr)/S\zeta r, \quad z+V\zeta\rho,$$

which will be denoted by

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6.$$

It will be found that by the transformation

$$z'=\zeta, \quad r'=\rho, \quad \zeta'=z, \quad \rho'=r,$$

the twelve vectors are merely transformed amongst themselves, each a into the corresponding α with the same suffix. Let this transformation be symbolized by A.

From the fundamental equation for the deformation of the z surface

$$d\zeta+Vdzr=0,$$

we at once obtain by aid of the formula

$$V\alpha V\beta\gamma = \gamma S\alpha\beta-\beta S\alpha\gamma,$$
$$Vd\zeta\rho = rS\rho dz-dzS\rho r;$$

but from the definition of ρ,

$$S\rho dz=0 \quad \text{and} \quad S\rho r=1;$$

and therefore $$dz+Vd\zeta\rho=0;$$

that is $$d\zeta'+Vdz'r'=0.$$

The transformation A therefore transforms z into ζ, a surface for which the deformation problem is also solved.

From $$d\zeta+Vdzr=0 \quad \text{and} \quad dz+Vd\zeta\rho=0,$$

we deduce $$d(\zeta+Vzr)+Vdrz=0,$$

Consider next the transformation B defined by

$$z'=r, \quad r'=z, \quad \zeta'=\zeta+Vzr, \quad \rho'=\rho/Sz\rho.$$

It will be found that this transforms a_i into α_{i+1} and α_{i+1} into a_i, where we make the conventions

$$a_{i+j}=a_k, \quad \alpha_{i+j}=\alpha_k, \quad \text{if} \quad i+j=k \pmod 6,$$

and leaves unaltered the fundamental equation

$$dz+Vd\zeta\rho=0.$$

By continued applications of the transformations A and B we thus obtain Darboux's twelve surfaces for each of which the deformation problem is solved when it is solved for one.

7. For the z surface ζdw_0 is the linear, and $r dw_0$ the angular displacement; we say then that ζ is the linear, and r the angular velocity. But ζ is the linear velocity, not of the origin of coordinates which is fixed, but of the extremity of the vector z.

These velocities, linear and angular, may therefore be replaced by the motion defined by a linear velocity $\zeta + Vzr$ of the origin and an angular velocity r at it.

The central axis of the velocity of displacement of the z surface has its direction parallel to the vector r, and it passes through the extremity of the vector

$$V(\zeta + Vzr)/r.$$

The angular velocity along this axis is the tensor of r, and the pitch of the screw-motion is

$$S(\zeta + Vzr)/r = S\zeta/r.$$

8. The geometrical relations between the twelve surfaces in the notation of this paper may be expressed as follows:—

The surfaces a_i and a_k

correspond orthogonally,	if $i+k = 2 \pmod 6$;
have their radii vectores and linear displacements parallel,	,, $= 5 \pmod 6$;
have their normals parallel,	,, $= 3 \pmod 6$;
are polar reciprocal,	,, $= 4 \pmod 6$;
have their central axes parallel and the pitch of either screw $= 1/Sa_{k+2}a_{i+2}$,	,, $= 1 \pmod 6$;
have their polar reciprocals corresponding orthogonally,	,, $= 0 \pmod 6$.

The asymptotic lines correspond on

$$a_1, \quad a_6, \quad a_4, \quad a_3,$$

and to these correspond conjugate lines with equal point invariants on

$$a_2, \quad a_4, \quad a_5, \quad a_1,$$

and conjugate lines with equal tangential invariants on

$$a_3, \quad a_5, \quad a_6, \quad a_2.$$

These geometrical relations may all be proved by the quaternion method of this paper. Thus suppose we wish to prove that the asymptotic lines

on the z surface correspond to conjugate lines with equal point invariants on the ρ surface.

Take the parametric lines to be asymptotic on the z surface, and let suffixes have their earlier meaning of differentiation. We have

$$S\rho_1 z_1 = S\rho_2 z_2 = 0,$$

since the coefficients of du^2 and dv^2 in the equation of the asymptotic lines must be zero, and from the definition of ρ,

$$S\rho z_1 = S\rho z_2 = 0.$$

It follows that $$z_1 = aV\rho\rho_1, \quad z_2 = bV\rho\rho_2,$$

where a and b are some scalars. From

$$z_1 + V\rho\zeta_1 = 0, \quad z_2 + V\rho\zeta_2 = 0,$$

we deduce $$\zeta_1 = a\rho_1 + c\rho, \quad \zeta_2 = b\rho_2 + d\rho,$$

where c and d are scalars. From

$$S\rho r = 1, \quad Sr\zeta_1 = Sr\zeta_2 = 0, \quad V\zeta_1\rho_2 = V\zeta_2\rho_1,$$

we deduce that c and d are zero and $a + b = 0$. The equations

$$\zeta_1 = a\rho_1, \quad \zeta_2 = -a\rho_2,$$

now give that the parametric lines are conjugate lines with equal point invariants both on the ρ and on the ζ surface.

A GENERAL THEOREM ON INTEGRAL FUNCTIONS OF FINITE ORDER

By J. E. LITTLEWOOD.

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1. In a memoir* published in 1903, Prof. Wiman put forward the following conjecture.

Let $F(z)$ be any integral function of order ρ less than $\frac{1}{2}$; then, if ϵ be an arbitrarily small positive number, there are values of r as large as we please, such that, for all points z of the circle $|z| = r$,

$$|F(z)| > \exp(r^{\rho-\epsilon}).$$

In the present paper I give a proof of an analogous theorem, expressing a relation between the maximum and the minimum modulus of $F(z)$ on the circle $|z| = r$. This theorem is given in § 4; the two articles now following are devoted to the proofs of certain results which are required in the proof of the main theorem.

2. LEMMA.—Let $\beta_1, \beta_2, \ldots$ be a sequence of real positive numbers such that for all values of s, $\beta_{s+1} \geqslant \beta_s$, and such that

$$\operatorname*{Lt}_{s=\infty} \beta_s s^{-2} = \infty.$$

We define the function $\beta(x)$, for values of x greater than 1, to be $\beta_s + (\beta_{s+1} - \beta_s)(x-s)$, when $s \leqslant x \leqslant s+1$. Then $\beta(x)$ is a continuous non-decreasing† function of x. It is easily seen, moreover, that

$$\operatorname*{Lt}_{x=\infty} \beta(x)\, x^{-2} = \infty. \tag{1}$$

Let us denote the curve $y = \beta(x)$ by C. This curve C starts at the point $[1, \beta(1)]$.

We shall show that, if any positive number h, however large, be

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* A. Wiman, *Arkiv. för Math. Astr. och Fysik*, Band I., 1903. "Ueber die angenäherte Darstellung von ganzen Funktionen."

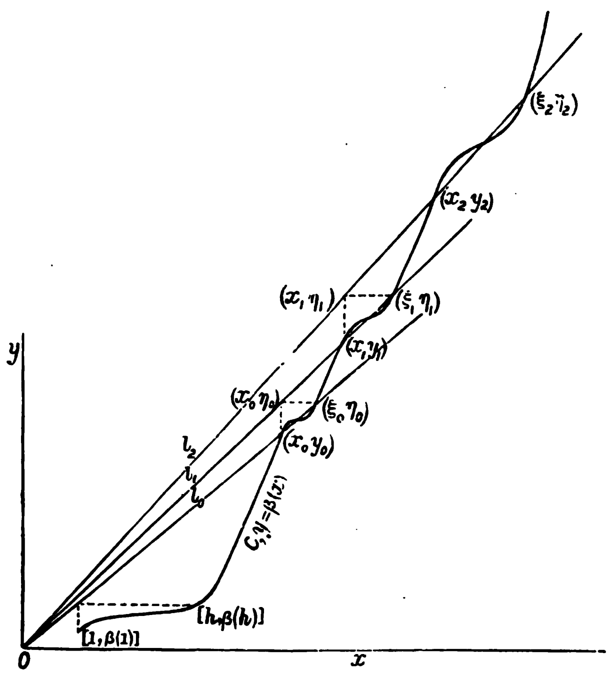
† By this phrase I mean that $\beta(x+y) \geqslant \beta(x)$, when $y > 0$.

assigned, there is a ray l through the origin lying in the first quadrant, such that if (X, Y) be that intersection of C and l which is nearest to the origin, the following state of things obtains.

(i.) $X > h$.

(ii.) When $x < X$, $Yx/X > \beta(x)$.

(iii.) When $x > X + \frac{1}{2}$, $Yx/X < \beta(x)$.



In the first place, if l, $y = mx$, be any ray* which passes above the point $[1, \beta(1)]$, there is at least one intersection of l and C. For the distant part of C is clearly above l [since $\underset{x=\infty}{\mathrm{Lt}}\, \beta(x)\, x^{-2} = \infty$, and consequently $\beta(x) > mx$ when x is sufficiently large], while the point $[1, \beta(1)]$ of C is below l.

Now let us suppose that, for a given h, there is no ray l such that (i.), (ii.), and (iii.) hold simultaneously. Then it is clear from the figure that the following must be the state of things.

Let l be any ray which passes above $[1, \beta(1)]$, and let (X, Y) be the

* I use the word "ray" throughout to mean a ray through the origin, lying in the first quadrant, and not coincident with the axis of y.

intersection of l and C which is nearest to the origin. Then, if $X > h$, there is a second intersection (X', Y') of l and C such that $X' > X+\frac{1}{2}$.

This result is evident when we observe that l is the ray through (X, Y), so that Yx/X is the ordinate of the point of l whose abscissa is x; that the part of the curve C for which $1 \leqslant x < X$ is below the corresponding part of l, while the distant part of C is above the corresponding part of l; and when we remember that if (i.) and (ii.) are true, on our present hypothesis (iii.) must be false.

We shall now show that the existence of this point $(X'Y')$, which we shall call the result (A), leads to a consequence incompatible with (1).

Let l_0 be the ray through $[1, \beta(h)]$. Since $\beta(x)$ is a non-decreasing function, it is evident from the figure that if $(x_0 y_0)$ be the intersection of l_0 and C which is nearest to the origin, we have $x_0 > h$.

Then, by the result (A), there must be a second intersection (ξ_0, η_0) of l_0 and C, such that $\xi_0 - x_0 > \frac{1}{2}$.

Let l_1 be the ray through $(x_0 \eta_0)$, and let $(x_1 y_1)$ be its intersection with C which is nearest to the origin.

It is again evident, from the figure and from the fact that $\beta(x)$ is a non-decreasing function, that $x_1 > \xi_0$.

Hence, by (A), there is a second intersection $(\xi_1 \eta_1)$ of l_1 and C, such that $\xi_1 - x_1 > \frac{1}{2}$.

Let l_2 be the ray through $(x_1 \eta_1)$, and let $(x_2 y_2)$ be its intersection with C nearest to the origin, and so on.

Then we have the following relations :—

$$\xi_n - x_n > \tfrac{1}{2}, \quad \xi_n > x_n > \xi_{n-1}, \tag{2}$$

and, since $(x_{n-1}\eta_{n-1})$, $(x_n y_n)$, $(\xi_n \eta_n)$ lie on the ray l_n,

$$\frac{\eta_{n-1}}{x_{n-1}} = \frac{y_n}{x_n} = \frac{\eta_n}{\xi_n}. \tag{3}$$

Hence we have

$$\begin{aligned}
\frac{\eta_n}{\xi_n} &= \frac{\eta_{n-1}}{\xi_{n-1}} \frac{\xi_{n-1}}{x_{n-1}} \\
&= \frac{\eta_{n-2}}{\xi_{n-2}} \frac{\xi_{n-2}}{x_{n-2}} \frac{\xi_{n-1}}{x_{n-1}} \\
&\dots \quad \dots \quad \dots \quad \dots \\
&= \frac{\eta_0}{\xi_0} \frac{\xi_0}{x_0} \cdots \frac{\xi_{n-2}}{x_{n-2}} \frac{\xi_{n-1}}{x_{n-1}} \\
&= \frac{\eta_0}{\xi_0 x_0} \frac{\xi_0}{x_1} \frac{\xi_1}{x_2} \cdots \frac{\xi_{n-2}}{x_{n-1}} \xi_{n-1} \\
&< \frac{\eta_0}{\xi_0 x_0} \xi_n, \quad \text{by means of (2).}
\end{aligned}$$

Thus $$\eta_n < [\eta_0/(\xi_0 x_0)]\, \xi_n^2. \tag{4}$$

Now $$\xi_n > x_n + \tfrac{1}{2} > \xi_{n-1} + \tfrac{1}{2} > \dots > \frac{n-1}{2},$$

so that by sufficiently increasing n we can make ξ_n as large as we please.

Again, x_0, ξ_0, η_0 are finite constants depending on h, and $(\xi_n \eta_n)$ is a point of the curve C.

But then the result (4) is incompatible with (1). Hence the result (A) is impossible, and the Lemma is proved.

3. We shall require the two following definite integrals:—

I. If $0 < \lambda < 1$,

$$\int_0^1 \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\,\frac{\lambda\,dx}{x} = 1/\lambda - \pi\cot(\pi\lambda).$$

II. If $0 < \lambda < 1$,

$$\int_0^1 \log(1+x)(x^\lambda + x^{-\lambda})\,\frac{\lambda\,dx}{x} = \pi\operatorname{cosec}(\pi\lambda) - 1/\lambda.$$

The integrals are easily seen to be convergent.

Consider the first integral. We have

$$I_1 = \int_0^1 \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\,\frac{\lambda\,dx}{x} = \int_0^{1-\theta} \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\,\frac{\lambda\,dx}{x} + \epsilon(\theta), \tag{1}$$

where θ is a small positive number, and where

$$\operatorname*{Lt}_{\theta=0} |\epsilon(\theta)| = 0.$$

The first term in the above expression

$$= \int_0^{1-\theta}\left[\sum_{s=1}^{\infty}\frac{x^{\lambda+s-1}}{s} + \sum_{s=1}^{\infty}\frac{x^{-\lambda+s-1}}{s}\right]\lambda\,dx$$

$$= \sum_{s=1}^{\infty}\int_0^{1-\theta}\frac{x^{\lambda+s-1}}{s}\lambda\,dx + \sum_{s=1}^{\infty}\int_0^{1-\theta}\frac{x^{-\lambda+s-1}}{s}\lambda\,dx$$

(since the infinite series are uniformly convergent when $x \leqslant 1-\theta$)

$$=\sum_{s=1}^{\infty}\frac{\lambda}{s(s+\lambda)}(1-\theta)^{\lambda+s}+\sum_{s=1}^{\infty}\frac{\lambda}{s(s-\lambda)}(1-\theta)^{-\lambda+s}.$$

Now $\Sigma\,\lambda/s(s\pm\lambda)$ is convergent.

Hence, by a well known theorem of Abel, the above expression

$$=\sum_{s=1}\frac{\lambda}{s(s+\lambda)}+\sum_{s=1}^{\infty}\frac{\lambda}{s(s-\lambda)}+\epsilon'(\theta), \tag{2}$$

where $$\operatorname*{Lt}_{\theta=0}|\epsilon'(\theta)|=0.$$

From (1) and (2) we see that we must have

$$\begin{aligned}I_1&=\sum_{s=1}^{\infty}\left[\frac{\lambda}{s(s+\lambda)}+\frac{\lambda}{s(s-\lambda)}\right]\\&=-\sum_{s=1}^{\infty}\left(\frac{1}{s+\lambda}+\frac{1}{\lambda-s}\right)\\&=\frac{1}{\lambda}-\pi\cot(\pi\lambda).\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}I_2&=\sum_{s=1}^{\infty}(-)^{s-1}\left[\frac{1}{s+\lambda}+\frac{1}{\lambda-s}\right]\\&=\pi\operatorname{cosec}\pi\lambda-\frac{1}{\lambda}.\end{aligned}$$

4. THEOREM.—*Let $F(z)$ be an integral function of order ρ, where $\tfrac{1}{2}>\rho\geqslant 0$. Let $M(r)$, $m(r)$ denote the maximum and the minimum moduli of $F(z)$ on the circle $|z|=r$.*

Then there is a sequence $r_1, r_2, \ldots,$ where $r_1<r_2<\ldots,$ and $\operatorname*{Lt}_{s=\infty} r_s=\infty$, with the following properties.

If ϵ be any assigned positive number there is a finite μ, such that when $s>\mu$, we have
$$m(r_s)>[M(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Moreover the sequence $r_1, r_2, \ldots$ is independent of the arguments of the zeros of $F(z)$, depending only on their moduli. Thus, if $F_1(z)$ be any other integral function of order ρ, the sequence of the moduli of whose zeros is the same as the corresponding sequence for $F(z)$, we have, when $s>$ a finite μ',
$$m_1(r_s)>[M_1(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Further, when $s >$ a finite μ'', we have

$$m(r_s) > [M_1(r_s)]^{\cos(2\pi\rho)-\epsilon}, \qquad m_1(r_s) > [M(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Suppose first that $$\tfrac{1}{2} > \rho > 0.$$

Let $$F(z) = Cz^p . \prod_{s=1}^{\infty} (1+z/a_s),$$

when $a_1, a_2, \ldots$ are arranged in order of non-decreasing moduli. Let $|a_s| = \alpha_s$. Let ϵ_1 be a (small) positive number less than $\frac{1}{2}-\rho$, which we shall presently choose suitably.

Let $$\beta_s = \alpha_s^{2(\rho+\epsilon_1)} = \alpha_s^{2\rho'}.$$

Since $F(z)$ is of order ρ, we have

$$\operatorname*{Lt}_{s=\infty} [\alpha_s s^{-1/(\rho+\epsilon_1)}] = \infty .$$

Hence $$\operatorname*{Lt}_{s=\infty} \beta_s s^{-2} = \infty .$$

Moreover β_s is clearly a non-decreasing function of s. We define the function $\beta(x)$ as in the Lemma, and then the result of the Lemma holds for this function.

Let $$\alpha(x) = [\beta(x)]^{1/2(\rho+\epsilon_1)} = [\beta(x)]^{1/2\rho'}.$$

Then $\alpha(s) = \alpha_s$, and $\alpha(x)$ is a non-decreasing function.

Now, let h be a (large) positive number to be chosen presently. Then by the Lemma there is a ray $y = \mu x$, such that, if (X, Y) be the intersection with $y = \beta(x)$ which is nearest to the origin, we have

$$\beta(x) < x\beta(X)/X, \quad \text{when } x < X;$$

and $$\beta(x) > x\beta(X)/X, \quad \text{when } x > X+\tfrac{1}{2}.$$

Choose $$r = |z| = [X''\beta(X)/X]^{1/2\rho'},$$

where $$X+\tfrac{1}{2} \geqslant X'' \geqslant X,$$

and where we shall determine X'' more precisely later.

Let m be the greatest integer not greater than X. Then we have

$$\log F(z) = \log|C| + p\log r + \sum_{s=1}^{\infty} \log\left|1+\frac{z}{a_s}\right|,$$

and $$\log F(z) = P+R+S+T_1+T_2+T_3+[\log|C|+p\log r],$$

where*
$$\left.\begin{aligned}
P &= \log\left|\frac{z^m}{a_1 a_2 \dots a_m}\right| = \log\left[\frac{r^m}{a_1 a_2 \dots a_m}\right]\\
R &= \sum_{s=3}^{\infty} \log\left|1+\frac{z}{a_{m+s}}\right|\\
S &= \sum_{s=1}^{m-1} \log\left|1+\frac{a_s}{z}\right|\\
T_1 &= \log\left|1+\frac{z}{a_{m+1}}\right|\\
T_2 &= \log\left|1+\frac{z}{a_{m+2}}\right|\\
T_3 &= \log\left|1+\frac{a_m}{z}\right|
\end{aligned}\right\} \tag{1}$$

We proceed to find a lower limit for P, and upper limits for R, $|R|$, S, $|S|$, ...

By the result of the Lemma, when $s \leqslant m < X$, we have
$$\beta_s = \beta(s) < \frac{s}{X}\beta(X).$$

Hence
$$a_s < \left(\frac{s}{X''}\right)^{1/(2\rho')}\left[\frac{X''}{X}\beta(X)\right]^{1/(2\rho')} < r\left(\frac{s}{X''}\right)^{\sigma'}, \tag{2}$$
where
$$\sigma' = 1/(2\rho').$$

Again, when $s \geqslant m+3 \geqslant X+\frac{1}{2}$, we have $\beta_s > s\beta(X)/X$, and hence
$$a_s > r\,[s/X'']^{\sigma'}. \tag{3}$$

First consider P. We have
$$\begin{aligned}
P &= \sum_{s=1}^{m} \log\left(\frac{r}{a_s}\right)\\
&> \sum_{s=1}^{m} \log\left(\frac{X''}{s}\right)^{\sigma'} \quad \text{[from (2)]}\\
&> \sigma'\left[m\log m - \log(m!)\right] \quad (\text{since } X'' > m)\\
&> \sigma'\left[m\log m - \{m\log m - m\,[(1+\epsilon(m)]\}\right]^{\dagger}\\
&> \sigma'.m\,[1+\epsilon(m)].
\end{aligned} \tag{4}$$

* The idea of dividing $\log F(z)$ into (practically) the three parts P, R, S, was employed by Wiman (*loc. cit.*) in the particular case $F(z) = \Pi(1+z/n^\rho)$.

† I shall always use $\epsilon(m)$, $\epsilon(x)$, ... for functions which tend to zero as their argument tends to its limit. I shall, moreover, use the same symbol for all the functions of this type with the same argument. The symbol $\epsilon(m)$ may be considered as an abbreviation for "some function which tends to zero with $1/m$."

Next consider R and S. When $s \geqslant 3$,

$$\left|\frac{z}{a_{m+s}}\right| = \frac{r}{a_{m+s}} < \left[\frac{X''}{m+s}\right]^{\sigma'} \quad \text{[from (3)]}$$

$$< \left[\frac{m+2}{m+s}\right]^{\sigma'}.$$

Hence
$$R = \sum_{s=3}^{\infty} \log \left|1+\frac{z}{a_{m+s}}\right|$$

$$\leqslant \sum_{s=3}^{\infty} \log \left[1+\left|\frac{z}{a_{m+s}}\right|\right] \text{ (algebraically), since } \left|\frac{z}{a_{m+s}}\right| < 1,$$

$$< \sum_{s=3}^{\infty} \log \left[1+\left(\frac{m+2}{m+s}\right)^{\sigma'}\right]$$

$$< \sum_{s=3}^{\infty} \left\{\int_{s-1}^{s} \log \left[1+\left(\frac{m+2}{m+x}\right)^{\sigma'}\right] dx\right\}$$

$$< \int_{2}^{\infty} \log \left[1+\left(\frac{m+2}{m+x}\right)^{\sigma'}\right] dx$$

$$< \int_{0}^{1} \log (1+t)\, \frac{m+2}{\sigma'}\, t^{-1-1/\sigma'}\, dt, \text{ on putting } \left(\frac{m+2}{m+x}\right)^{\sigma'} = t. \qquad (5)$$

Again, when $s \leqslant m-1$,

$$\left|\frac{a_s}{z}\right| = \frac{a_s}{r} < \left[\frac{s}{X''}\right]^{\sigma'} < \left[\frac{s}{m}\right]^{\sigma'}.$$

Hence
$$S = \sum_{s=1}^{m-1} \log \left[1+\frac{a_s}{z}\right]$$

$$\leqslant \sum_{s=1}^{m-1} \log \left[1+\frac{a_s}{r}\right] \quad \text{(algebraically)}$$

$$< \sum_{s=1}^{m-1} \log \left[1+\left(\frac{s}{m}\right)^{\sigma'}\right]$$

$$< \sum_{s=1}^{m-1} \left\{\int_{s}^{s+1} \log \left[1+\left(\frac{x}{m}\right)^{\sigma'}\right] dx\right\}$$

$$< \int_{0}^{m-1} \log \left[1+\left(\frac{x}{m}\right)^{\sigma'}\right] dx$$

$$< \int_{0}^{1} \log (1+t)\, \frac{m}{\sigma'}\, t^{-1+1/\sigma'}\, dt, \text{ on putting } \left(\frac{x}{m}\right)^{\sigma'} = t,$$

$$< \frac{m+2}{\sigma'} \int_{0}^{1} \log (1+t)\, t^{-1+1/\sigma'}\, dt. \qquad (6)$$

From (5) and (6) we have

$$R+S < (m+2)\int_0^1 \log(1+t)(t^{1/\sigma'}+t^{-1/\sigma'})\,\frac{1/\sigma'\,dt}{t}$$

$$< (m+2)\left(\pi\,\mathrm{cosec}\,\frac{\pi}{\sigma'}-\sigma'\right)^*, \tag{7}$$

by the second result of § 3, since $1/\sigma' < 1$.

Again, we have

$$R > \sum_{s=3}^{\infty}\left\{-\log\left(1-\frac{r}{a_{m+s}}\right)\right\} \quad \text{(algebraically)}$$

$$> \sum_{s=3}^{\infty}\left\{-\log\left[1-\left(\frac{m+2}{m+s}\right)^{\sigma'}\right]\right\}$$

$$> \sum_{s=3}^{\infty}\left\{\int_{s-1}^{s}\log\left[1-\left(\frac{m+2}{m+x}\right)^{\sigma'}\right]^{-1}dx\right\}$$

$$> -\int_0^1 \log(1-t)^{-1}\,\frac{m+2}{\sigma'}\,t^{-1/\sigma'-1}\,dt, \quad \text{on putting } \left(\frac{m+2}{m+x}\right)^{\sigma'} = t.$$

Similarly $$S > -\int_0^1 \log(1-t)^{-1}\,\frac{m}{\sigma'}\,t^{1/\sigma'-1}\,dt,$$

and hence $$R+S > -(m+2)\int_0^1 \log(1-t)^{-1}(t^{1/\sigma'}+t^{-1/\sigma'})\,\frac{1/\sigma'\,dt}{t}$$

$$> -(m+2)\left(\sigma'-\pi\cot\frac{\pi}{\sigma'}\right), \tag{8}$$

by the second result of § 3.

We shall now show that it is possible to choose X'' subject to the limitations so far imposed, and such that

$$[T_1+T_2+T_3+|\log|C||+p\log r]/P = \epsilon(m). \tag{9}$$

We have $$m \leqslant X < X+\tfrac{1}{2} < m+2. \tag{10}$$

Now, so far we have only restricted X'' to lie between X and $X+\frac{1}{2}$. Thus r, or $[X''\beta(X)/X]^{\sigma'}$, may vary between

$$[\beta(X)]^{\sigma'} \quad \text{and} \quad [1+1/(2X)]^{\sigma'}[\beta(X)]^{\sigma'}.$$

Moreover a_m and a_{m+2} do not lie between these limits, though a_{m+1} may possibly do so. These results follow at once from (10).

* The results (7) and (8) were suggested by the form of the asymptotic expansion of Barnes's function $P_p(s)$.

Hence it is possible to choose X'', within the limits already prescribed, such that

$$|r-a_{m+p}| > \tfrac{1}{2}\left\{[\beta(X)]^{\sigma'}\left(1+\frac{1}{2X}\right)^{\sigma'}-[\beta(X)]^{\sigma'}\right\} \quad (p=0,\,1,\,2)$$

$$> \frac{\sigma'}{4X}[\beta(X)]^{\sigma'} \quad (\text{since } \sigma' > 1). \tag{11}$$

Suppose that this is done. Then, since $|a_m/z| < 1$, we have

$$|T_3| = \left|\log\left|1+\frac{a_m}{z}\right|\right| \leqslant -\log\left(1-\frac{a_m}{r}\right)$$

$$\leqslant \log\left(\frac{r}{r-a_m}\right)$$

$$< \log\left[\left(\frac{X''}{X}\right)^{\sigma'}\frac{4X}{\sigma'}\right] \quad [\text{from (11), substituting for } r]$$

$$< \log\left[\frac{4(m+1)}{\sigma'}\left(1+\frac{1}{2m}\right)^{\sigma'}\right]$$

$$= m\epsilon(m).$$

Hence, from (4), $$|T_3|/P = \epsilon(m). \tag{12}$$

Next consider $$|T_1| = \left|\log\left|1+\frac{z}{a_{m+1}}\right|\right|.$$

(i.) Suppose $$\left|\frac{z}{a_{m+1}}\right| \leqslant \tfrac{1}{2}.$$

Then $$|T_1| \leqslant \log\left(\frac{1}{1-\frac{1}{2}}\right) \leqslant \log 2$$

$$= m\,.\,\epsilon(m).$$

(ii.) Suppose $$\tfrac{1}{2} < \left|\frac{z}{a_{m+1}}\right| < 1.$$

Then $$|T_1| \leqslant -\log\left(1-\frac{r}{a_{m+1}}\right)$$

$$\leqslant \log\left(\frac{a_{m+1}}{a_{m+1}-r}\right) < \log\left(\frac{2r}{a_{m+1}-r}\right)$$

$$< \log\left[2\left(\frac{X''}{X}\right)^{\sigma'}\frac{4X}{\sigma'}\right] \quad [\text{by (11)}]$$

$$= m\,.\,\epsilon(m),$$

as in the case of $|T_3|$. Hence in cases (i.) and (ii.),

$$|T_1|/P = \epsilon(m). \tag{13$_1$}$$

(iii.) The relation $|z/a_{m+1}| = 1$ is impossible on account of (11). Suppose, then, that $|z/a_{m+1}| > 1$. Then

$$|T_1| \leqslant \log\left|\frac{z}{a_{m+1}}\right| + \left|\log\left|1+\frac{a_{m+1}}{z}\right|\right|.$$

By means of (11) we prove, as in the case of $|T_3|$, that

$$\left|\log\left|1+\frac{a_{m+1}}{z}\right|\right| \Big/ P = \epsilon(m).$$

Also
$$\log\left|\frac{z}{a_{m+1}}\right| \Big/ P = \frac{\log\left|\dfrac{z}{a_{m+1}}\right|}{\sum\limits_{s=1}^{m} \log\left|\dfrac{z}{a_s}\right|} < \frac{1}{m} = \epsilon(m).$$

Hence
$$|T_1|/P = \epsilon(m), \tag{13$_2$}$$

and hence, in any case,
$$|T_1|/P = \epsilon(m). \tag{13}$$

Consider now
$$|T_2| = \log\left|1+\frac{z}{a_{m+2}}\right|.$$

(i.) If $\left|\dfrac{z}{a_{m+1}}\right| \leqslant \frac{1}{2}$,

$$|T_2| \leqslant \log 2 \quad \text{and} \quad |T_2|/P = \epsilon(m).$$

(ii.) If $\frac{1}{2} < \left|\dfrac{z}{a_{m+2}}\right| < 1$, then, as in the case (ii.) for $|T_1|$, we obtain, by means of (11),
$$|T_2|/P = \epsilon(m).$$

Hence, in any case,
$$|T_2|/P = \epsilon(m). \tag{14}$$

Finally, when r is large,

$$\begin{aligned} P = \log\left(\frac{r^m}{a_1 a_2 \dots a_m}\right) &> \log\left(\frac{r^{p/\epsilon}}{a_1 a_2 \dots a_{p/\epsilon}}\right) \\ &> \frac{p}{\epsilon}\log r - \log(a_1 a_2 \dots a_{p/\epsilon}) \\ &> \frac{p}{\epsilon}\log r\,[1+\epsilon(r)], \end{aligned}$$

however small ϵ may be.

Hence $$\{|\log|C||+p\log r\}/P=\epsilon(r)=\epsilon(m). \tag{15}$$

From (12), (13), (14), and (15) we obtain (9).

The first part of our theorem, for $\rho\neq 0$, now follows easily.

Denote by $[f(z)]_M$, $[f(z)]_m$, the maximum and the minimum algebraic values of the (real) expression $f(z)$, for all points z on the circle $|z|=r$. Then we have

$$\log M(r)=P+[R+S]_M+[T_1+T_2+T_3+\log|C|+p\log r]_M,$$

$$\log m(r)=P+[R+S]_m+[T_1+T_2+T_3+\log|C|+p\log r]_m.$$

From (7) and (8) we have

$$[R+S]_M<\quad(m+2)\left(\pi\operatorname{cosec}\frac{\pi}{\sigma'}-\sigma'\right),$$

$$[R+S]_m>-(m+2)\left(\sigma'-\pi\cot\frac{\pi}{\sigma'}\right).$$

Hence, by means of (10), noticing that $\left(\pi\operatorname{cosec}\frac{\pi}{\sigma'}-\sigma'\right)$ and $\left(\sigma'-\pi\cot\frac{\pi}{\sigma'}\right)$ are positive, we have

$$\frac{\log m(r)}{\log M(r)}>\frac{1-\dfrac{m+2}{P}\left(\sigma'-\pi\cot\dfrac{\pi}{\sigma'}\right)+\epsilon(m)}{1+\dfrac{m+2}{P}\left(\pi\operatorname{cosec}\dfrac{\pi}{\sigma'}-\sigma'\right)+\epsilon(m)}$$

$$>\frac{1-\dfrac{1}{\sigma'}\left(\sigma'-\pi\cot\dfrac{\pi}{\sigma'}\right)+\epsilon(m)}{1+\dfrac{1}{\sigma'}\left(\pi\operatorname{cosec}\dfrac{\pi}{\sigma'}-\sigma'\right)+\epsilon(m)}$$

[on replacing P by its minimum given by (4)]

$$>\cos\frac{\pi}{\sigma'}+\epsilon(m). \tag{16}$$

We can make m as large as we please, and consequently r as large as we please, by choosing h sufficiently large. Choose h so that m is so large that $\epsilon(m)<\epsilon_1$. Then

$$\frac{\log m(r)}{\log M(r)}>\cos\left(\frac{\pi}{\sigma'}\right)-\epsilon_1>\cos[2\pi(\rho+\epsilon_1)]-\epsilon_1.$$

Choose ϵ_1 so that

$$\cos[2\pi(\rho+\epsilon_1)]-\epsilon_1>\cos(2\pi\rho)-\epsilon.$$

(This can always be done, since $\rho < \frac{1}{2}$.) Then

$$\frac{\log m(r)}{\log M(r)} > \cos(2\pi\rho) - \epsilon,$$

and
$$m(r) > [M(r)]^{\cos(2\pi\rho)-\epsilon}. \tag{17}$$

We have therefore proved that when ϵ is assigned we can find a value of r, as large as we please, such that the above inequality holds.

Now, in the above analysis, the number r was determined quite independently of the arguments of $a_1, a_2, \ldots$, although it may depend on the constant C.

We may therefore determine our sequence $r_1, r_2, \ldots$ as follows.

Determine, by the above methods, a number r_n, such that

$$r_n > r_{n-1} + 1,$$

and such that (17) holds for $r = r_n$ when $\epsilon = 2^{-n}$.

Then $\quad \operatorname*{Lt}_{n=\infty} r_n = \infty \quad$ and $\quad m(r_s) > [M(r_s)]^{\cos(2\pi\rho)-\epsilon}$

when $s >$ some finite μ.

Again, P is independent of the arguments of z and of the zeros, while the limits which we found for $[R+S]_M$ and $[R+S]_m$ are independent of the arguments of the zeros. Moreover, the relation (9) always holds.

Then it follows, by considerations similar to those which gave us the result (16), that if

$$F_1(z) = Cz^p . \prod_{s=1}^{\infty} \left(1 + \frac{z}{a_s'}\right)$$

be a function with the same constant C, and with the same sequence of moduli of zeros as that of $F(z)$, then $\dfrac{\log m_1(r)}{\log M(r)}$ and $\dfrac{\log m(r)}{\log M_1(r)}$ are greater than $\cos\dfrac{\pi}{\sigma'} + \epsilon(m)$.

It follows, then, that

$$\left.\begin{aligned} m_1(r_s) &> [M(r_s)]^{\cos(2\pi\rho)-\epsilon} \\ m(r_s) &> [M_1(r_s)]^{\cos(2\pi\rho)-\epsilon} \end{aligned}\right\}, \tag{18}$$

when $s > \mu$.

Finally, if the C and the p which occur in the product-forms for $F_1(z)$ be replaced by C' and p', it is easily seen that the inequalities (18) still hold when $s >$ some finite μ', which is possibly different from μ.

Thus we have proved our theorem for the case when $\rho \neq 0$.

Although the above analysis breaks down when $\rho = 0$, it is easy to

see that the theorem remains valid for the limiting case, the main inequality taking the form*

$$m(r) > [M(r)]^{1-\epsilon}.$$

We have proved implicitly that the theorem stated at the beginning of the article holds when $F(z)$, instead of being of order ρ (ρ not zero), is of order less than ρ. Now a function of zero order is of order less than ϵ, and we can therefore determine a sequence of circles of radii $r_1, r_2, \ldots$, such that

$$m(r_s) > [M(r_s)]^{\cos(2\pi\epsilon)-\epsilon}, \quad \text{when} \quad s > \mu,$$

with the corresponding relations between $m_1(r_s)$, $M_1(r_s)$, ...

The theorem for the limiting case then follows immediately.

5. When $\rho < \frac{1}{4}$, we have $\cos 2\pi\rho > 0$.

It follows that, if $F(z)$ be an integral function of order less than $\frac{1}{4}$, then, on circles of radii as large as we please,

$$m(r) > [M(r)]^c,$$

where

$$c > 0.$$

Now, if any number p be assigned,

$$\underset{r=\infty}{\text{Lt}}\ r^{-p/c} M(r) = \infty.$$

Hence $r^{-p} m(r)$ has an upper limit infinity, and as z tends to infinity along any radius vector through the origin, $|z^{-p} F(z)|$ must have infinity for its upper limit.

6. The theorem of § 4 is only provisional. I am convinced that it remains true when the condition $0 \leqslant \rho < \frac{1}{2}$ is replaced by $0 \leqslant \rho < 1$, and when the index $\cos(2\pi\rho)$ is replaced by $\cos(\pi\rho)$.

But, although it would seem that the expression which I have called P must play a prominent part in the proof, I believe that to establish the result an entirely new point of view is required.

The line of proof of this paper is based essentially on the Lemma. It might be thought that by further refinement the Lemma might be extended into a form which should enable us to deal with the case $\frac{1}{2} \leqslant \rho < 1$. This, however, is not the case: if, with the notation of the statement of the Lemma, $\beta(x)$ tends to infinity like x^{2-k}, no more valuable result is true than that X' is of order X^{1+k}, and this result is clearly useless for our purpose.

* I have already given the theorem for this limiting case, *Proc. London Math. Soc.*, Ser. 2, Vol. 5, p. 365.

7.* The extended form of the theorem, if it be true, is chiefly interesting as showing that when $\rho < \frac{1}{2}$, $m(r) > [M(r)]^{c}$, where c is positive. For it is easy to deduce, by a well-known device, from the theorem established in § 4, a theorem applicable to all functions of finite order, and analogous to the result which we have obtained for the case $\frac{1}{4} < \rho < \frac{1}{2}$. This theorem is as follows.†

Let $F(z)$ be any integral function of finite (apparent) order ρ. Then there exists a sequence of circles of radii $r_1, r_2, \ldots$ tending to infinity, and depending only on the moduli of the zeros of $F(z)$, such that

$$m(r_s) > [M(r_s)]^{-c(\rho)},$$

where $c(\rho)$ is a constant depending only on ρ.

Let λ be the least integer greater than 2ρ, and let ω be a primitive λ-th root of unity.‡

Let $$F(z)\, F(\omega z) \ldots F(\omega^{\lambda-1} z) = G(z),$$

and let $$\zeta = z^{\lambda}, \qquad |\zeta| = t.$$

Now $G(z)$ is clearly an integral function of z^{λ} or ζ, $H(\zeta)$ suppose.

Since $$|H(\zeta)| < [M(r)]^{\lambda} < \exp(\lambda r^{\rho+\epsilon}) < \exp(\lambda t^{(\rho+\epsilon)/\lambda}),$$

for large values of r or t, $H(\zeta)$ is of order in ζ not greater than ρ/λ.§

Let $\mathcal{M}(t)$ be the maximum modulus of $H(\zeta)$ on the circle $|\zeta| = t$.

Since $\rho/\lambda < \frac{1}{2}$, there exists a sequence of circles $|\zeta| = t$, of radii $r_1^{\lambda}, r_2^{\lambda}, \ldots$, tending to infinity, and depending only on the *moduli* of the zeros of $H(\zeta)$, *i.e.*, depending only on the *moduli* of the zeros of $F(z)$, such that, on each circle

$$\begin{aligned} |H(\zeta)| &> [\mathcal{M}(r_s^{\lambda})]^{\cos(2\pi\rho/\lambda) - [1-\cos(2\pi\rho/\lambda)]} \\ &> [\mathcal{M}(r^{\lambda})]^{-1} \\ &> [\{M(r)\}^{\lambda}]^{-1}. \end{aligned}$$

In particular, if z_1 be the point of the circle $|z| = r_s$ for which $|F(z_1)| = m(r_s)$, we obtain

$$m(r_s)\,[M(r_s)]^{\lambda-1} \geqslant |G(z_1)| > [M(r_s)]^{-\lambda}.$$

* The remainder of the paper was added December 24th.

† I must here withdraw the statement which I put forward in a previous paper (*Proc. London Math. Soc.*, *loc. cit.*, § 5), that no theorem such as the above could exist.

‡ The method here employed is substantially that used in extending Hadamard's theorem that $m(r) > \exp(-r^{\rho+\epsilon})$, from the case when $\rho < 1$, to the case when ρ is general.

§ $H(\zeta)$ is actually of order ρ/λ, when ρ is not an integer.

Hence $m(r_s) > [M(r_s)]^{-(2\lambda-1)}$, and if we take

$$c(\rho) = 2\lambda - 1,$$

it is seen that the theorem is true.

8. It is interesting to notice that, if $F(z)$ and $F_1(z)$ be two integral functions of the same apparent order ρ, with the same sequence of moduli of zeros, it is *not* necessarily true that we can find a sequence of circles of radii tending to infinity, such that

$$m_1(r_s) > [M(r_s)]^{-c}.$$

We have proved that this inequality holds when $\rho < \frac{1}{2}$, and I believe that it holds when ρ is not an integer; but a simple example, due to M. Borel, shows that when, *e.g.*, $\rho = 1$, the inequality need not hold.

If $$F(z) = \sin \pi z, \qquad F_1(z) = z^{-1}[\Gamma(z)]^{-2},$$

we have

$$M(r) = \exp[\{1+\epsilon(r)\}\pi r], \qquad m_1(r) = \exp[-\{1+\epsilon(r)\}2r \log r],$$

and we cannot have $$m_1(r) > [M(r)]^{-c}.$$

NOTE ON WEIERSTRASS' E-FUNCTION IN THE CALCULUS OF VARIATIONS

By A. E. H. LOVE.

[Received December 14th, 1907.—Read December 12th, 1907.]

THE object of this Note is not to present rigorous formal proofs of theorems in the Calculus of Variations, but to simplify the customary methods of introducing the E-function by substituting for them a more intuitional method.*

The E-function has been developed chiefly in connection with integrals of the form

$$\int F(x, y, y')\, dx,$$

where y' denotes dy/dx, and it is understood that the integral is taken along a curve joining two fixed points. For many purposes it is convenient to use Weierstrass' method of parametric representation, in which the coordinates x, y are regarded as functions of a parameter θ; and then the integral takes the form

$$\int f(x, y, \dot{x}, \dot{y})\, d\theta,$$

where $\dot{x}$ and $\dot{y}$ denote $dx/d\theta$ and $dy/d\theta$, and the function f is homogeneous of the first degree, though not necessarily rational, in $\dot{x}$, $\dot{y}$. It is known that the integral cannot be a maximum or a minimum unless the curve along which it is taken is one of the curves determined by the differential equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

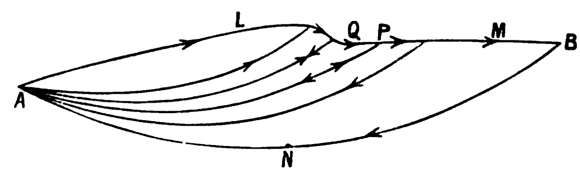
Such curves have been called "stationary" curves or "extremals" of the integral. The question arises—Is the integral taken along a stationary

* Reference may be made to the treatises by A. Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, and O. Bolza, *Lectures on the Calculus of Variations*, Chicago, 1904.

curve a maximum or a minimum? It is in connection with this question that the E-function is introduced.

Denote the fixed end points by A, B, the value of the integral $\int F dx$ taken along the stationary curve joining two points P, Q by (P, Q), using a round bracket, the value of the integral taken along any other assigned curve joining the same points by $[P, Q]$, using a square bracket. Let any assigned curve other than the stationary curve AB be drawn joining the fixed end points A, B. Then the variation of the integral is expressed by $[A, B]-(A, B)$. If this expression has the same sign for all curves which join the same two points, and lie in a sufficiently small neighbourhood of the stationary curve AB, the integral taken along that stationary curve is certainly a maximum or a minimum (a minimum if the sign is $+$, a maximum if it is $-$).

In the figure $ALQPMB$ represents the varied curve, and ANB the stationary curve joining A, B. The expression $[A, B]-(A, B)$, or the



variation of the integral, is equal to the line integral of $F dx$ taken round the closed contour $ALQPMBNA$. This line integral can be transformed into a line integral taken along the varied curve $ALQPMB$ only, and can then be written $\int E\, ds$, where ds denotes the element of arc of the varied curve. The function E is Weierstrass' E-function. Since the integral of this function yields the *excess* of the integral of $F dx$ taken along the varied curve above the integral of $F dx$ taken along the stationary curve, I have elsewhere called it the "excess-function."

The transformation of the line integral taken round the closed contour is effected by the familiar device of breaking up the included region into compartments, and replacing the line integral by a sum of line integrals taken round the boundaries of the compartments, the common boundary of two adjacent compartments being traversed twice in opposite senses, as is indicated by the arrows in the figure. The compartments are bounded by two stationary curves such as AQ, AP joining the point A to points Q, P on the varied curve, and by the intercepted arc QP of that curve. The line integral of $F dx$ taken round the boundary of one compartment is denoted by $(A, Q)+[Q, P]-(A, P)$, and the variation of the integral is denoted by

$$\Sigma\{(A, Q)+[Q, P]-(A, P)\},$$

where the summation extends to all the compartments in a series determined by a series of points, such as Q, P, taken in order on the arc of the varied curve. When QP is made very short the expression $(A, Q)+[Q, P]-(A, P)$ passes over into the product $E\,ds$, where ds denotes the differential element of arc of the curve $AQPB$, and E is a certain function, which depends upon the position of P, the direction of the tangent at P to the stationary curve AP, and the direction of the tangent at P to the varied curve $AQPB$. The line integral of $F\,dx$ taken round the contour $ALQPMBNA$ is therefore transformed into the integral of E taken along the varied curve from A to B. It remains to evaluate E.

Now the expression $(A, Q)-(A, P)$ may be evaluated simply by means of Lagrange's rule for the variation of an integral in the case of variable limits. Let ψ denote the angle of slope of the varied curve at P, in the sense from Q to P, and let ϕ denote the angle of slope of the stationary curve AP at P in the sense from A to P; also let δs denote the arc PQ, supposed very short. Then Lagrange's rule gives the result

$$(A, P)-(A, Q)$$
$$= \delta s \cos\psi \left\{ F(x, y, \tan\phi)+(\tan\psi-\tan\phi)\,\frac{\partial F(x, y, p)}{\partial p} \right\}_{p=\tan\phi},$$

which is correct to the first order in δs. Also we have, correctly to the same order,

$$[Q, P] = F(x, y, \tan\psi)\,\delta s \cos\psi,$$

and therefore, to the same order,

$$(A, Q)+[Q, P]-(A, P) = E\,\delta s,$$

where

$$E = \cos\psi$$
$$\times \left\{ F(x, y, \tan\psi)-F(x, y, \tan\phi)-(\tan\psi-\tan\phi)\,\frac{\partial F(x, y, p)}{\partial p} \right\}_{p=\tan\phi}.$$

It may be observed parenthetically that Lagrange's rule is proved very easily by the aid of the parametric method. In the integral

$$\int f(x, y, \dot{x}, \dot{y})\,d\theta,$$

let x be changed into $x+\epsilon u$, and y into $y+\epsilon v$, where ϵ is a constant, and u and v are functions of θ which, with their first differential coefficients, are continuous, and are independent of ϵ. The expressions ϵu, ϵv are equivalent to Lagrange's δx, δy. When the first variation of the

ON THE UNIFORM APPROACH OF A CONTINUOUS FUNCTION TO ITS LIMIT

By W. H. YOUNG, Sc.D., F.R.S.

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[illegible] I introduced the term "uniform divergence at a [illegible] incidentally that various theorems involving uniformity [illegible] function to its limit still held whether or no that limit [illegible]. In the paper in question I was concerned more [illegible] the distribution of particular points, and the behaviour [illegible] function in the neighbourhood of those points. In a large [illegible] we are concerned with the behaviour of the limiting [illegible] throughout an interval, and the question naturally forces itself [illegible] how we are to characterise uniformity of approach throughout [illegible] interval, when the sense of the words is the generalised one in [illegible]. In the present note I give the formulation in the case in which [illegible] functions are continuous (but not necessarily bounded). [illegible] illustration of the use of this formulation, I prove various theorems [illegible] up to the following generalisation of Weierstrass' theorem that [illegible] continuous (bounded) function can be expressed as the sum of a uniformly converging series of polynomials:—

THEOREM.—*An unbounded continuous function is expressible as the sum of a uniformly converging and diverging series of polynomials or rational fractions according as in the extension of the definition of continuity to unbounded functions, the two infinities* $+\infty$ *and* $-\infty$ *are regarded as distinct or not.*

The paper concludes with a formulation of the property of uniform continuity throughout an interval in the case of unbounded continuous functions.

Uniform Convergence and Divergence at a Point.

2. The definition of uniform divergence at a point given in my paper on the subject published lately in the *Proceedings of the London Mathematical Society* (p. 36) was as follows:—

The series of functions $f_1, f_2, \ldots$ is said to diverge uniformly at a point

P where it has no finite limit, if, given any quantity A, an interval d_P can be described, having P as internal point, so that for all points x within the interval d_P,

$$f_n(x) > A,$$

for all values of $n > m_P$, where m_P is an integer, independent of x, which can always be determined.

It is also said to diverge uniformly at P, if in this condition we alter the inequality to

$$f_n(x) < A.$$

In this definition the values $+\infty$ and $-\infty$ are distinguished, in accordance with the extended view of continuity adopted, where a function which is not finite is still regarded as continuous, if it is infinite with determinate sign at a point P, and is the only limit of values in the neighbourhood.

It is to be remarked that, just as in extending the idea of continuity to non-finite functions it is not necessary to distinguish the two infinities, so it is not necessary to do so in defining uniform divergence. The first point of view is equivalent to regarding the axis of y, where y is the dependent variable, as a segment with two end-points, the points $+\infty$ and $-\infty$. The second point of view is that of regarding the axis of y as a loop, without any end-point. The definition of uniform divergence when the two infinities are identified will only differ from the above in the two inequalities, which are replaced by the single inequality

$$|f_n(x)| > A.$$

In either case the limiting function will have a point of continuity at such a point of uniform divergence, *whether or no* $f_1, f_2, \ldots$ *are continuous at the point.* This indicates that the definition is open to objection except when $f_1, f_2, \ldots$ are continuous functions. We shall, however, confine our attention to this latter case, which is by far the most important one in practice.

3. The definition so given, though analogous to the recognised definition of uniform convergence at a point, labours under the disadvantage that the inequality employed is different in form according as there is convergence, or divergence, at the point considered. Moreover, the analogous definition of uniform convergence itself presents certain disadvantages, to obviate which it was shewn in the paper referred to that it might be replaced by another in the case when the functions $f_1, f_2, \ldots$ were *continuous.* This new definition had the advantage of being the

same in form whether the series converged, or diverged, at the point considered. It depended on the definition of the peak and chasm functions, uniform convergence, or divergence, taking place at any point where these are equal, and at such points only.

I propose, first of all, to transform the original definition of uniform convergence and divergence at a point in such a way that, without using the peak and chasm functions, its form is the same whether the series converges, or diverges, at the point considered. The new definition is as follows:—

Let $f_1, f_2, \ldots$ be a series of continuous functions which converges, or diverges to a definite limit $F(x)$ at every point of an interval. The series is said to approach uniformly to its limit at a point P of this interval if, corresponding to any segment on the y-axis containing the point*

$$y = F(P)$$

as internal point,† we can find an interval d_P containing the point P of the x-axis, and determine an integer m_P, such that the points

$$y = F(x) \quad \textit{and} \quad y = f_n(x)$$

lie for all values of $n \geqslant m_P$ inside the given segment, provided x lies inside the interval d_P.

It is evident that if this is the case, there is uniform convergence, or uniform divergence, at the point P, according as the point $y = F(P)$ is, or is not, at infinity. We have, in fact, in the former case, only to choose the segment of length $2e$ with the point $y = F(P)$ as middle point, and in the latter case to choose as segment all the part of the y-axis beyond the point $y = A$, on one side or the other, if the two infinities are distinguished, while, if the two infinities are identified, we have only to choose as segment all the part of the y-axis at a distance greater than A from the origin.

To shew conversely that when the series is uniformly convergent, or divergent, at the point P, this property holds, we proceed as follows. First, let P be such that the corresponding point of the y-axis,

$$p = F(P),$$

* In the generalised sense, $+\infty$ and $-\infty$ being distinguished, or identified.

† In the exceptional case when the point $y = F(P)$ is an end-point of the range of y (whether this range is finite, or infinite with a finite end-point, or the whole straight line with $+\infty$ distinguished from $-\infty$), $y = F(P)$ is to be included as an "internal point" in any segment having it as end-point. A similar remark applies to the range of x if this has one, or two, end-points.

is not at infinity. Let the distance of p from the nearest end-point of the given segment be $3e$. Then, since the series is uniformly convergent at P, we can find an interval d_P containing P as internal point, and determine an integer m_P, such that

$$| F(x) - f_n(x) | \leqslant e,$$

provided only the point x lies in the interval d_P, and $n \geqslant m_P$.

Also, since F is known to be continuous at P, we can so choose the interval d_P, that

$$| F(P) - F(x) | \leqslant e.$$

From these two inequalities it follows that

$$| F(P) - f_n(x) | \leqslant 2e,$$

or, in other words, the point $y = f_n(x)$

lies in the same segment as the point $p = F(P)$.

Secondly, let the point p be at infinity, and suppose first that the two infinities are identified. Then the point p is, as before, internal to the given segment.

Let that one of the two end-points of the segment which is nearest to the origin be denoted by

$$y = A.$$

Then, since there is uniform divergence at P, we can determine an interval d_P containing P as internal point, and an integer m_P, such that

$$|f_n(x)| > A,$$

provided the point x lies in the interval d_P and $n \geqslant m_P$. Thus the point

$$y = f_n(x)$$

lies in the segment containing the point p.

Further, since F is known to be continuous, we can secure that the interval d_P is such that

$$|F(x)| > A,$$

so that the point $y = F(x)$

also lies in the segment. We have therefore the same relation as when the point p was not at infinity.

The argument when the two infinities are distinguished is precisely analogous. For definiteness take

$$p = +\infty.$$

Then the given segment consists of all the points

$$y \geqslant A,$$

including the point p, which, though an end-point, is now to be regarded as tantamount to an internal point.

Then, since there is uniform divergence at P, we can determine an interval d_P, containing P as internal point, and an integer m_P, such that

$$f_n(x) > A,$$

provided the point x lies in the interval d_P and $n \geqslant m_P$. Thus the point

$$y = f_n(x)$$

lies in the given segment.

Further, since F is continuous, we can secure that this interval d_P is such that

$$F(x) > A,$$

so that the point

$$y = F(x)$$

also lies in the given segment.

4. *If a series of functions converges, or diverges, uniformly at every point of an interval, or of a set of points, it is said to converge uniformly throughout the interval, or set.*

It now follows that *the necessary and sufficient condition that a series of functions $f_1(x)$, $f_2(x)$, ... should approach uniformly to a limiting function throughout a closed interval, or set, is that however we divide up the range of the dependent variable $y = F(x)$ into a finite number of segments, we can find a corresponding division of the range of the independent variable x into a finite number of intervals, and a fixed integer m, such that, if the points x and x' belong to the same interval, n being any integer $\geqslant m$, the points*

$$y = F(x), \qquad y' = f_n(x')$$

lie in the same segment of the y-axis, or in the same two adjacent segments.

First this condition is necessary. For, taking any particular division of the y-axis, each point P of the range on the axis of x determines a segment on the axis of y, viz., that part, or that pair of adjacent parts, inside which the corresponding point $y = F(P) = p$ lies. This segment, provided the given series is uniformly convergent or divergent at P, determines, as explained in the preceding section, an interval d_P containing the point P, and an integer m_P. By the Heine-Borel theorem a finite number of these intervals suffice to cover every point of the range of x, provided that range is a closed interval or set. Let m be the greatest of the corresponding integers m_P, and let the intervals be $d'_1, d'_2, \ldots, d'_i$. Then, provided $n \geqslant m$,

and that the points x and x' both belong to one of these intervals, say d'_r, the points

$$y = F(x), \qquad y' = f_n(x'),$$

both lie in the same segment, or the same pair of adjacent segments, determined by the interval d'_r. These intervals $d'_1, d'_2, \ldots$ however, overlap; if we now replace d'_2 by the part of it not internal to d'_1, and each succeeding interval in turn by the part of it not internal to the preceding intervals, we get a finite number of non-overlapping intervals $d_1, d_2, \ldots, d_r$, each of which determines a segment, or a pair of adjacent segments, on the y-axis, inside which the points

$$y = F(x), \qquad y' = f_n(x')$$

both lie, whenever x and x' both lie in the corresponding interval d_1 and $n \geqslant m$. Thus the given condition is necessary.

It is, moreover, sufficient, for, supposing it true, however we divide the y-axis, then given any point P of the x-axis, this determines a point

$$p = F(P)$$

of the y-axis. Taking any segment containing p, let us make any convenient division of the y-axis in which that part which contains p, as well as the adjacent part, or parts, lie inside that segment. By hypothesis this division determines a finite number of non-overlapping intervals containing all the points x, and determines also an integer m. If P belongs to only one interval d, then, corresponding to the chosen segment of the y-axis, we have found a d and an m, such that, if x is any point belonging to d, and $n \geqslant m$, the points

$$y = F(x), \qquad y = f_n(x),$$

lie in the same part, or pair of parts, as p and $y = f_n(P)$, so that they lie inside the chosen segment of the y-axis. If P belongs to two adjacent intervals, then, taking together these two intervals, we get a d and, as before, an m. Thus in either case the criterion for uniform convergence at P, given in § 3, is satisfied. Thus every point P is a point of uniform convergence, so that the condition is not only necessary but sufficient.

5. THEOREM.—*Let $f_{i,1}, f_{i,2}, \ldots, f_{i,n}, \ldots$ be a series of continuous functions which approaches uniformly throughout an interval S to a limiting function f_i, for each integral value of i. Also, let $f_1, f_2, \ldots, f_i, \ldots$ approach to a limiting function F. Then we can find a series of the continuous functions $f_{i,n}$ which approach throughout the interval S to the limiting function F.*

Let $R_1, R_2, \ldots$ be a countable set of points dense everywhere on the

y-axis *e.g.*, the rational points. The first i points $R_1, R_2, \ldots, R_i$, determine uniquely a division of the y-axis into a finite number of segments i or $i+1$, in number according as we identify or distinguish $+\infty$ and $-\infty$. The characteristic of the i-th division, performed in this way, is that if R_j and R_k are the end-points of the same segment, there is no point R_n inside that segment, whose index $n \leqslant i$, *a fortiori*, whose index $n < j$ or $< k$. At the n-th and at all subsequent divisions such a point R_n will be itself the end-point of two adjacent segments, whose other end-points may at first be R_j and R_k, but will, if not from the first, certainly from and after some subsequent stage, always lie inside the segments (R_j, R_n) and (R_n, R_k) respectively.

Hence it follows that if a series of segments, one from each successive division, is given, say (R_1, R_1'), (R_2, R_2'), ..., in such a way that points $P_1, P_2, \ldots$, one from each segment, have only one limiting point P, then the same will be true of $R_1, R_2, \ldots$ and of $R_1', R_2', \ldots$; and therefore of any other set of points $Q_1, Q_2, \ldots$ lying in the same segments, or, indeed, by similar reasoning, in either of the adjacent segments at each stage.

This being premised, let us determine, corresponding to the i-th division, the integer m_i, such that, whatever x may be, the points

$$y = f_i(x), \qquad y = f_{i,n}(x),$$

always lie in the same segment, or in adjacent segments, provided $n \geqslant m_i$. This we can do, since the functions $f_{i,n}$ converge, or diverge, uniformly throughout the interval S. Then, since, by hypothesis, the points

$$P_i = y = f_i(x),$$

for fixed x, have the single limiting point

$$P = F(x),$$

it follows, from what has been pointed out, that the points

$$Q_i = f_{i, m_i}(x)$$

have the same single limiting point. Thus the series of continuous functions

$$f_{1, m_1}, f_{2, m_2}, f_{3, m_3}, \ldots$$

has at each point x the limit $F(x)$, which proves the theorem.

COR.—*If we know further that the series $f_1, f_2, \ldots$ approaches uniformly at the point P to its limit $F(P)$, then the series*

$$f_{1, m_1}, f_{2, m_2}, \ldots$$

approaches uniformly at P to the same limit.

We shall suppose, for convenience of wording, that P is not one of the points $R_1, R_2, \ldots$. The argument is, however, the same when this is not the case, only that the point p, or $y = F(P)$, determines then two adjacent segments, instead of a single segment.

Take any segment d containing the point p of the y-axis. Then we can determine the integer i so that that segment in which p lies at the i-th division by means of the points $R_1, R_2, \ldots, R_i$, together with the adjacent segment or segments, all lie inside the given segment d. Now, since the series $f_1, f_2, \ldots$ converges, or diverges, uniformly at P, we can, by § 3, find an interval d_P containing the point P of the x-axis, and determine an integer m_P greater than i, so that, if x is any point of d_P and k any integer $\geqslant m_P$, the points

$$y = F(x) \quad \text{and} \quad y = f_k(x)$$

lie in that segment of the i-th division in which the point p lies. Now the points

$$y = f_k(x) \quad \text{and} \quad y = f_{k, m_k}(x)$$

lie in the same segment at the k-th division, and therefore, since $k > i$, in the same segment at the i-th division. Thus the points

$$y = F(x) \quad \text{and} \quad y = f_{k, m_k}(x),$$

both lie in the given segment d; for the former lies in the same segment as p at the i-th division, and the latter in the same, or, if $y = f_k(x)$ is an end-point of this segment, in one of the adjacent segments, which, by our choice of i, all lie in the given segment d.

Thus the criterion for uniform convergence, or divergence, at P, as given in § 3, is satisfied, which proves the theorem.

6. We now proceed to the extension of Weierstrass' theorem.

LEMMA.—*If $F(x)$ is a continuous function which is always positive (or always negative) but not necessarily finite, then F is the limit of a series of bounded positive continuous functions approaching its limit uniformly throughout the interval considered.*

For, if n be any positive integer, the points x at which

$$F(x) \leqslant n,$$

form a closed set, including no infinity of $F(x)$. Thus the infinities (F being positive) are internal to the black intervals of this set, and, since the infinities form a closed set, to a finite number of those black in-

tervals. In each of the intervals so determined put

$$f_n(x) = n,$$

and at the remaining points $f_n(x) = F(x)$.

Then, since at the end-points of each of the intervals in question

$$F(x) = n,$$

and $F(x)$ is finite and continuous outside the intervals, $f_n(x)$ is a finite and continuous function.

Now, if $m < n$, the closed set

$$F(x) \leqslant n,$$

contains the closed set $F(x) \leqslant m$;

and therefore the black intervals of the former set lie inside those of the latter set. Thus, throughout the intervals in which, by definition,

$$f_m(x) = m,$$

we have for all values of $n > m$,

$$f_n(x) \geqslant m,$$

and, throughout the intervals complementary to the intervals just mentioned, we have

$$f_n(x) = f_m(x) = F(x).$$

This shews that at every infinity of $F(x)$, the series $f_1(x), f_2(x), \ldots$ diverges uniformly to $F(x)$, while at every other point it converges uniformly to $F(x)$, which proves the theorem.

THEOREM.—*Any function which, without being always finite, is continuous when $+\infty$ is distinguished from $-\infty$, is expressible as the limit of a series of polynomials, which approaches its limit uniformly for every value of x for which the function is defined.*

Let $F(x)$ be the function, and A any positive finite number. Then we define two new functions $U(x)$ and $V(x)$, as follows:—

$U(x) = F(x)+A$, wherever $F(x)$ is positive, and elsewhere $U(x) = A$.

$V(x) = F(x)-A$, wherever $F(x)$ is negative, and elsewhere $V(x) = -A$.

Then, at every point $F(x) = U(x)+V(x)$.

But, by the preceding Lemma, $U(x)$ is the limit of a series of bounded continuous functions $u_1(x), u_2(x), \ldots$ approaching its limit uniformly.

By the known theorem of Weierstrass, $u_i(x)$ is the limit of a uniformly

convergent series of polynomials. Since this is true for each value of i, we can apply the corollary to the theorem of § 5, and state that $U(x)$ itself is the limit of a suitably chosen series of the polynomials, say

$$u_1(x),\ u_2(x),\ \ldots,$$

approaching its limit uniformly.

Similarly $V(x)$ is the limit of a series of polynomials

$$v_1(x),\ v_2(x),\ \ldots,$$

approaching its limit uniformly.

Since $U(x)$ and $V(x)$ have no common infinities, their sum $F(x)$ is the limit of the sum of corresponding polynomials, say

$$f_i(x) = u_i(x) + v_i(x),$$

and the series $f_1(x), f_2(x), \ldots$ approaches uniformly to its limit $F(x)$, which proves the theorem.

7. Before proceeding to the second case we shall prove the following theorem :—

THEOREM.—*If $f_1, f_2, \ldots$ is a series of functions of x which approaches uniformly a limiting function $F(x)$, each function being continuous (but not necessarily finite) at any point P, and $g(x)$ any other function continuous also at P, then the series $g[f_1(x)], g[f_2(x)], \ldots$ approaches uniformly $g[F(x)]$ as limit at the point P.*

For, taking three axes corresponding to variables x, y, and z, and taking the point P of the x-axis, let us choose any segment on the z-axis containing the point

$$z = g(p),$$

where

$$p = F(P),$$

we can, since g is continuous, find a segment d_p of the y-axis, containing the point

$$y = p = F(P),$$

such that, whatever point y be taken in this interval d_p, the corresponding point

$$z = g(y)$$

of the z-axis lies inside the chosen segment.

But, since the series of continuous functions $f_1, f_2, \ldots$ approaches uniformly $F(x)$ as limit, we can, corresponding to the segment d_p of the y-axis, find an interval d_P of the x-axis, containing the point P, and determine an integer m_P, such that, for all points x inside the interval d_P, and for

all values of $n \geqslant m_P$, the points

$$y = f_n(x)$$

lie in the interval d_p of the y-axis, and therefore the points

$$z = g(y) = g[f_n(x)]$$

lie inside the chosen interval. But this is the condition for uniform convergence, or divergence, of the series $g[f_1(x)]$, $g[f_2(x)]$, ... at the point P of the x-axis to the limit $g[F(x)]$.

One of the most important applications of the preceding theorem consists in the process of inverting a given series. In other words, if the series

$$f_1(x),\ f_2(x),\ \ldots$$

converges, or diverges, uniformly at a point P, so does the series

$$\frac{1}{f_1(x)},\ \frac{1}{f_2(x)},\ \ldots.$$

This process was not allowable in dealing with uniformly convergent, but not divergent series, a point where the series had the limit zero leading to a point of divergence of the inverted series.

Moreover, it is only allowable if we adopt the definition of continuity and divergence which depends on the two infinities being identified.

8. Weierstrass' theorem requires modification in the case when the two infinities are identified. We have, in fact, the following theorem :—

THEOREM.—*A function which is continuous if, and only if, the two infinities are identified, cannot* be expressed as the limit of a series of bounded continuous functions, which converges, or diverges, uniformly, and this, whether or no the two infinities are identified in defining uniform divergence.*

For, if P be a point of uniform divergence, we can assign an interval d_P and an integer m_P, such that, for all values of $n \geqslant m_P$, and all points x of the interval d_P,

$$|f_n(x)| > A\,;$$

thus $f_n(x)$ never vanishes in the interval d_P, and therefore, being a continuous bounded function is throughout the interval d_P of one sign. Thus there is either one series of continually increasing integers n such that $f_n(x)$

* Except in the trivial case when it is always infinite and indeterminate as to sign, *e.g.*,

$$f_n(x) = (-)^n\, n \operatorname{cosec} x.$$

is always positive in the interval d_P, and another series always negative, or else any such series of integers always determine the same sign. In the latter case the infinity at P will have the same sign, and will be a point of continuity without identifying the two infinities. In the former case, however, at each point x of the interval d_P the one series of functions $f_n(x)$ will give rise to a limit which is positive, and the other to a limit which is negative. These two limits must, however, coincide, and are therefore both infinite at every point of the interval d_P. In this trivial case the limiting function is indeterminately infinite at every point of a closed interval, since, by the above, the end-points of an interval throughout which the function was indeterminately infinite could not be points of uniform divergence without the function being indeterminately infinite at these points also. Apart from this trivial case, the theorem is therefore true.

9. On the other hand, a function which is continuous if, and only if, the two infinities are identified, may be expressed as the limit of a uniformly converging, and diverging, series of rational functions.

To prove this we remark first, as in proving the Lemma, that the infinities of the function F lie in a finite number of the black intervals of the closed set of points at which

$$|F| \leqslant A.$$

Let these be (B_1, C_1), (B_2, C_2), (B_3, C_3), ..., (B_n, C_n), and let the whole interval considered be (B, C). Then in each of these partial intervals

$$F \neq 0,$$

at each of their end-points $F = A$ or $-A$,

while in the remaining partial intervals F is finite and continuous.

We now define n functions $f_1(x), f_2(x), \ldots, f_n(x)$, as follows:—

$$\begin{aligned} f_i(x) &= F(x) \text{ in the interval } (B_i, C_i) \\ &= F(B_i) \text{ in the interval } (B, B_i) \\ &= F(C_i) \text{ in the interval } (C_i, C). \end{aligned}$$

Then each of these functions $f_i(x)$ $(i = 1, 2, \ldots, n)$, is continuous and numerically never less than A; their reciprocals are therefore finite and continuous, so that, by Weierstrass' theorem, we may express each of these reciprocals as the limit of a polynomial,

$$\frac{1}{f_i(x)} = \operatorname*{Lt}_{r=\infty} P_{i,r}(x),$$

which converges uniformly throughout the interval (B, C) to its limit. Therefore (the two infinities being now identified)

$$f_i(x) = \frac{1}{\underset{r=\infty}{\mathrm{Lt}}\, P_{i,r}(x)} = \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{i,r}(x)},$$

the convergence, or divergence, of the rational function to its limit being uniform (§ 7).

Now, by their definition, no two of the functions $f_i(x)$ have an infinity at the same point; therefore their sum is, like each of them, continuous throughout the whole interval (B, C); in each of the intervals (B_i, C_i) it differs from $F(x)$ only by a constant, say K_i, and in each of the remaining intervals it is constant, the value in the interval (C_{i-1}, B_i) being, since the function is continuous,

$$F(B_i)+K_i = F(C_{i-1})+K_{i-1}, \text{ or say } K_i'.$$

Thus, if we define another function $f_{n+1}(x)$ in the following manner:—

$$\begin{aligned} f_{n+1}(x) &= F(x)-K_1' \text{ in the first interval } (B, B_1) \\ &= -K_1 \text{ in the second interval } (B_1, C_1) \\ &= F(x)-K_2' \text{ in the third interval } (C_1, B_2) \\ &= -K_2 \text{ in the next interval } (B_2, C_2), \end{aligned}$$

and so on, this function will be continuous throughout the whole interval (B, C), and will be finite, since $F(x)$ is finite and continuous in each of the intervals in which $f_{n+1}(x)$ is not constant. Hence, by Weierstrass' theorem, we may write

$$f_{n+1}(x) = \underset{r=\infty}{\mathrm{Lt}}\, P_{n+1,r}(x),$$

the convergence of the polynomial to its limit being uniform.

The sum of $f_{n+1}(x)$ to the sum of the n functions $f_i(x)$, will then be $F(x)$ at every point, thus

$$F(x) = \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{1,r}(x)} + \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{2,r}(x)} + \ldots + \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{n,r}(x)} + \underset{r=\infty}{\mathrm{Lt}}\, P_{n+1,r}(x).$$

Since, in this sum of limits, no two of the limits are infinite at the same point, the sum of the limits is the limit of the sum. Also, since each rational function converges, or diverges, uniformly throughout the whole interval (B, C), the same is true of the sum. Thus, finally,

$$F(x) = \underset{r=\infty}{\mathrm{Lt}} \left(\frac{1}{P_{1,r}(x)} + \frac{1}{P_{2,r}(x)} + \ldots + \frac{1}{P_{n,r}(x)} + P_{n+1,r}(x) \right).$$

the convergence, or divergence, being uniform, which proves the theorem.

10. We conclude by pointing out that the formulation of uniform approach to a limit throughout an interval given in § 4, gives us a corresponding formulation of the property of uniform continuity applicable to any continuous non-finite function. For continuity at a point P is neither more nor less than uniform convergence, or divergence, of $f(x+h)$ to $f(x)$ at the point P. The continuous variable h which approaches in any manner the limit 0, takes the place now of the discontinuous variable n approaching its limit $+\infty$.

Thus, if $f(x)$ is continuous at every point x of a finite closed interval, we can, corresponding to any given division of the y-axis into a finite number of segments, find a value of h, for which and all smaller values, the points

$$y = f(x) \quad \textit{and} \quad y = f(x+h),$$

lie in the same segment, or in the same pair of adjacent segments of the y-axis, this segment, or pair of segments, being determined only by the particular point x chosen.

We can, if we please, further modify the wording so as to permit of the point $x = \infty$ entering as an internal or end-point into the closed interval of the x-axis in which the function is continuous.

ON THE PROJECTIVE GEOMETRY OF SOME COVARIANTS OF A BINARY QUINTIC

By E. B. ELLIOTT.

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1. The representation of a binary quintic here attended to is that by five coplanar points in association with the unique conic which passes through all of them. The connectors of the points and any origin on the conic, and any projections of these connectors, have for Cartesian equations with regard to axes through their intersection the results of making zero linear transformations of the quintic. Covariants of the quintic are marked by sets of points on the conic, in projective association with the five points of such a character as to be symmetrical in its reference to the five—*i.e.*, as not to have reference only to the five arranged in some particular order or in an order chosen from a subgroup of possible orders. Conversely, a set of points on the conic, thus associated with the five, which can be linearly constructed or exactly specified as the intersections of a conic with conics or cubics themselves rationally specified by means of the five points, will have for connectors with the chosen origin on the conic sets of lines with equations rational in the coefficients of the quintic, and will mark covariants of the quintic. In linear constructions it will not be necessary to regard the conic as drawn, the five given points being all that pure geometry needs in order to obtain desired second intersections, poles, and polars.

It is to be remarked that the quintic stands alone among binary quantics in being exactly specified by its appropriate number 5 of points taken at will in a plane. The five points uniquely determine the conic which has to be taken with them. Four points do not; and to specify a quartic, we have to specify four marking points and a chosen conic through them. On the other side, six points are too many to be chosen at will and lie on a conic; so that to specify a sextic we must choose six marking points through which a conic passes.

It will often be convenient to take as standard case, from the facts in which the general statement of facts can be deduced by projection, that in which the conic has been projected into the parabola $x = y^2$. With this reference the quintic points have coordinates (t_1^2, t_1), (t_2^2, t_2), ..., (t_5^2, t_5);

and the quintic pencil may be taken as whichever is most convenient of the two:

$$\prod_1^5 (x-ty) = 0, \qquad \prod_1^5 (y-t) = 0,$$

which connect the points with the vertex and the point at infinity on the axis respectively.

Linear covariants of the quintic have special interest. What points on the conic have, taken singly, symmetrical projective relationship to the five points, and also rational specifiability by means of them? Any point whatever on the conic has the desired symmetrical relationship: for, by Pascal's theorem, we can pass from any point P on the conic to the same point again by the following linear construction, in which $ABCDE$ mean the quintic points arranged in any order whatever:—Let PA, CD meet in X, AB, DE in Y, and BC, XY in Z: then EZ passes through P. Accordingly there is a certain propriety in the statement that every point on the conic marks a linear covariant of the quintic: but the question of the geometrical specification of such points as mark rational linear covariants remains open.

One fact is at once clear: that, if we can construct three, we have the means of specifying geometrically an infinite number. For there will be a fourth having with the three, arranged in a definite order, any anharmonic ratio we like to assign as a number or an absolute invariant; and this fourth, like the three, regards the quintic points symmetrically. If the anharmonic ratio is that of the elements in order of any given range or pencil, the fourth can be linearly constructed.

Of course, when two only are constructed, we have the certification that an infinite number, algebraically specified, exist. Express the two as of the same degree in the coefficients by invariant factors; for instance, if (7, 1), (11, 1) are the two, take (11, 1) and (4, 0)(7, 1). The infinite system is, for that case,

$$(11, 1)+\lambda (4, 0)(7, 1)$$

for numerical values of λ. The marking points densely cover every arc of the conic, however small: indeed, if we allow irrational as well as rational values of λ, they cover the whole conic continuously.

To realize the importance of rational specifiability in general, consider a sextic instead of a quintic. Any point on the conic will have symmetrical projective relationship to sets of any five whatever of the six marking points, abstracting the sixth altogether, and so to the six points; *i.e.*, but for the requirement of rationality, it marks a linear covariant of the sextic. Now we know that a sextic has no rational linear covariants.

2. In any complete list of twenty-three irreducible concomitants of a quintic (1, 5) there have to figure three quadratic covariants (2, 2), (6, 2), (8, 2), and four linears (5, 1), (7, 1), (11, 1), (13, 1). Of the linears, the first two are unique, but two (11, 1)'s which differ by a numerical multiple of (4, 0)(7, 1) are equally allowable, and so are two (13, 1)'s which differ by an invariant multiple of (5, 1). By (m, n) is always meant a covariant of order n, with coefficients of degree m in the coefficients of (1, 5).

A problem which has long interested geometers is that of the construction of four points on the conic which severally mark (5, 1), (7, 1), an (11, 1), and a (13, 1).

The construction of (7, 1) and (5, 1) has been effected by Prof. Morley.* If $P_1P_2P_3P_4P_5$ are the points marking the quintic, he first shows with remarkable ingenuity that the connector of the two points on the conic—imaginary points if the P's are all real—which mark the unique (2, 2), may be obtained as follows. Construct (linearly, by use of two of the line-pair conics through $P_2P_3P_4P_5$) the point Q_1 which is conjugate to P_1 (any one of the five quintic points) with regard to all conics through the other four: then construct (also linearly) the polar of Q_1 with regard to the harmonic triangle of the quadrangle $P_2P_3P_4P_5$: this line and the tangent at P_1 intersect on the required connector, which is accordingly given by any two of five constructible points, the colinearity of which is an interesting geometrical fact. After this he, in effect, specifies two points which mark linear covariants of a given quintic and quadratic, obtaining a linear construction for them which is real in a case, such as the one on which he is going to fix attention, when the quintic points are real and the quadratic points imaginary on a real connector. The construction which I give below (§ 3) is based on his, but is perhaps easier to grasp. The marking points of the (7, 1) and the (5, 1) of the quintic $P_1P_2P_3P_4P_5$ he thus obtains as those linear covariant points of that quintic (1, 5) and its (2, 2) which are afforded by his construction.

In connexion with the first of Morley's succession of constructions, it is interesting to notice incidentally a fact as to a quartic. He shows that what he calls the conjugate polar of P_1 with regard to the quadrangle $P_2P_3P_4P_5$—*i.e.*, the polar of Q_1, found as above, with regard to the harmonic triangle of $P_2P_3P_4P_5$—is, wherever P_1 be, the polar of P_1 with regard to a certain conic associated with the quadrangle. This conic is the imaginary one with regard to which the pencils of four lines at the vertices of the harmonic triangle, in the figure of the complete quadrangle, reciprocate into the ranges of four points on the opposite sides respec-

* "A Construction by the Ruler only of a Point Covariant with Five given Points," *Math. Ann.*, Bd. XLIX., s. 496.

tively of that triangle in the figure. It can be shown that the four imaginary points in which that conic cuts the conic through $P_2P_3P_4P_5$ and a chosen origin are the points which mark for that origin, or any origin on its conic, the Hessian of the quartic marked by $P_2P_3P_4P_5$. A second (imaginary) quadrangle $P_2'P_3'P_4'P_5'$ on the conic has the same Hessian quadangle as $P_2P_3P_4P_5$, and is apolar with $P_2P_3P_4P_5$.

Morley anticipated that the next step towards the construction of an (11, 1) and a (13, 1) must be the construction of the marking points of the canonizant cubic (3, 3). It seems more practicable to look either for another quadratic covariant or for a quintic one, and, having found either, to apply the construction for linear covariants of a quintic and quadratic to the new quintic or quadratic and the old quadratic or quintic.

Two quadratics at once suggest themselves as ready at hand, viz., the quadratic (5, 1) (7, 1) itself, and the quadratic of common harmonic conjugates of this pair and (2, 2). Taken with (1, 5), however, they provide linear covariants which present themselves with high degrees in the coefficients, 25 and 37 in the one case and 29 and 43 in the other, which it is not easy to examine.

We shall see, however, that a quintic covariant of the needful simplicity is available.

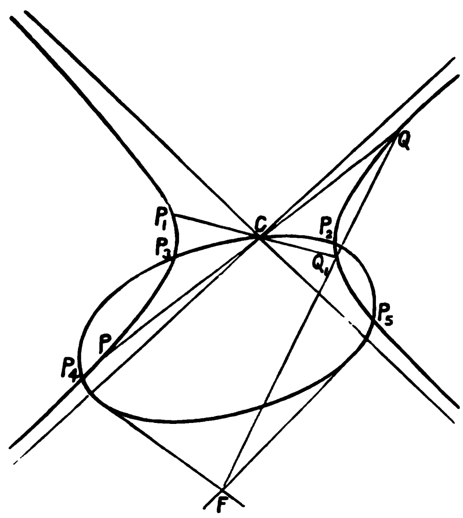
3. Before obtaining and applying this quintic, let us exhibit a construction, alternative to Morley's, for his two linear covariant points of a quintic marked by $P_1P_2P_3P_4P_5$ and a quadratic marked by AB on the conic through these points—on the P-conic, let us say. Take C the pole of AB for the conic. A conic passes through and is determined by $CP_2P_3P_4P_5$. If A, B are real, as well as the P's, the points D, E, where CA, CB meet this conic again, can be linearly constructed, as we know C and four other points on the conic; and so can F, the pole of DE, with regard to this conic. If, on the other hand, as happens in the cases of most importance, A, B are imaginary on a real connector with a real pole C for the P-conic, we can still find F by a real linear construction: for, through C we can linearly construct any number of pairs of conjugate lines with regard to the P-conic—two pairs suffice, *e.g.*, construct the conjugates of CP_2, CP_3 — and these meet the conic $(CP_2P_3P_4P_5)$ in pairs of an involution, also constructible; and the pole of this involution is F. Now, having F, in either case, take Q_1, where CP_1 meets again the P-conic, and let FQ_1 meet this conic again in Q. This point and P, where CQ meets the P-conic again, are the two covariant points required on that conic.

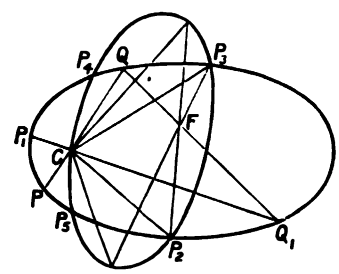
The directly obtained geometrical theorem is that the Q and P thus

obtained from P_1 and $(CP_2P_3P_4P_5)$ are equally obtained in the same way from P_2 and $(CP_1P_3P_4P_5)$, and from the other three separations of the five P's into one and four.

Notice the further geometrical conclusion involved in the identical character of the passage from P_1 to P with that from P to P_1. Not only is P the second covariant point of AB and $P_1P_2P_3P_4P_5$, but every one of the six points $PP_1P_2P_3P_4P_5$ is the second covariant point of AB and the quintic marked by the other five.

In the first of the two figures drawn, A, B are taken real and at infinity. In the second they are taken imaginary and nearly on a directrix of the P-conic.





To prove the construction, we project the tangent at B to infinity, and AB and the tangent at A into rectangular axes, so that xy is the quadratic and $x = y^2$ is the P-conic, the quintic being $\prod_1^5 (x - ty) = 0$, as in § 1. The collineations being always the same, the apparent treatment of the quadratic points as real is immaterial. The two linear covariants are

$$\frac{\partial^4}{\partial x^2 \partial y^2} \Pi (x - ty) \quad \text{and} \quad \Pi \left(t \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (xy)^3.$$

If the former is $x - \tau y$, the latter is $x + \tau y$; and the two are harmonic conjugates with regard to xy, *i.e.*, are reflections of one another in the axis. τ is the ratio of the sum of the products of t_1, t_2, t_3, t_4, t_5 three together to the sum of the products of them two together; and, if

s_1, s_2, s_3, s_4 denote the sums of the products of a chosen four of them, t_2, t_3, t_4, t_5, one, two, and three together, this fact may be written

$$s_3+(t_1-\tau)s_2-t_1\tau s_1 = 0,$$

an equality of which one interpretation is that the point $(s_3/s_1, s_2/s_1)$ lies on the connector of the points τ and $-t_1$ on the parabola. If then the point $(s_3/s_1, s_2/s_1)$ can be constructed, the connector of it with the point $-t_1$, *i.e.*, with the reflection of the point t_1, *i.e.*, of P_1, in the axis $y = 0$, will determine the first covariant point τ as its second intersection with the parabola. Reflection in the axis will then give the second covariant point $-\tau$.

Now the conic through t_2, t_3, t_4, t_5 on the parabola and C the point at infinity on the tangent at the origin, *i.e.*, the pole of the axis AB, is

$$x^2-s_1xy+s_2x-s_3y+s_4 = 0;$$

and the polar of $(s_3/s_1, s_2/s_1)$ with regard to this is

$$s_3x-s_1s_3y+s_4 = 0.$$

But the equation of the conic may be written

$$(s_3x-s_1s_3y+s_4)(s_1x+s_3) = (s_3^2+s_1^2s_4-s_1s_2s_3)x;$$

so that $s_3x-s_1s_3y+s_4 = 0$ is the line joining the points where CA $(x=0)$ and CB (the line at infinity) are cut by the conic, as well as at C.

Accordingly, the geometry generalized by projection at the outset of this article is justified.

Of the two linear covariants constructed, τ, *i.e.*, Q, is of degrees 1 in the coefficients of the quintic and 2 in those of the quadratic, while $-\tau$, *i.e.*, P, is of degrees 1 and 3. Applying the construction to a quintic (1, 5) and its covariant (2, 2), the Q obtained is then the unique (5, 1), and the P the unique (7, 1), of (1, 5).

4. With a view to further constructions it is desirable to look first for covariants which can be broken up into linear factors rational in the roots of our quintic (1, 5). The marking points of such covariants we may hope to be able to construct.

A linear factor of a covariant of order ϖ which has this property, and is not a product of other rational covariants, must have for the coefficient of x in it a function of the differences of t_1, t_2, ..., t_5 which is ϖ-valued for permutations of those letters.

Now no functions exist which are lower than 5-valued for permutations of five letters, except one-valued or symmetric functions. It is useless, then, to look for covariants of orders between 1 and 5 with the property in question.

One of order 6 will be introduced presently. For our immediate purpose, one of odd order is desired; and one of order 5 is at once obtained from the 5-valued function

$$(t_1-t_2)(t_1-t_3)(t_1-t_4)+(t_1-t_2)(t_1-t_3)(t_1-t_5)$$
$$+(t_1-t_2)(t_1-t_4)(t_1-t_5)+(t_1-t_3)(t_1-t_4)(t_1-t_5),$$

which is $$\frac{10}{a}(at_1^3+3bt_1^2+3ct_1+d),$$

if $(a, b, c, d, e, f)(x, y)^5$ is the quintic (1, 5).

The linear covariant of the five linear forms $t_1, t_2, \ldots, t_5$ which this leads, viz.,

$$\sum_{2345}(t_1-t_2)(t_1-t_3)(t_1-t_4)(x-t_5 y),$$

a factor of the quintic covariant of (1, 5) which we are investigating, is the linear polar of t_1 with regard to the other four. The product of the five such, being of order 5 with leading coefficient of weight 15, is of degree $\frac{1}{5}(2.15+5)=7$, and is accordingly

$$a^2\prod_1^5(at^3+3bt^2+3ct+d).$$

To identify it, let us find the terms free from c, d in its expression in terms of the coefficients. These are given by

$$a^2\prod_1^5(at^3+3bt^2)=f^2\prod_1^5(at+3b)=\frac{1}{a}f^2(a, b, 0, 0, e, f)(3b, -a)^5$$
$$=f^2(-a^4f+15a^3be-162b^5).$$

In terms, then, of the complete system of concomitants exhibited in my *Algebra of Quantics*, § 235, for the semi-canonical form

$$(a, b, 0, 0, e, f)(x, y)^5,$$

the covariant specified is

$$81(7, 5)-(4, 0)(3, 5)+22(2, 2)(5, 3).$$

It has just as much right to be taken as the irreducible covariant of degree 7 and order 5 in a complete system of irreducibles as has the more usual (7, 5) itself.

The marking points of the linear factors of the (7, 5) thus found can be easily constructed as follows. Construct the harmonic conjugate of the tangent at P_1 with regard to P_1P_2, P_1P_3, and also that with regard to P_1P_4, P_1P_5: then construct the harmonic conjugate of the same tan-

gent with regard to these two harmonic conjugates, and let it cut the conic in R_1. Similarly construct R_2, R_3, R_4, R_5 from P_2, P_3, P_4, P_5 and the other sets of five, taken in pairs in any way in each case. $R_1 R_2 R_3 R_4 R_5$ marks the covariant specified.

To prove this, project the conic into a parabola with P_1 at infinity, thus getting ∞, t_2', t_3', t_4', t_5' for t_1, t_2, t_3, t_4, t_5. The t_1 factor of the covariant becomes

$$4x-(t_2'+t_3'+t_4'+t_5')\,y = 0,$$

which cuts $x = y^2$ on $\quad y = \frac{1}{8}(t_2'+t_3'+t_4'+t_5'),$

i.e., on the parallel to the axis through the centroid of P_2', P_3', P_4', P_5'; and a construction for this has been given in projective form above.

R_1 is also the point of contact of the second tangent to the conic from the point Q_1, constructed as in § 1, which is conjugate to P_1 with regard to every conic through $P_2 P_3 P_4 P_5$.

5. By Morley's construction, or that of § 3, we can now find the two marking points Q', P' of two linear covariants of the (7, 5) which has been constructed and the (2, 2). The degrees of these linear covariants in the coefficients of (1, 5) will be $7+2.2 = 11$ for Q', and $7+3.2 = 13$ for P'. The two are harmonic conjugates with regard to (2, 2). We need to be sure that Q' and P' do not coincide with the P and Q before obtained, respectively: they cannot coincide respectively with Q and P, for coincidence would mean algebraical identity but for an invariant factor, and no invariant of degree 6 exists. When we have shown either that the (11, 1) marked by Q' is not merely (4, 0)(7, 1), or that the (13, 1) marked by P' is not merely an invariant multiple of (5, 1), the other fact will follow, and we shall know that Q, P, Q', P' mark four linear covariants which are entitled to places in a complete system of twenty-three irreducible concomitants of the quintic.

For the examination of such questions there is great convenience in the use of Hammond's* so called (a, b, c) canonical form of a quintic. This canonical form is the one arrived at when we apply such a linear transformation to the quintic as to reduce the canonizant (3, 3) of the quintic to the ordinary canonical form $k\,(x^3+y^3)$. As the canonizant and the quintic are apolar forms, the latter must assume such a form as to be annihilated by $(\partial/\partial y)^3-(\partial/\partial x)^3$. Whence $d = a$, $e = b$, $f = c$.

The forms in Hammond's complete system of concomitants (*loc. cit.*)

* *Proc. London Math. Soc.*, Vol. XXVII., p. 393.

are not in all cases quite the same as those of Cayley* and Salmon, or as those of the list referred to in my *Algebra of Quantics*; and it is necessary to have before us a partial table of equivalences in the three notations.

CAYLEY.	ELLIOTT.	HAMMOND.
(4, 0)	(4, 0)†	$-\ 9\,(4, 0)$
(8, 0)	(8, 0)	$-27\,(8, 0)$
(12, 0)	(12, 0)	$-27\,(12, 0)$
(18, 0)	$-(18, 0)$	$729\,(18, 0)$
(5, 1)	(5, 1)	$9\,(5, 1)$
(7, 1)	(7, 1)	$-27\,(7, 1)$
(11, 1)	$-(11, 1)$	$-81\,(11, 1)$
(13, 1)	$-6(13, 1)-2(8, 0)(5, 1)$	$486\,(13, 1)$
(2, 2)	(2, 2)	$-3\,(2, 2)$
(6, 2)	(6, 2)	$9\,(6, 2)$
(8, 2)	(8, 2)	$27\,(8, 2)$
(3, 3)	(3, 3)	(3, 3)
(5, 3)	(5, 3)	$9\,(5, 3)$
(3, 5)	(3, 5)	$-3\,(3, 5)$
(7, 5)	$(7, 5)-(2, 2)(5, 3)$	$-9\,(7, 5)+27\,(2, 2)(5, 3)$

Thus the (7, 5) constructed in the last article is, in Hammond's notation, after division by -27,

$$27\,(7, 5)+(4, 0)(3, 5)+22\,(2, 2)(5, 3).$$

For his canonical form

$$(1, 5) \equiv (a, b, c, a, b, c)(x, y)^5,$$

the expressions for those of Hammond's concomitants which we require are, writing a', b', c', k for $bc-a^2$, $ca-b^2$, $ab-c^2$, $3abc-a^3-b^3-c^3$ respectively,

* Salmon, *Higher Algebra*, 4th ed., p. 237. Cayley's *Collected Works*, Vol. II., p. 282.

† *Algebra of Quantics*, p. 309. On p. 307 the sign is different, and a coefficient has dropped out. Read there $(af-3be+2cd)^2-4\,(ae-4bd+3c^2)(bf-4ce+3d^2)$.

$$(4,\ 0) \equiv 4a'c'-b'^2,$$
$$(8,\ 0) \equiv k^2b',$$
$$(12,\ 0) \equiv k^4,$$
$$(18,\ 0) \equiv k^4(a'^3-c'^3),$$
$$(5,\ 1) \equiv k\,(a'x+c'y),$$
$$(7,\ 1) \equiv k\,\{-(2c'^2+a'b')\,x+(2a'^2+b'c')\,y\},$$
$$(11,\ 1) \equiv k^3(a'x-c'y),$$
$$(13,\ 1) \equiv k^3(c'^2x+a'^2y),$$
$$(2,\ 2) \equiv c'x^2-b'xy+a'y^2,$$
$$(6,\ 2) \equiv k^2xy,$$
$$(8,\ 2) \equiv k^2(c'x^2-a'y^2),$$
$$(3,\ 3) \equiv k\,(x^3+y^3),$$
$$(5,\ 3) \equiv k\,(b'x^3-2a'x^2y+2c'xy^2-b'y^3),$$
$$(3,\ 5) \equiv (2a'y-b'x)(a,\ b,\ c,\ a,\ b)(x,\ y)^4-(2c'x-b'y)(b,\ c,\ a,\ b,\ c)(x,\ y)^4,$$
$$(7,\ 5) \equiv k^2(ax^5+3bx^4y+2cx^3y^2-2ax^2y^3-3bxy^4-cy^5).$$

6. The Q', P' to the construction of which we have been led mark respectively the (11, 1) obtained by operating with $(2, 2)^2$ on, and the (13, 1) obtained by operating on $(2, 2)^3$ with, the quintic

$$27\,(7,\ 5)+(4,\ 0)(3,\ 5)+22\,(2,\ 2)(5,\ 3),$$

the operator in each case having $\partial/\partial y$, $-\partial/\partial x$ in it for x and y. We seek first the latter, P', by use of Hammond's canonical form.

After some tedious simple algebra we find that the coefficient of x in (7, 5) on $(2, 2)^3$ is $360k^3(2c'^2-a'b')$, while that of y is the result of interchanging a and c in this. Thus 27 (7, 5) on $(2, 2)^3$ is

$$27\,.\,360\,\{2(13,\ 1)-(8,\ 0)(5,\ 1)\}.$$

Again, (3, 5) on $(2, 2)^3$ is

$$(a,\ b,\ c,\ a,\ b)\left(\frac{\partial}{\partial y},\ -\frac{\partial}{\partial x}\right)^4\{-3\,(4,\ 0)\,x\,(c'x^2-b'xy+a'y^2)^2\}$$
$$+(b,\ c,\ a,\ b,\ c)\left(\frac{\partial}{\partial y},\ -\frac{\partial}{\partial x}\right)^4\{-3\,(4,\ 0)\,y\,(c'x^2-b'xy+a'y^2)^2\},$$

which can only be a numerical multiple of $(4, 0)(5, 1)$. To find what numerical multiple it suffices to compare the coefficients of any particular term. The coefficient of $(4, 0)\, a^5 x$ is that in

$$-3\,(4,\ 0)(a,\ 0,\ 0,\ a,\ 0)\left(\frac{\partial}{\partial y},\ -\frac{\partial}{\partial x}\right)^4 (a^4 x y^4),$$

i.e., it is $-3.4.3.2.1 = -72$, while in $(4, 0)\, k(a'x+c'y)$ it is $+1$. Thus $(4, 0)(3, 5)$ on $(2, 2)^3$ is $-72\,(4, 0)^2\,(5, 1)$.

Lastly, $(2, 2)$ on $(2, 2)^3$ is $9\,(4, 0)(2, 2)^2$; and the leading coefficient of $(5, 3)$ on $(2, 2)^2$ is that in

$$k\left\{b'\frac{\partial^3}{\partial y^3}+2a'\frac{\partial^3}{\partial x\,\partial y^2}+2c'\frac{\partial^3}{\partial x^2\partial y}+b'\frac{\partial^3}{\partial x^3}\right\}$$
$$\times\left\{c'^2x^4-2b'c'x^3y+(b'^2+2a'c')\,x^2y^2-2a'b'xy^3+a'^2y^4\right\},$$

i.e., it is $4ka'(4a'c'-b'^2)$, which leads $4\,(4, 0)(5, 1)$. Thus $22\,(2, 2)(5, 3)$ on $(2, 2)^3$ is $22.9.4\,(4, 0)^2\,(5, 1)$.

It follows that the constructed $(13, 1)$, P', is

$$27.360\,\{2(13,\ 1)-(8,\ 0)(5,\ 1)\}-72\,(4,\ 0)^2\,(5,\ 1)+22.36\,(4,\ 0)^2\,(5,\ 1),$$

i.e., after division by 360,

$$54\,(13,\ 1)-27\,(8,\ 0)(5,\ 1)+2\,(4,\ 0)^2\,(5,\ 1). \qquad (P')$$

The constructed $(11, 1)$, Q', has its expression obtained by writing down the harmonic conjugate of this with regard to $(2, 2)$. In the canonical notation used it is at once given by

$$\left[54k^3\left(c'^2\frac{\partial}{\partial y}-a'^2\frac{\partial}{\partial x}\right)-\{27k^2b'-2(4a'c'-b'^2)^2\}\,k\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)\right]$$
$$\times(c'x^2-b'xy+a'x^2),$$

in which the coefficient of x is

$$54k^3(-b'c'^2-2a'^2c')-27k^3b'(-a'b'-2c'^2)+2(4a'c'-b'^2)\,k(-a'b'-2c'^2),$$

i.e., $$-27\,(4a'c'-b'^2)\,k^3a'+2\,(4a'c'-b'^2)\,k\,(-a'b'-2c'^2),$$

from which the invariant $4a'c'-b'^2 = (4, 0)$ divides out, as it should, and the other factor is minus the leading coefficient in

$$27\,(11,\ 1)-2\,(4,\ 0)(7,\ 1), \qquad (Q')$$

which is accordingly the harmonic conjugate Q' required.

In Cayley's notation the expressions for P' and Q', affected by suitable numerical factors, are respectively

$$81\{(13, 1)+(8, 0)(5, 1)\}+2(4, 0)^2(5, 1),$$

and
$$81(11, 1)+2(4, 0)(7, 1).$$

7. There is a quite different procedure by which we can construct a (13, 1) of a given (1, 5). We are able to construct the linear polar of a linear form with regard to a given sextic. The following is an immediate method: another has been described by Mr. C. F. Russell (see reference below).

We want the polar one point of a given point A on a conic with regard to six given points $R_1, R_2, \ldots, R_6$ on that conic. Construct the pole G of the tangent at A with regard to the triangle $R_1R_2R_3$: this is merely a matter of joining points and finding harmonic conjugates. Also construct H the pole of the same tangent with regard to the triangle $R_4R_5R_6$. Then construct AR the harmonic conjugate of the same tangent with regard to AG and AH. The point R where this meets the conic again is the polar point required.

To prove this, consider the tangent at A to have been projected to infinity, so that, $x = y^2$ being the conic, we want the linear polar of $y = 0$ with regard to a given sextic pencil

$$\prod_1^6 (x-ty) = 0.$$

This is
$$\frac{\partial^5}{\partial x^5} \prod_1^6 (x-ty) = 0,$$

i.e., it is
$$6x-(t_1+t_2+\ldots+t_6)\, y = 0,$$

which meets $x = y^2$ on
$$6y = t_1+t_2+\ldots+t_6,$$

i.e., on the parallel to $y = 0$ through the centroid of $R_1, R_2, \ldots, R_6$, *i.e.*, on the parallel to the axis through the middle point of the connector of the centroids of $R_1R_2R_3$, $R_4R_5R_6$.

It may be remarked that always the construction of the linear polar of a linear form for a binary n-ic is obtained by expressing projectively a construction for the line in a given direction which passes through the centroid of n points.

Now we have ready for use a constructed sextic covariant of a given quintic $P_1P_2P_3P_4P_5$. I have shown* how to construct by points an im-

* "A Pascalian Theorem as to Pentagons," *Quarterly Journal*, Vol. XXXVIII., p. 265.

portant (6, 6), reducible in Hammond's notation as $9(1, 5)(5, 1)-25(3, 3)^2$, which is that of which the leading coefficient is the product of the roots of the sextic resolvent of the quintic equation. Let P_2P_5, P_3P_4 meet in Y_1, and Y_1P_1 cut the conic again in P_1': this can be found linearly in virtue of Pascal's theorem. Cyclically let P_3P_1, P_4P_5 meet in Y_2, and Y_2P_2 cut the conic again in P_2'. Further, let P_1P_2, $P_1'P_2'$ meet in Z, and ZP_4 cut the conic again in X_1. This and the five other points constructed in like manner, taking the P's in the cyclical arrangements 12453, 12534, 12543, 12435, 12354, are the marking points $X_1, X_2, \ldots, X_6$ of the (6, 6).

The linear polar of (5, 1) with regard to this (6, 6) can be constructed as above. We proceed to exhibit it as a (13, 1). What we need is the result of operating with $(5, 1)^5$ on

$$9(1, 5)(5, 1)-25(3, 3)^2.$$

Since (5, 1) as an operator annihilates (5, 1), the result of operating with $(5, 1)^5$ on (1, 5)(5, 1) is only an invariant multiple of (5, 1). This multiple is, in Hammond's canonical form,

$$k^5\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)^5(a,\ b,\ c,\ a,\ b,\ c)(x,\ y)^5,$$

i.e., it is 120 times

$$\begin{aligned}
&k^6(a'^5c-5a'^4c'b+10a'^3c'^2a-10a'^2c'^3c+5a'c'^4b-c'^5a)\\
&\quad\equiv k^4\{a'^2(a'^3-10c'^3)(a'b'-c'^2)-5a'c'(a'^3-c'^3)(a'c'-b^2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+c'^2(10a'^3-c'^3)(b'c'-a'^2)\}\\
&\quad\equiv k^4(a'^3-c'^3)\{a'^2(a'b'-c'^2)-5a'c'(a'c'-b'^2)+c'^2(b'c'-a'^2)-9a'^2c'^2\}\\
&\quad\equiv k^4(a'^3-c'^3)\{-k^2b'-(4a'c'-b'^2)^2\}\\
&\quad\equiv -(18, 0)\{(8, 0)+(4, 0)^2\}.
\end{aligned}$$

Again the result of operating with $(5, 1)^5$ on $(3, 3)^2$ is

$$\begin{aligned}
k^7\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)^5(x^6+2x^3y^3+y^6) &\equiv 720k^7(a'^3-c'^3)(c'^2x+a'^2y)\\
&\equiv 720(18, 0)(13, 1).
\end{aligned}$$

Rejecting then the invariant factor $-360(18, 0)$, the linear covariant which we have just constructed is

$$50(13, 1)+3\{(8, 0)+(4, 0)^2\}(5, 1). \qquad (P'')$$

The harmonic conjugate of this (13, 1) with regard to (2, 2) is not an (11, 1) but a (15, 1), namely, as ascertained by the method of § 6,

$$25\,(4,0)(11,1)-\{28\,(8,0)+3\,(4,0)^2\}\,(7,1). \qquad (Q'')$$

In fact every (11, 1) is included in

$$(11,1)+\lambda\,(4,0)(7,1),$$

for some numerical value of λ; and those (13, 1)'s which are harmonic conjugates of (11, 1)'s with regard to (2, 2) form the restricted system

$$2\,(13,1)-(8,0)(5,1)-\lambda\,(4,0)^2\,(5,1).$$

8. There is a way, independent of Morley's at its outset, by which linear covariants of a quintic can be constructed, which I have not followed out in detail, but to which I will now allude.

Mr. C. F. Russell* has indicated a finite succession of linear processes by which we can arrive at the point of a conic which accompanies $n-1$ given points in forming a system apolar with n other given points: in other words, he has shown that we can construct a linear covariant of an $(n-1)$-ic and an n-ic—one of partial degrees 1, 1 in the coefficients of the $(n-1)$-ic and n-ic. In particular a linear covariant of a quintic and sextic is thus given. There would be failure of the construction if the quintic and sextic were themselves apolar forms, but this case does not arise when, for instance, we take (1, 5) and my (6, 6) of the last article. Taking them we arrive at the linear (7, 1). Again, taking the (7, 5) of § 4 and the same (6, 6), we are led to the construction of a (13, 1).

To proceed from these to (5, 1) as a companion of (7, 1), and to a companion of the constructed (13, 1), which may prove to be an (11, 1) or a (15, 1), the natural course is to proceed with Morley and construct (2, 2), then obtaining the harmonic conjugates with regard to it of (7, 1) and the (13, 1).

However, the sextic made use of being $9\,(1,5)(5,1)-25\,(3,3)^2$, and (1, 5) and $(3,3)^2$ being readily seen to be apolar forms, Mr. Russell's point for the sextic and (1, 5) is also his point for (1, 5)(5, 1) and (1, 5). Thus, algebraically, (7, 1) is obtained by operating with (1, 5) on (1, 5)(5, 1). It is also true that (5, 1), multiplied by an invariant, is given by operating with (1, 5) on (1, 5)(7, 1); and, in fact, using Hammond's canonical form, it is easy to see that the result of operating with (1, 5) on the product of

* "On the Geometrical Interpretation of Apolar Binary Forms," *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 342.

(1, 5) and any linear form is the harmonic conjugate of that form with regard to (2, 2). Assuming, then, that in Russell's sequence of constructions no indeterminateness presents itself when his five points are also five of his six points, or, as will no doubt be the case, that a determinate and simplified sequence of constructions will be applicable under such circumstances, we have a means, without the construction of (2, 2), for obtaining the (5, 1) point when the (7, 1) point is known, and the (11, 1) or (15, 1) point which is the conjugate of the (13, 1) point when this is available.

9. I will conclude with a few remarks on the geometrical grouping of related quadratic and linear covariants. Geometrically, as well as in the algebraical theory of irreducibles, quadratic covariants of a given quintic form triads, and linear covariants tetrads associated with these triads. It seems a desirability to exhibit a fundamental triad and tetrad having an association of the greatest possible geometrical simplicity. It is an interesting, and perhaps a remarkable, fact that algebraical irreducibility and geometrical simplicity of relationship do not go together. The (2, 2), (6, 2) and (8, 2) of an irreducible system have not the compactness as a geometrical triad, and the symmetrical relationship to a tetrad of linears, which are possessed, for instance, by (2, 2), (8, 2) and the (10, 2) which is reducible as (4, 0)(6, 2)−(8, 0)(2, 2), or by (6, 2), (8, 2) and the (14, 2) reducible as (8, 0)(6, 2)+(12, 0)(2, 2). Each of these last two triads consists of three pairs of elements of which every two pairs are harmonically conjugate. Associated with every such triad a best tetrad of linears to fix upon consists of either (5, 1) or (7, 1) and its harmonic conjugates with respect to the three pairs of the triad. We then have the figure of an inscribed quadrangle and its harmonic triangle. With the first of the two self-conjugate triads named above there thus goes the tetrad of linears (5, 1), (7, 1), (13, 1) and the (15, 1) which is reducible as

$$(4, 0)(11, 1)-(8, 0)(7, 1);$$

and with the second goes, for instance, the tetrad (5, 1), (11, 1), (13, 1) and the reducible (8, 0)(11, 1)+(12, 0)(7, 1). Hammond's canonical orms for this second triad and tetrad have marked simplicity, being

$$\begin{array}{ll} k^2xy, & k(a'x+c'y), \\ k^2(c'x^2-a'y^2), & k^3(a'x-c'y), \\ k^4(c'x^2+a'y^2), & k^3(c'^2x+a'^2y), \\ & k^5(c'^2x-a'^2y). \end{array}$$

Having any tetrad of linears we construct the associated triad

of quadratics by drawing the sides of the harmonic triangle of the quadrangle of marking points of the tetrad, the intersections of these with the conic being the pairs of marking points of the quadratics. This can be applied to tetrads of linears which we have constructed. For instance, with the tetrad of linears (5, 1), (7, 1), Q', P' goes the triad of quadratics

$$(2,\ 2),$$

$$2\{(12,\ 0)-(8,\ 0)(4,\ 0)\}\{27(6,\ 2)-2(4,\ 0)(2,\ 2)\}+27(8,\ 0)(5,\ 1)^2,$$

$$54(18,\ 0)(2,\ 2)-27(4,\ 0)(5,\ 1,)(11,\ 1)+2(4,\ 0)^2(5,\ 1)(7,\ 1),$$

the notation being Hammond's, as it has been throughout where the contrary has not been stated.

ON THE INEQUALITIES CONNECTING THE DOUBLE AND REPEATED UPPER AND LOWER INTEGRALS OF A FUNCTION OF TWO VARIABLES

By W. H. YOUNG, Sc.D., F.R.S.

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1. When $f(x, y)$ is a bounded function of two variables, the inequalities

$$\overline{\iint} f(x, y)\, dx\, dy \geqslant \overline{\int} dy \overline{\int} f(x, y)\, dx \geqslant \underline{\int} dy \underline{\int} f(x, y)\, dx \geqslant \underline{\iint} f(x, y)\, dx\, dy,$$

or, using obvious abbreviations,

$$\text{upper double} \geqslant \text{upper-upper} \geqslant \text{lower-lower} \geqslant \text{lower double},$$

between the upper and lower double (proper) integrals and the two extreme repeated (proper) integrals are well known.*

That these inequalities do not necessarily hold when the integrand is an unbounded function, in which case the integrals concerned are improper integrals, may be inferred from the study of stray examples given incidentally by previous writers.

Thus, in the example given by Dr. Hobson,† $f(x, y)$ has an improper double integral, whose value is zero, while its integral with respect to y is infinite for a set of values of x everywhere dense, and is elsewhere zero. Integrating, first with respect to y and then with respect to x, we have, therefore,

$$\text{double integral} = \text{lower integral of integral} = 0,$$

while

$$\text{upper integral of integral} = \infty,$$

so that the inequality at the head of this article is violated.

It might be contended that the idiosyncrasies of this function are of an extreme character. Here $f(x, y)$ is a discontinuous function as well as an unbounded one; moreover, its integral with respect to y is infinite at an everywhere dense set of points on the y-axis.

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* Prof. Pierpont has given conditions under which these inequalities still hold for integration with respect to a set of points. "On Improper Multiple Integrals," 1906, *Trans. of the American Math. Soc.*, Vol. VII., pp. 155–174.

† "On Absolutely Convergent Improper Integrals," 1906, *Proc. London Math. Soc*, Ser. 2, Vol. 4, p. 156.

Now, in the first place, the admission of infinite values for the integral constantly produces abnormalities. For example, as I have lately pointed out elsewhere,* the theorem that the integral of an unbounded function of a single variable is a continuous function of its upper limit is no longer always true if $+\infty$ or $-\infty$ be allowed as values which the integral may assume, and this is the case even when the integrand is continuous in the extended sense.

In the second place, the doubt might arise whether, if we took as integrand a *continuous* unbounded function $f(x, y)$, the usual inequalities might not inevitably hold good.

I have therefore been at pains to construct an example of an unbounded continuous function such that (1) its double integral is finite, (2) its integral with respect to x is a bounded non-integrable function of y, (3) this function of y has for *lower* integral the double integral.

This example (§ 4), beside setting at rest the doubts in question, is found to throw considerable further light on the various possibilities which may arise with regard to the inequalities which form the main subject of the paper. No systematic investigation of these possibilities appears to exist, and the theorems given in the paper are, I believe, stated for the first time. I shew that (I.) *for functions with a finite upper bound,*

$$\textit{upper double} \geqslant \textit{upper-upper};$$

(II.) *for functions with a finite lower bound,*

$$\textit{lower-lower} \geqslant \textit{lower double};$$

(III.) *for any functions whatever,*

$$\textit{upper double} \geqslant \textit{lower-upper},$$

and (IV.)

$$\textit{upper-lower} \geqslant \textit{lower double}.$$

In case (I.) the sign of equality holds when the integrand is upper semi-continuous, and it holds in case (II.) when the integrand is lower semi-continuous.

The example already referred to (§ 4) shews that the inequality (I.) may be violated if the restriction as to the finitude of the upper bound is removed. Similarly, of course, (II.) may be violated when the corresponding restriction is removed.

It is then shown by means of two examples (§ 8) that a connexion between the inequalities (III.) and (IV.) cannot in general be established; even in the case of bounded functions, the lower-upper may be either greater or less than the upper-lower.

* "On a Test for Continuity," 1908, *Proc. of the Royal Society of Edinburgh*, Vol. XXVIII., § 8, pp. 254, 255.

From all this it follows, in particular, that the double integral of an integrable function may be found by successive upper and lower integration whenever one bound of the function is finite. If, however, the bounds are both infinite, the double integral will not, in general, be capable of calculation by this method.

One additional result may be noticed. It is known that, if $f(x, y)$ is a bounded continuous function of the ensemble (x, y), its integral with respect to x between fixed limits, say $\int_a^b f(x, y)\,dx$, is a continuous function of y. It is shewn in §§ 1–3 that, when f, remaining continuous, is unbounded even at one point, the integral is in general an upper or lower semi-continuous function of y according as f has a finite upper or lower bound in the interval considered. We may, of course, divide the segment of the axis of x under consideration into a finite number of segments in each of which one of the bounds is finite, since f is continuous.

It should be added, in conclusion, that the paper has been so worded that the definition employed for improper double integrals may be taken to be that of de la Vallée-Poussin. The definition given by myself in my paper quoted below, presented to the Cambridge Philosophical Society, leads more naturally to the results obtained, but I have not explicitly employed it, with the object of rendering the paper more readily comprehensible to those acquainted with the existing literature of the subject.

The range of integration I have always taken to be a finite rectangle, and I have not thought it necessary to enter into the obvious generalisations which arise when the number of independent variables is more than two.

1. The following preliminary theorem is fundamental:—

THEOREM 1.—*If $f(x, y)$ is an upper (lower) semi-continuous function of the ensemble (x, y) having a finite upper (lower) bound,* (1) *its upper (lower) integral with respect to one variable x is an upper (lower) semi-continuous function of the other variable y, and* (2) *its upper (lower) double integral is its upper-upper (lower-lower) integral.*

It will be sufficient to prove these theorems when $f(x, y)$ is *upper* semi-continuous.

Since $f(x, y)$ has a finite upper bound, it may* be expressed as the limit of a monotone descending sequence of bounded continuous

* "On Monotone Sequences of Continuous Functions," § 5, Cor., *Proc. Camb. Phil. Soc.*, Lent Term, 1908.

functions $$f_1(x, y) \geqslant f_2(x, y) \geqslant f_3(x, y) \geqslant \dots.$$

Keeping y constant and integrating from a to b with respect to x, and denoting the integral of f_n by $F_n(y)$, the functions

$$F_1(y) \geqslant F_2(y) \geqslant F_3(y) \geqslant \dots$$

form a monotone descending sequence of continuous bounded functions.

The limit of this sequence is therefore an upper semi-continuous function of y. By a known theorem,* however, this limit is the upper integral with respect to x of the upper semi-continuous function $f(x, y)$. This proves the statement (1).

Again, the double integrals of $f_1(x, y)$, $f_2(x, y)$, ... also form a monotone descending sequence whose limit is again, by a known theorem, the upper double integral of $f(x, y)$.

Since $f_n(x, y)$ is bounded and continuous,

$$\iint f_n(x, y)\, dx\, dy = \int dy \int f(x, y)\, dx,$$

so that $$\overline{\iint} f(x, y)\, dx\, dy = \underset{n=\infty}{\mathrm{Lt}} \int dy \int f_n(x, y)\, dx = \underset{n=\infty}{\mathrm{Lt}} \int F_n(y)\, dy.$$

But, since it has been shewn that $F_1(y)$, $F_2(y)$, ... form a monotone decreasing sequence of continuous bounded functions of y, whose limit $F(y)$ is therefore an upper semi-continuous function of y, the last-mentioned limit is the upper integral of $F(y)$, that is, by what has been proved

$$\overline{\iint} f(x, y)\, dx\, dy = \overline{\int} dy \overline{\int} f(x, y)\, dx.$$

This proves the statement (2).

2. On account of the fundamental character of the above theorem, we now give an instructive alternative proof of the first result.

We require the following lemma :—

LEMMA.—*If* $f(x, y)$ *is an upper (lower) semi-continuous function of the ensemble* (x, y), *and* U_y and L_y *are the upper and lower bounds of* $f(x, y)$ *on any particular parallel to the axis of* x *between fixed limits for* x, *then* U_y *and* L_y *are both upper (lower) semi-continuous functions of* y.

We give the proof for an upper semi-continuous function.

* For the upper integral of an upper semi-continuous function is its generalised or Lebesgue integral, and generalised integration, term by term, is allowable in the case of a monotone sequence. It is easy to give an independent proof of the case of this theorem used more than once in our investigation. For the general theorem, *cp.* Beppo Levi, *Atti di Torino*, 1907.

For, if $y_1, y_2, \ldots$ is a sequence of values of y having y_0 as limit, there will be a point P_n on the line $y = y_n$ where $f(x, y_n)$ *assumes* its upper bound U_{y_n}. These points P_n for all values of n have one or more limiting points lying on $y = y_0$, and at such a limiting point, $f(x, y)$ being upper semi-continuous with respect to the ensemble (x, y), the value of $f(x, y_0)$ is not less than any limit approached by the quantities U_{y_n}; *a fortiori*, the same is true of U_{y_0}, which shews that U_y is an upper semi-continuous function of y.

Again, on the line $y = y_0$ there will be a point P_0 where $f(x, y)$ has a value less than L, where L is any quantity greater than L_y. Since $f(x, y)$ is upper semi-continuous with respect to y, its values on all neighbouring lines $y = y_n$ on the ordinate of P_0 are also less than L, and therefore the same is true of the corresponding lower bounds L_{y_n}. That is to say, L_y is an upper semi-continuous function of y.

COR. 1.—*If $f(x, y)$ is a continuous function of the ensemble (x, y), its upper and lower bounds U_y and L_y on any parallel to the axis of x between fixed limits for x, are continuous functions of y.*

COR. 2.—*If $f(x, y)$ be upper (lower) semi-continuous with respect to y only, then the lower (upper) bound only is an upper (lower) semi-continuous function of y.*

Alternative proof of (1) in Theorem 1:—

Since $f(x, y)$ has a finite upper bound, it may be shewn* that its upper integral with respect to x is the lower limit of its upper summations.

Now, taking any fixed division of the segment (a, b) into a finite number of segments, and for fixed y taking the upper limit of $f(x, y)$ in each segment and summing, so as to form one of these upper summations, we get, by the preceding theorem, an upper semi-continuous function of y.

Taking any monotone sequence of such upper summations, descending to the lower limit, it follows that that limit is an upper semi-continuous function of y, which proves the required result, viz., that the upper integral of the upper semi-continuous function $f(x, y)$ with respect to x is an upper semi-continuous function of y when $f(x, y)$ has a finite upper bound.

3. As a special case of Theorem 1, we note the following:—

THEOREM 2.—*If $f(x, y)$ is a continuous function of the ensemble (x, y) with a finite upper (lower) bound, its double integral is the upper (lower) integral with respect to y of the integral with respect to x.*

---

* This theorem, which is easily proved, is probably contained in Severini's *Thèse*, which, however, I have not been able to procure.

Further, the integral with respect to x of such a function is an upper (lower) semi-continuous function of y.

The following example shews that the result just stated is all that can, in general, be predicated:—

Construction of a Continuous Unbounded Positive Function $f(x, y)$, whose Integral

$$F(y) = \int_0^1 f(x, y)\, dx,$$

with respect to x, is a Bounded Lower Semi-Continuous Function of y, not Continuous.

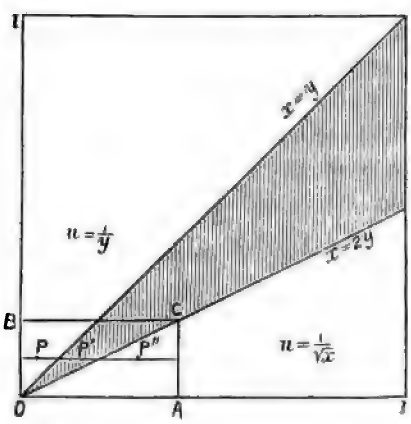

FIG. 1.

Take the unit square (Fig. 1) and divide it by the lines

$$x = y, \qquad x = 2y,$$

into three triangles.

In the isosceles right-angled triangle put

$$u = f(x, y) = \frac{1}{y}. \tag{1}$$

In the other right-angled triangle put

$$u = f(x, y) = x^{-\frac{1}{2}}. \tag{2}$$

In the remaining triangle, shaded in the figure, put

$$u = f(x, y) = \frac{2y-x}{y^2} + \frac{x-y}{\sqrt{2}\, y^{\frac{3}{2}}} = \frac{2}{y} - \frac{1}{\sqrt{2y}} - x\left(\frac{1}{y^2} - \frac{1}{\sqrt{2}\, y^{\frac{3}{2}}}\right), \tag{3}$$

so that in this triangle, for any constant value of y,

$$\frac{1}{y} \geqslant f(x, y) \geqslant x^{-\frac{1}{2}}, \tag{4}$$

the sign of equality holding only at the respective extreme points, and $f(x, y)$ decreasing in a monotone manner from the former to the latter value. Thus, for constant y, $f(x, y)$ is a monotone decreasing continuous function of x; it is always positive, and, except on the axis of y, always finite. On the axis of x it is $x^{-\frac{1}{2}}$.

The formulæ (1), (2) and (3) being continuous functions, it follows that, at any point of the unit square not on one of the dividing lines

$$x = y, \qquad x = 2y,$$

$f(x, y)$ is a continuous function of the ensemble (x, y). The same is true on the dividing lines, with the possible exception of the origin, since the expressions for $f(x)$ in the two triangles having that line as boundary agree on that line. The only doubt remains at the origin, where $f(x, y)$ is infinite. But, if we draw any rectangle, as $OACB$ in the figure, having the corner opposite the origin on the line

$$x = 2y,$$

and in this rectangle draw any line parallel to the x-axis, since $f(x, y)$ decreases monotonely along this line, its value at any point, such as P, or P' or P'' in the figure, is greater than that on the bounding ordinate AC, that is, greater than at A, where it is $x^{-\frac{1}{2}}$. Thus, by taking A sufficiently near to O, so that

$$OA < \frac{1}{k^2},$$

all the values of $f(x, y)$ in the rectangle are greater than k, so that f is continuous at its infinity, the origin.

Integrating, we have

$$F(0) = \int_0^1 x^{-\frac{1}{2}} dx = 2, \tag{5}$$

and when y is not zero,

$$F(y) = \int_0^y f(x, y)\, dx + \int_y^{2y} f(x, y)\, dx + \int_{2y}^1 f(x, y)\, dx,$$

or, using the formulæ (1), (2) and (3), and integrating

$$F(y) = 1+2-\sqrt{\frac{y}{2}} - \frac{3}{2}\left(1-\sqrt{\frac{y}{2}}\right)+2(1-\sqrt{2y}) = \tfrac{7}{2}\left(1-\sqrt{\frac{y}{2}}\right). \tag{6}$$

From (5) and (6), we see that

$$F(0) = 2 < \tfrac{7}{2} < \underset{y=0}{\mathrm{Lt}}\, F(y),$$

so that $F(y)$, though elsewhere continuous, is only lower semi-continuous at the origin.

4. *Construction of a Continuous Unbounded Positive Function* $f(x, y)$, *whose Integral*
$$F(y) = \int_0^1 f(x, y)\, dx,$$
with respect to x *is a Bounded Non-Integrable Lower Semi-Continuous Function of* y.

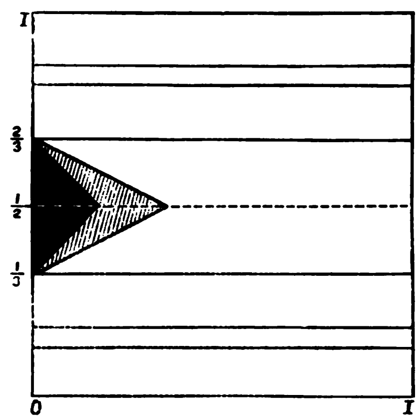

FIG. 2.

Ex. 2.—We define $f(x, y)$ in the unit square as follows :—

On the y-axis take the typical ternary perfect set of positive content* (lying between $\frac{1}{3}$ and $\frac{2}{3}$), say G. On each black interval of G erect an isosceles right-angled triangle (black in the figure), and an isosceles triangle whose altitude is equal to its base (shaded outside the black triangle).

Let a_i be the lower, and b_i the upper end of the i-th black interval, and m_i its middle point. Then the line
$$y = m_i$$
divides the strip bounded by the parallels through a_i and b_i symmetrically.

In each half-strip we have one black and one shaded triangle and a white part. We define $f(x, y)$ in the lower half-strip as follows, and then change $(y-a_i)$ into (b_i-y), so as to get the corresponding formulæ in the upper half-strip.

In the black triangle $$f(x, y) = \frac{1}{y-a_i}; \tag{1}$$

* Young's *Theory of Sets of Points*, Ex. 1, p. 78. The construction is by means of division into 3, 3^2, 3^3, ..., 3^n, ... equal parts. The complementary set of "black intervals" consists of the central third of the segment (0, 1), the central ninth of (0, $\frac{1}{3}$) and ($\frac{2}{3}$, 1), and so on.

in the white part $$f(x, y) = x^{-\frac{1}{2}}; \tag{2}$$

in the shaded triangle $$f(x, y) = \frac{2y'-x}{y'^2} + \frac{x-y'}{y'\sqrt{2y'}}, \tag{3}$$

where, for shortness, we write $y' = y - a_i$. (3a)

This function is, by the preceding example, continuous throughout the half strip, and therefore throughout the whole strip, since the values of the given expressions in the two half strips agree on the median line. On the two extreme lines

$$y = a_i, \qquad y = b_i,$$

we have $$f(x, y) = x^{-\frac{1}{2}}, \tag{4}$$

which also expresses the function on any parallel through a point of the perfect set G. The function $f(x, y)$, so defined for the whole unit square is then clearly a positive continuous function of the ensemble (x, y), and is finite everywhere except at the points of G, where it is infinite.

Integrating, we have

$$F(y) = 2 \text{ at all the points of the perfect set } G,$$

while elsewhere

$$F(y) = \tfrac{7}{2}[1-(y-a_i)^{\frac{1}{2}}] \quad \text{or} \quad \tfrac{7}{2}[1-(b_i-y)^{\frac{1}{2}}],$$

according as y lies in the lower or the upper half of the black interval (a_i, b_i) or d_i. Thus $F(y)$ is lower semi-continuous at every point of the perfect set G and is elsewhere continuous; it is therefore a non-integrable function of y.

Since $F(y)$ is lower semi-continuous, its lower integral is its generalized integral, that is

$$2I + \Sigma 2 \int_{a_i}^{m_i} F(y)\, dy = 2I + 7\Sigma [\tfrac{1}{2}d_i - \tfrac{2}{3}(\tfrac{1}{2}d_i)^{\frac{3}{2}}] = \tfrac{7}{2} - \tfrac{3}{2}I - \frac{7\sqrt{2}}{6}\Sigma d_i^{\frac{3}{2}}$$

(where I is the content of G), which is the value of the double integral of $f(x, y)$ over the unit square.

It may be noticed that the upper integral of $F(y)$ only differs from the above by taking $\frac{7}{2}$ instead of 2 at all the points of G, so that

$$\overline{\int_0^1} F(y)\, dy = \frac{7}{2} - \frac{7\sqrt{2}}{6}\Sigma d_i^{\frac{3}{2}}.$$

Thus the upper integral of the integral is greater than the double integral, contrary to what can happen when the integrand is bounded, in which

case the double integral is always greater than or equal to the upper-upper integral, whether the integrand is continuous or not.

If we integrate $f(x, y)$ with respect to y from 0 to 1, we get a function of x which is clearly finite and continuous for all values of x other than zero, and is always greater than $Ix^{-\frac{1}{2}}$, so that it has the limit $+\infty$ at the origin. The value at the origin is also $+\infty$. Thus the function is a continuous unbounded function, having a single infinity at the origin. The integral of this function is, of course, the double integral of $f(x, y)$ over the unit square, and has therefore the value already found.

THEOREM 3.—*Given any function $f(x, y)$ whatever with a finite upper bound, its upper-upper integral is less than or equal to its upper double integral, that is*

$$\overline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

Let $\phi(x, y)$ be the associated upper limiting function of $f(x, y)$, that is the function got by taking at each point (x, y) the highest value which can be approached as limit by $f(x, y)$ in the neighbourhood of the point in question. Then ϕ is an upper semi-continuous function of the ensemble (x, y), and it has the same finite upper bound as f itself. Hence, by Theorem 1, its upper double integral is its upper-upper integral. Since, however, as is easily seen, f and ϕ have the same upper double integral, this proves that

$$\overline{\int} dy \overline{\int} \phi(x, y)\, dx = \overline{\iint} f(x, y)\, dx\, dy. \qquad (1)$$

Now the associated upper limiting function of $f(x, y)$ when y is constant is evidently less than or equal to $\phi(x, y)$ at each point (x, y). Hence the upper integral of $f(x, y)$ with respect to x, y being constant, which is the same as the upper integral of that associated upper limiting function, is less than or equal to the upper integral of ϕ with respect to x, that is

$$\overline{\int} f(x, y)\, dx \leqslant \overline{\int} \phi(x, y)\, dx. \qquad (2)$$

Hence, by (1) and (2),

$$\overline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

A similar argument proves the alternative theorem :—

Given any function $f(x, y)$ whatever with a finite lower bound, the lower-lower integral is greater than or equal to the lower double integral, that is

$$\underline{\iint} f(x, y)\, dx\, dy \leqslant \underline{\int} dy \underline{\int} f(x, y)\, dx.$$

COR. 1.—*If $f(x, y)$ is any bounded function of the ensemble (x, y),*

$$\underline{\iint} f(x, y)\, dx\, dy \leqslant \underline{\int} dy \underline{\int} f(x, y)\, dx \leqslant \overline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

COR. 2.—*If $f(x, y)$ is an integrable bounded function, then in the preceding inequality the sign of equality must be taken throughout.*

6. THEOREM 4.—*If $f(x, y)$ be any function of x and y, its lower-upper integral is less than or equal to its upper double integral, that is*

$$\underline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

Let ϕ be the upper limiting function of f. Then ϕ is an upper semi-continuous function of the ensemble (x, y), and therefore* can be expressed as the limit of a monotone descending sequence of continuous functions each having a finite lower bound,

$$\phi_1(x, y) \geqslant \phi_2(x, y) \geqslant \ldots,$$

and the upper double integral of ϕ is the limit of the double integral of ϕ_n. Thus, if the upper double integral of ϕ is finite, we can find n so that

$$\iint \phi_n(x, y)\, dx\, dy - \overline{\iint} \phi(x, y)\, dx\, dy < e, \tag{1}$$

e being any positive quantity previously chosen at will.

But, by Theorem 1, since ϕ_n has a finite lower bound, and is continuous,

$$\begin{aligned} \iint \phi_n(x, y)\, dx\, dy &= \underline{\int} dy \int \phi_n(x, y)\, dx\, dy \\ &= \underline{\int} dy \overline{\int} \phi_n(x, y)\, dx\, dy \geqslant \underline{\int} dy \overline{\int} f(x, y)\, dx, \end{aligned} \tag{2}$$

since at every point $\phi_n \geqslant \phi \geqslant f$.

Since f and ϕ have the same upper double integral, it now follows from (1) and (2) that

$$\overline{\iint} f(x, y)\, dx\, dy + e \geqslant \underline{\int} dy \overline{\int} f(x, y)\, dx,$$

which, e being at our disposal, proves the theorem.

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* Theorem of § 5, Case 3, of my paper, "Note on Monotone Sequences of Continuous Functions."

7. Summing up our results so far, we have shewn that for functions with a finite upper bound,

$$\text{upper double} \geqslant \text{upper-upper};$$

for functions with a finite lower bound,

$$\text{lower-lower} \geqslant \text{lower double};$$

for any functions,

$$\text{upper double} \geqslant \text{lower-upper},$$

$$\text{upper-lower} \geqslant \text{lower double}.$$

The example just given shews that it *may* happen for functions with an infinite upper bound, that

$$\text{upper double} > \text{upper-lower};$$

and for functions with an infinite lower bound, that

$$\text{lower-upper} > \text{lower double}.$$

Thus for a bounded function,

$$\text{upper double} \geqslant \text{upper-upper} \geqslant \text{lower-upper},$$

$$\text{upper-lower} \geqslant \text{lower-lower} \geqslant \text{lower double},$$

while for an unbounded function the upper double may be displaced so as to occupy the second position in its own sequence, but not the third, while in the other sequence it may be displaced so as to occupy the second place, between upper-lower and lower-lower, but no further. Similar remarks apply, of course, to the lower double. Thus in the case of a function which has infinite upper and lower bounds, the following is a possible inequality:—

$$\text{upper-lower} > \text{upper double} > \text{lower double} > \text{lower-upper},$$

or, more fully,

$$\begin{aligned}\text{upper-upper} > \text{upper-lower} > \text{upper double} &> \text{lower double}\\ &> \text{lower-upper} > \text{lower-lower}.\end{aligned}$$

One point only rests in doubt, namely, as to the possible relative positions of the lower-upper and the upper-lower. It is clear that in the case of an unbounded function they may change positions, that is, we cannot say, *a priori*, which of the two is greater. It remains, however, still to discuss their relative position when the function is bounded. In this case also either position is possible, as is shewn by Examples 3 and 4.

8. The following simple example shews that in the case of a bounded function the following relative position is possible,

$$\text{upper-lower} > \text{lower-upper}.$$

Ex.—Take in the segment (0, 1) of the y-axis a perfect set nowhere dense of positive content I. For every value of y belonging to this perfect set, let

$$f(x, y) = 1,$$

and elsewhere

$$f(x, y) = 0,$$

the region of integration being the unit square.

Then $f(x, y)$ is, for every value of y, a continuous function of x and therefore integrable, so that

$$\underline{\int} f(x, y)\, dx = \overline{\int} f(x, y)\, dx,$$

and has the value 1 or 0 according as y does or does not belong to the perfect set. The function of y so defined is therefore non-integrable, having the upper integral $= I$, and the lower integral $= 0$, so that

$$I = \text{upper-lower} > \text{lower-upper} = 0.$$

The following example, on the other hand, shews that in the case of a bounded function the relative position may be reversed, so that we may have

$$\text{lower-upper} > \text{upper-lower}.$$

Ex.—Take in the segment (0, 1) of the x-axis a perfect set nowhere dense and of positive content I, and on each of the ordinates through its points place a similar set. We thus get a plane perfect set nowhere dense of content I^2, such as is given in Ex. 5 and Fig. 24 of my *Theory of Sets of Points*, pp. 173, 174.

At every point of this plane perfect set let

$$f(x, y) = 0,$$

and elsewhere have the value 1.

Then for every value of y belonging to the perfect set of content I on the y-axis, we have

$$\overline{\int} f(x, y)\, dx = 1,$$

but

$$\underline{\int} f(x, y)\, dx = 1 - I,$$

while for other values of y both the upper and the lower integrals have

the value 1. Thus the upper integral with respect to x is an integrable function of y, while the lower integral with respect to x is a non-integrable function of y, hence

lower-upper = upper-upper > upper-lower.

In fact $$1 = \underline{\int} dy \overline{\int} f(x, y)\, dx > \overline{\int} dy \underline{\int} f(x, y)\, dx = 1 - I.$$

9. From the above results we can at once deduce the following:—

(1) *If an integrable function have a finite upper bound,*

upper double = upper-upper = upper-lower = lower double.

(2) *If an integrable function have a finite lower bound,*

upper double = lower-upper = lower-lower = lower double.

(3) *In the case of an integrable function having an infinite upper and an infinite lower bound, the method of repeated upper and lower integration will totally fail in general to give the value of the double integral.*

[*Added March* 14*th*, 1908.—In the above paper I have deliberately avoided the use of the concept of Lebesgue integration. I hoped in this way to appeal to a larger public. I should like to point out, however, what is indeed obvious to any one acquainted with the Lebesgue theory, that the reasoning by which the inequalities (III.) and (IV.) were obtained, really gives us a slightly more extended result, viz.,

(V.) upper double $\geqslant$ middle-upper.

(VI.) middle-lower $\geqslant$ lower double.

where the word "middle" is used to denote the generalised, or Lebesgue, integral, which, as is well known, lies in general between the upper and lower integrals, and may be equal to either or both. When the function is lower semi-continuous the Lebesgue integral is equal to the lower integral, which I write, symbolically,

$$\overline{\int} f(x)\, dx = \dot{\int} f(x)\, dx,$$

or, perhaps,

$$\overline{\int} f(x)\, dx = \dot{\int} f(x)\, dx.$$

Hence, in § 6, equation (2) may be written, since $\int \phi_n(x, y)\,dx$ is a lower semi-continuous function of y,

$$\begin{aligned}\iint \phi_n(x, y)\,dx\,dy &= \underline{\int} dy \int \phi_n(x, y)\,dx \\ &= \dot{\int} dy \int \phi_n(x, y)\,dx \\ &= \dot{\int} dy\, \overline{\int} \phi_n(x, y)\,dx \\ &\geqslant \dot{\int} dy\, \overline{\int} f(x, y)\,dx.\end{aligned}$$

Hence, for a function unbounded above and below, the most general inequality will be

upper-upper $>$ upper-lower $>$ upper double $>$ middle-upper
$>$ middle-lower $>$ lower double $>$ lower-upper $>$ lower-lower.]

GENERALISATION OF A THEOREM IN THE THEORY OF DIVERGENT SERIES

By G. H. HARDY.

[Received July 23rd, 1907.—Read November 14th, 1907.]

1. In a paper recently printed in these *Proceedings** I proved the following theorem† :—*If*

(1) Σa_n *is a series summable by Césaro's method of mean values, i.e., if*

$$(s_0+s_1+\ldots+s_n)/(n+1),$$

where $$s_n = a_0+a_1+\ldots+a_n,$$

tends to a finite limit as n tends to infinity;

(2) f_n *is a function of n which, together with its first and second differences*

$$f_n-f_{n+1}, \quad f_n-2f_{n+1}+f_{n+2},$$

is positive for all values of n;

then the series $\Sigma a_n f_n$ is also summable.

Further, if f_n is also a function of a variable x, and the condition (2) *is satisfied throughout a certain interval of values of x, say* (0, 1), *and f_0 has a finite upper limit throughout this interval,‡ then the series $\Sigma a_n f_n$ is* **uniformly** *summable throughout the interval: and if every f_n is a continuous function of x, the sum of the series is also a continuous function of x.*

I also stated (*l.c.*, p. 267) that I had no doubt of the truth of an obvious generalisation of this theorem. Suppose that the first of the quantities

$$s_n^1 = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

$$s_n^2 = \frac{s_0^1+s_1^1+\ldots+s_n^1}{n+1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 247.

† *L.c.*, p. 256.

‡ It is obvious that the same is true of f_n.

which tends to a limit as n tends to infinity, is s_n^k. Then the series Σa_n may be said to be *summable* (Hk).*

Then it is natural to suppose that the theorem may be generalised by supposing Σa_n to be summable (Hk), and the $k+1$ sets of differences

$$\Delta f_n,\ \Delta^2 f_n,\ \ldots,\ \Delta^{k+1} f_n,$$

to be positive. But when I wrote my former paper I had not been able to overcome the considerable algebraical difficulties which appeared to be involved in the proof of this theorem.

On the other hand, the theorem which I had proved was not sufficient to deal with all the interesting particular cases which actually arise when we try to make applications of it (*v.* p. 264 of my former paper). I was therefore led to consider in greater detail the most interesting particular case, viz., that in which the f_n's are such that $\Sigma a_n f_n$ is *convergent* for all points of (0, 1) except $x = 0$, and

$$\lim_{x \to 0} f_n = 1,$$

for all values of n; and I obtained three theorems† which were sufficiently general for the purposes of the applications which I had in view. Mr. Bromwich then proved a more general theorem which included all these theorems and also some very similar theorems arrived at independently, for the case of $k = 1$, by Dr. C. N. Moore.‡

It is mainly owing to suggestions derived from these latter investigations that I have since been able to prove a theorem which, so far as I know, includes all the theorems which have been referred to. This theorem stands to the generalisation contemplated in my former paper in the same relation which Mr. Bromwich's theorem bears to the first of the theorems which I proved in the *Math. Annalen*: that is to say, the condition

$$f_n,\ \Delta f_n,\ \Delta^2 f_n,\ \ldots,\ \Delta^{k+1} f_n \geqslant 0$$

is replaced by the more general condition that

$$\Sigma n^k \mid \Delta^{k+1} f_n \mid$$

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* This extension of Césaro's method is due (implicitly) to Hölder, *Math. Annalen*, Bd. XX., p. 535.

† "Some Theorems concerning Infinite Series," *Math. Annalen*, Bd. LXIV., p. 77.

‡ Moore, *Trans. Amer. Math. Soc.*, Vol. VIII., p. 299; Bromwich, *Math. Annalen*, Bd. LXV., pp. 359 and 362.

is convergent, or (when f_n is a function of x) that

$$\sum_{\nu=0}^{n} \nu^k |\Delta^{k+1} f_\nu| < K$$

for all values of x and n.

2. There are two alternative definitions of the sum of a divergent series on mean value lines when Césaro's original definition fails. One is Hölder's definition stated above, which defines *summability* (Hk). But Césaro himself gave a somewhat similar definition.* Let

$$A_n^k = \frac{(n+1)(n+2)\dots(n+k)}{k!}$$

—which we may, in the ordinary continental notation, write in the form

$$A_n^k = \binom{n+k}{k},$$

—and let $$S_n^k = A_n^k a_0 + A_{n-1}^k a_1 + \dots + A_0^k a_n.$$

And suppose that, as n tends to infinity,

$$S_n^k / A_n^k$$

tends to a limit. Then we shall say that Σa_n is *summable* (Ck).

For $k=1$ Hölder's and Césaro's definitions are identical. That this is so for $k=2$ has been proved by Mr. Bromwich.† In all ordinary cases (as applied, *e.g.*, to the series $1^s - 2^s + 3^s - \dots$) the two definitions lead to the same result: and it has been proved by K. Knopp‡ that Césaro's definition *includes* Hölder's—*i.e.*, that *if* a series is summable (Hk) it is also summable (Ck), and the sums agree. It is not unlikely that Césaro's definition is more general: it is conceivable that the two always cover the same ground. But Césaro's definition should certainly be adopted as the standard one; for it is *at least* equally general, and is far more easy to work with in practice, owing to the fact that the expression of a_n in terms of the sums S_n^k is as simple as the reverse equation, whereas the expression of a_n in terms of s_n^k is complicated and clumsy. The contrast appears very clearly when Mr. Bromwich's work, with Césaro's definition, is contrasted with his own, or mine, with Hölder's.

* Bromwich, *Infinite Series*, pp. 311 *et seq.*

† See pp. 363–5 of his paper in the *Math. Annalen* already quoted.

‡ *Grenzwerte von Reihen u. s. w.*, Inaugural Dissertation, Berlin, 1907, p. 19.

3. The first part of the theorem is as follows :—

THEOREM A.—*If* Σa_n *is summable* (Ck) *and*

$$\Sigma n^k \mid \Delta^{k+1} f_n \mid$$

is convergent, then $\Sigma a_n f_n$ *is summable* (Ck). *Further, its sum is equal to that of the series*

$$\Sigma S_n^k \Delta^{k+1} f_n,$$

which is absolutely convergent.

We note as a matter of minor detail that, if Σa_n is summable (Hk), it is also summable (Ck), and so $\Sigma a_n f_n$ is summable (Ck): but we cannot affirm that the latter series is summable (Hk), except for $k = 1, 2$.

That $\Sigma S_n^k \Delta^{k+1} f_n$ is absolutely convergent follows at once from the fact that S_n^k / n^k tends to a limit as $n \to \infty$.

Some Algebraical Preliminaries.

4. We denote the sum

$$A_n^k a_0 f_0 + A_{n-1}^k a_1 f_1 + \ldots + A_0^k a_n f_n$$

formed from $\Sigma a_n f_n$, as S_n^k is formed from Σa_n, by T_n^k: and we proceed to express T_n^k in terms of

$$S_0^k,\ S_1^k,\ \ldots,\ S_n^k,$$

and the differences of the functions f_n. We have

$$\text{(1)} \qquad a_n = S_n^k - \binom{k+1}{1} S_{n-1}^k + \binom{k+1}{2} S_{n-2}^k - \ldots + (-)^{k+1} S_{n-k-1}^k.\text{*}$$

Thus

$$\text{(2)} \qquad T_n^k = \sum_{\nu=0}^{n} A_{n-\nu}^k f_\nu \sum_{r=0}^{k+1} (-)^r \binom{k+1}{r} S_{\nu-r}^k.$$

This expression, as it stands, involves a certain number of terms S_j^k with negative suffixes j: these must be considered to be defined as being equal to zero. In this formula for T_n^k the coefficient of S_j^k is

$$\sum_{\nu=j}^{j+k+1} (-)^{\nu-j} \binom{k+1}{\nu-j} A_{n-\nu}^k f_\nu,$$

or

$$\sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i}.$$

* It is easy to see (Bromwich, *Infinite Series*, *l.c.*) that

$$\Sigma S_n^k x^n \equiv (1-x)^{-(k+1)} \Sigma a_n x^n,$$

and

$$\Sigma a_n x^n \equiv (1-x)^{k+1} \Sigma S_n^k x^n.$$

If this expression contains any terms for which $j+i>n$, they may simply be omitted. Thus, with this proviso, (2) may be written in the form

$$T_n^k = \sum_{j=0}^{n} S_j^k \sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i} = \sum_{j=0}^{n} a_j S_j^k, \tag{3}$$

say.

5. Now

$$a_j = \sum_{i=0}^{k+1} (-)^i \beta_{j+i} f_{j+i}, \tag{4}$$

where $$\beta_{j+i} = \binom{k+1}{i} A_{n-j-i}^k.$$

From this it follows that

$$a_j = \sum_{i=0}^{k+1} (-)^i \gamma_{j+i} \Delta^{k+1-i} f_{j+i}, \tag{5}$$

where

$$\begin{aligned} \gamma_{j+i} &= \beta_{j+i} - \binom{k-i+2}{1} \beta_{j+i-1} + \binom{k-i+3}{2} \beta_{j+i-2} - \dots \\ &= \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \beta_{j+i-\nu}. \end{aligned} \tag{6}$$

To verify this result substitute for γ_{j+i} in the expression (5), and pick out the coefficient of $\beta_{j+\lambda}$. We find this coefficient to be $(-1)^\lambda$ times

$$\binom{k-\lambda+1}{0} \Delta^{k-\lambda+1} f_{j+\lambda} + \binom{k-\lambda+1}{1} \Delta^{k-\lambda} f_{j+\lambda+1}$$
$$+ \binom{k-\lambda+1}{2} \Delta^{k-\lambda-1} f_{j+\lambda+2} + \dots = \sum_{i=\lambda}^{k+1} \binom{k-\lambda+1}{i-\lambda} \Delta^{k-i+1} f_{j+i},$$

and it is easy to see that this reduces to $f_{j+\lambda}$.* Thus

$$\gamma_{j+i} = \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \binom{k+1}{i-\nu} A_{n-j-i+\nu}^k. \tag{8}$$

---

* The simplest proof is probably by means of symbolical operators. Let E denote the operation which, when performed on f_n, changes it into f_{n+1}. The expression above, on writing $i=\lambda+\mu$, becomes

$$\sum_{\mu=0}^{k+1-\lambda} \binom{k+1-\lambda}{\mu} \Delta^{k+1-\lambda-\mu} E^\mu f_{j+\lambda} = \left(1+\frac{E}{\Delta}\right)^{k+1-\lambda} \Delta^{k+1-\lambda} f_{j+\lambda} = (\Delta+E)^{k+1-\lambda} f_{j+\lambda}.$$

But $$(\Delta+E) f_n = f_n - f_{n+1} + f_{n+1} = f_n,$$

whence the result.

But this expression may be simplified considerably. For

$$\binom{k-i+1+\nu}{\nu}\binom{k+1}{i-\nu}A^k_{n-j-i+\nu}$$

$$=\frac{(k-i+1+\nu)!}{(k-i+1)!\,\nu!}\;\frac{(k+1)!}{(i-\nu)!\,(k-i+1+\nu)!}\;\frac{(n-j-i+\nu+k)!}{k!\,(n-j-i+\nu)!}$$

$$=\binom{k+1}{i}\binom{i}{\nu}\binom{n-j-i+\nu+k}{k},$$

and so

(9) $$\gamma_{j+i}=\binom{k+1}{i}\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}.$$

But $$\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}$$

is the coefficient of t^k in

$$\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}(1+t)^{n-j-i+\nu+k},$$

or $$(1+t)^{n-j-i+k}\{1-(1+t)\}^i,$$

or $$(-)^i\,t^i\,(1+t)^{n-j-i+k};$$

and is therefore equal to $(-)^i\binom{n-j-i+k}{k-i}$,

if $0\leqslant i\leqslant k$, and to zero if $i=k+1$. Thus

(10) $$\gamma_{j+i}=(-)^i\binom{k+1}{i}\binom{n-j-i+k}{k-i}.$$

Hence

(11) $$a_j=\sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

and

(12) $$T^k_n=\sum_{j=0}^{n}S^k_j\sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

with the proviso, we may repeat, that if $j+i>n$ we must write 0 for f_{j+i}. This formula is the end of our algebraical transformations.*

* It has been suggested to me that these transformations should be capable of being simplified, and I do not doubt that this is so; but I have not been able to effect any appreciable simplification.

6. Suppose, *e.g.*, that $k = 1$. Then (12) becomes

$$(12_1) \qquad T_n^1 = \sum_{j=0}^{n} S_j^1 \{(n-j+1)\Delta^2 f_j + 2\Delta f_{j+1}\},$$

which is easily verified. If $k = 2$, (12) becomes

$$(12_2) \quad T_n^2 = \sum_{j=0}^{n} S_j^2 \left\{\frac{(n-j+2)(n-j+1)}{2}\Delta^3 f_j + 3(n-j+1)\Delta^2 f_{j+1} + 3\Delta f_{j+2}\right\},$$

and so on.

7. We can now proceed to the proof of our theorem. We suppose that

$$\Sigma n^k \,|\, \Delta^{k+1} f_n \,|$$

is convergent. If this is so the same is true, as has been shown by Mr. Bromwich,* of all the series

$$\Sigma n^{k-\lambda} \,|\, \Delta^{k+1-\lambda} f_n \,| \quad (\lambda = 0, 1, \ldots, k).$$

We have to show that in these circumstances

$$\lim (T_n^k / A_n^k) = \sum_0^\infty S_n^k \Delta^{k+1} f_n .$$

8. We consider first the terms in T_n^k for which $i = 0$. These give

$${}_0T_n^k = \sum_{j=0}^{n} \binom{n-j+k}{k} S_j^k \Delta^{k+1} f_j.$$

Now
$$\binom{n-j+k}{k} - A_n^k = \frac{1}{k!}\{(n-j+1)(n-j+2)\ldots(n-j+k) - (n+1)(n+2)\ldots(n+k)\},$$
which is negative and numerically less than

$$Kn^{k-1},$$

where K is a constant. Thus

$$\frac{{}_0T_n^k}{A_n^k} = \sum_{j=0}^{n} S_j^k \Delta^{k+1} f_j + R_0,$$

where
$$|R_0| < \frac{K}{n}\sum_{j=0}^{n} |\Delta^{k+1} f_j| ;$$
and so

$$(13) \qquad \lim_{n\to\infty} \left(\frac{{}_0T_n^k}{A_n^k}\right) = \sum_{j=0}^{\infty} S_n^k \Delta^{k+1} f_j .$$

* *Math. Annalen*, *l.c.*, p. 361.

9. Next we consider

$$ {}_iT_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} S_j^k \Delta^{k+1-i} f_{j+i}. $$

Since

$$ \binom{n-j-i+k}{k-i} = (n-j+1)(n-j+2)\dots(n-j+k-i)/(k-i)! < Kn^{k-i}, $$

it follows that
$$ \left| \frac{{}_iT_n^k}{A_n^k} \right| < \frac{K}{n^i} \sum_{j=0}^{n} | \Delta^{k+1-i} f_{j+1} |; $$

and therefore

(14)
$$ \lim_{n\to\infty} \left(\frac{{}_iT_n^k}{A_n^k} \right) = 0. $$

From (13) and (14) it follows that

(15)
$$ \lim (T_n^k / A_n^k) = \sum_{j=0}^{\infty} S_j^k \Delta^{k+1} f_j, $$

which establishes the theorem.

10. THEOREM B.—*If, in addition,*

$$ \sum_0^n \nu^k | \Delta^{k+1} f_\nu | < K^* $$

for all values of n and x, then the series

$$ \Sigma S_j^k \Delta^{k+1} f_j $$

is **uniformly** *convergent.*

Let S be the sum (Ck) of the series Σa_n: and let $\Sigma a_n'$ be the series for which

$$ a_0' = a_0 - S, \quad a_n' = a_n \quad (n > 0), $$

so that $S' = 0$. Then

$$ \sum_{j=m}^{m'} S_j^k \Delta^{k+1} f_j = S \sum_m^{m'} \Delta^{k+1} f_j + \sum_m^{m'} S_j'^k \Delta^{k+1} f_j = \sigma_1 + \sigma_2, $$

say. Choose m so that for $j \geqslant m$,

$$ | S_j'^k / A_j^k | < \epsilon. $$

---

* Mr. Bromwich (*l.c.*, p. 361) has proved that the same is then true of

$$ \sum_0^n \nu^{k-\lambda} | \Delta^{k+1-\lambda} f_\nu | \quad (\lambda = 0, 1, \dots, k). $$

Then

(16) $$|\sigma_2| < 2\epsilon K.$$

Also

(17) $$\left|\sum_{m}^{m'} \Delta^{k+1} f_j\right| < \frac{1}{m^k}\sum_{m}^{m'} j^k \,|\, \Delta^{k+1} f_j \,|\, < \frac{K}{m^k},$$

and from (16) and (17) the theorem follows.

COROLLARIES.—(α) *If every f_n is continuous, the sum of the series $\Sigma a_n f^n$ is continuous.*

(β) *If all the differences*

$$f_n,\ \Delta f_n,\ \ldots,\ \Delta^{k+1} f_n$$

are positive, the condition $\sum_0^n n^k \Delta^{k+1} f_n < K$

is certainly satisfied, and the conclusions of the theorem apply.

The proof of this will be found in Lemma A of my paper in the *Math. Annalen* quoted above.

11. *Applications.*—I have already stated that the very general theorems proved by Messrs. Fejér, Moore, and Bromwich, and myself, with especial reference to a particular case, enable us to deal effectively enough with the majority of interesting special applications which occur naturally in analysis. It would therefore be futile to give any considerable number of illustrations here. In the paper cited above* I pointed out the kind of case in which a more general theorem of the kind here proved is necessary. A simple example is given by supposing

$$f_n = \frac{1}{(a+nx)^s} \quad (s > 0).$$

If the series Σa_n is summable (Ck) it follows that

$$\Sigma \frac{a_n}{(a+nx)^s}$$

is uniformly summable (Ck) in any interval $(0, \xi)$. Thus, *e.g.*,

$$\Sigma \frac{(-)^n n^t}{(a+nx)^s},$$

---

* *L.c.*, p. 85.

where $0 < t < k$, is uniformly summable (Ck) and is a continuous function of x for $x = +0$. In order to deal with this by my former theorems it was necessary to suppose $s > k+1$, while Mr. Bromwich's theorem required $s > k$—the series being then *convergent* except for $s = 0$.

Even in the theorem here proved, however, it must be observed that f_n is what Dr. Moore has called a *convergence factor*: its introduction into the series Σa_n makes that series, if not convergent, at any rate *more summable*. The series

$$\Sigma (-)^n n^t (a+nx)^s \quad (s > 0),$$

in which f_n is a *divergence* factor, and $\Sigma a_n f_n$ *less* summable than Σa_n, falls outside the scope of any theorem hitherto proved, though, of course, it may be dealt with easily enough by special devices.

The Theorems A, B, however, seem to me interesting less on account of any of their applications than as a contribution to the abstract theory of divergent series, and as marking something like the limit of what may reasonably be expected to be proved concerning the introduction of convergence factors into series summable by the method of mean values.

AN EXTENSION OF EISENSTEIN'S LAW OF RECIPROCITY

(SECOND PAPER.)

By A. E. WESTERN.

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1. This is a continuation of my former paper with the same title.* In the first part, a general method is given for determining all the reciprocal factors ψ or Ψ in the field of l^n-th roots of 1, l being an odd prime; and it is proved that, in general, a necessary and sufficient condition that ν should be primary is that the unit of Ψ_1 should be 1.

The second part contains results analogous to those of the first paper and Part I. of this paper, as regards the field of 2^n-th roots of 1, and the Eisensteinian law of reciprocity is established for this field.

The third part applies the general theory in Part II. to the fields of the 8-th and 16-th roots of 1. And the fourth part deals in like manner with the field of the 9-th roots of 1.

2. The laws of reciprocity given in these papers, including as a particular case Eisenstein's law, furnish complete solutions of the ancient problems in the theory of residues of powers of rational numbers, (1) to find the residue of $a^{(p-1)/x}$ mod p, without actual calculation; and more generally, (2), if a is given, to find what numbers p are such that $a^{(p-1)/x}$ has 1, or any other given residue mod p.

These problems for the case $x = 2$ led to the discovery of the quadratic law of reciprocity; and then, in order to deal with the case of $x = 4$, Gauss found that it was necessary to take into consideration the complex factors of the form $a+b\iota$ of which real primes of the form $4n+1$ are composed. From this, as its starting-point, there has been developed the theory of algebraic numbers, in which algebraic numbers are the primary elements, the data of the theory.

From that point of view, Eisenstein's law is incomplete, in that one

* *Proc. London Math. Soc.*, p. 16, *supra*. This is referred to in the present paper as the "first paper."

of the two numbers concerned must be rational; the law is merely a step (though an indispensable step) on the way to Kummer's general law of reciprocity between any two numbers of the field of l-th roots of 1. This incompleteness of Eisenstein's law, from the point of view of the theory of algebraic numbers, is probably the reason why neither Eisenstein himself nor any subsequent writer, so far as I can find, has called attention to its finality from the point of view of the ordinary theory of numbers.* In the following paragraph I indicate briefly how Eisenstein's law, as extended by me, furnishes the answers to the above-mentioned problems.

3. In the first place, these problems may be reduced to the case of x being the power of a prime. For, if $x = l^n l'^{n'} \ldots$, and $a^{(p-1)/x} \equiv r \pmod{p}$, then $a^{(p-1)/l^n} \equiv r^{x l^{-n}} \pmod{p}$, so that each factor l^n of x can be dealt with separately. Then let $x = l^n$, where l is 2 or an odd prime, and for brevity, let a be a prime q, different from l. And first assume that the prime factors of p in the field of l^n-th roots are ideals of the principal class, *i.e.*, are actual numbers, and let π be one of these factors in its primary form. Then

$$q^{(p-1)/l^n} \equiv \{q/\pi\} \pmod{\pi},$$

and, by the law of reciprocity,

$$\{q/\pi\} = \{\pi/q\},$$

and the latter depends upon the residues of π and its conjugates mod q, its explicit expression being given in the first paper, § 5, and in § 19 below. If, for instance, l is odd, and q is given, and p is to be found to satisfy

$$q^{(p-1)/l^n} \equiv 1 \pmod{p},$$

then the condition for p is that

$$\Pi\, \pi_t^{Q r^{-t}} \equiv 1 \pmod{q}, \quad (t = 0, 1, \ldots, f-1),$$

and another expression of this condition is

$$\psi_1 \ldots \psi_{q-1} \equiv 1 \pmod{q}.$$

Secondly, when the prime factors of p are non-actual ideals, let $\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_h$ be a set of h ideals, which respectively belong to the h classes of ideals existing in the field in question. Supposing such a set to be known, and that $\mathfrak{h}$ is that one of them which belongs to the opposite class

* See, for instance, H. J. S. Smith, *Report on the Theory of Numbers*, Art. 56. Kummer's law does not exist for certain values of l, such as 37, 59, and 67, and in these fields Eisenstein's law is the only law of reciprocity yet discovered.

to $\mathfrak{p}$, then $\mathfrak{h}\mathfrak{p}$ is an ideal of the principal class, say ν. Then ν being made primary, the law of reciprocity gives

$$\{q/\mathfrak{p}\} = \{\nu/q\}\{q/\mathfrak{h}\}^{-1},$$

and so the value of $\{q/\mathfrak{p}\}$ depends upon the residues of ν and its conjugates mod q.

And in either case, the congruence*

$$\{q/\mathfrak{p}\}^{q} \equiv \psi_1 \ldots \psi_{q-1} \pmod{q},$$

may be used for the calculation of $\{q/\mathfrak{p}\}$, that is, of $q^{(p-1)/l^n}$ (mod p).

PART I.

The Field of l^n-th Roots of 1, *l being an odd Prime.*

4. In the first paper, § 2, I defined ψ_{l^n-1}, in accordance with the definition of ψ_g there given, to be $-p$. But for the present purpose, it is more convenient to define ψ_{l^n-1}, and ψ_{l^n} by means of the property

$$\psi_g = \Sigma \zeta^{x+gy}$$

(see first paper § 10); whence we obtain

$$\psi_{l^n-1} = \psi_{l^n} = m-1+m\Sigma\zeta^t \quad (t = 1, 2, \ldots, l^n-1),$$

or, in the reduced form, $\psi_{l^n-1} = \psi_{l^n} = -1$.

Then the equation (14) in the first paper, § 10, holds for the exceptional cases, in which $g \equiv 0$ or -1 (mod l^{n-1}).

The following known properties of reciprocal factors

$$\psi_{-g-1} = \psi_g \tag{1}$$

and

$$\psi_{g'}(\zeta^g) = \psi_g, \tag{2}$$

where

$$gg' \equiv 1 \pmod{l^n},$$

are easily proved from either definition of ψ_g.

And, writing $\psi_g = \Sigma A_v^g \zeta^v$, in its unreduced form $(v = 0, 1, \ldots, l^n-1)$,

we obtain

$$A_v^{-g-1} = A_v^g, \tag{3}$$

and

$$A_{gv}^{g'} = A_v^g. \tag{4}$$

These results, which appear to have been discovered by Jacobi, will be useful in determining the units of the reciprocal factors in the field corresponding to given values of l and n.

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* First paper, § 7.

5. The method proposed in the first paper, § 19, for finding the unit of X is incomplete as there given. For the purpose of completing it, some additional results are needed connecting numbers respectively belonging to the fields of l^{n-1}-th and l^n-th roots of 1.

I denote ideals and numbers of the field $k(\zeta^l)$ by an asterisk (*).

$(x)^*$ means the least positive residue of x mod l^{n-1}, and

$[x]^*$ the greatest number not greater than xl^{-n+1}.

L means $l^{n-2}(l-1)$.

$\mathfrak{p}^*$ is that prime ideal factor of p which is divisible by $\mathfrak{p}$, so that $\mathfrak{p}^*$ satisfies the congruence

$$R^{-lm} \equiv \zeta^l \quad (\text{mod } \mathfrak{p}^*).$$

The primitive root r mod l^n is also a primitive root mod l^{n-1}, and will be used for the field $k(\zeta^l)$.

If $\mathfrak{p}_u$ is a factor of $\mathfrak{p}^*$, then

$$R^{-lm} \equiv \zeta^{lr^u} \quad (\text{mod } \mathfrak{p}_u),$$

so that

$$r^u \equiv 1 \quad (\text{mod } l^{n-1}),$$

that is, $u = xL$, where $x = 0, 1, \ldots, l-1$.

Therefore, if ν^* denotes the similar product of $\mathfrak{p}^*, \mathfrak{p}'^*, \ldots$ that ν is of $\mathfrak{p}, \mathfrak{p}', \ldots$, we obtain

$$\epsilon\nu_t^* = \Pi\nu_{t+xL} \quad (x = 0, 1, \ldots, l-1),$$

where ϵ is a unit of the field $k(\zeta^l)$. The result of putting $\nu' = \zeta^z\nu$ for ν in this equation is to introduce on the right the unit $\zeta^{zr^t\Sigma r^{xL}}$; now

$$(r^L-1)\,\Sigma r^{xL} = r^{lL}-1,$$

and r^L-1 is divisible by l^{n-1} and no higher power of l, so that

$$\Sigma r^{xL} = lv,$$

where v is prime to l. This substitution therefore enables us to make ϵ become a real unit, say ϵ'. And since putting $\epsilon'\nu_t^*$ for ν_t^* will not alter the expression of ψ_g^* in terms of ν^*, except perhaps as to sign,† we may omit ϵ', and write

$$\nu_t^* = \Pi\nu_{t+xL} \quad (x = 0, 1, \ldots, l-1). \tag{5}$$

The same reasoning shews that this equation specifies the power of ζ appropriate to ν, except as to a power of $\zeta^{l^{n-1}}$.

† First paper, § 12.

6. Let ρ be a primitive l-th root of 1, and $x_0, x_1, \ldots$ any algebraical quantities. Then

$$\Pi\,(x_0+x_1\rho^y+x_2\rho^{2y}+\ldots+x_{l-1}\rho^{(l-1)y}) \quad (y=0,\,1,\,\ldots,\,l-1)$$

is a rational integral function of the variables, with rational integral coefficients. It is, when $l>3$, a non-symmetric function. The number of ways in which any term $x_0^{a_0}x_1^{a_1}\ldots$ (other than $x_0^l, x_1^l, \ldots$) occurs in the expansion of the product is the multinomial coefficient

$$P\,(a_0,\,a_1,\,\ldots)=l!/a_0!\,a_1!\,\ldots,$$

which is a multiple of l. Suppose that ρ occurs as the coefficient of this term in c of these P ways; then, since the total coefficient is rational, every other power of ρ (except 1) must each occur as coefficient just c times, and the coefficient will be 1 in the remaining $P-c\,(l-1)$ times. Adding together all these coefficients the total coefficient of the term is $P-cl$. So in the expanded product the coefficient of every term except $x_0^l, x_1^l, \ldots$ is a multiple of l;† that is,

$$\Pi\,(x_0+x_1\rho^y+\ldots+x_{l-1}\rho^{(l-1)y}) \equiv x_0^l+x_1^l+\ldots+x_{l-1}^l \pmod{l}. \tag{6}$$

This supplies an elementary proof that the norm of any number prime to l of the field $k\,(\rho)$ is $\equiv 1 \pmod{l}$; for, if $x_0, x_1, \ldots$ are rational numbers, the left side is the product of Σx and the norm of

$$x_0+x_1\rho+\ldots+x_{l-1}\rho^{l-1},$$

and the right side is $\equiv \Sigma x \pmod{l}$. It may also be noted that the same reasoning applies to any other symmetric function of the l linear expressions occurring in this product.

7. Writing ρ for $\zeta^{l^{n-1}}$, $\nu=\Sigma a_v\zeta^v$, and

$$\sigma_h=\Sigma a_{lv+h}\,\zeta^{lv+h} \quad (v=0,\,1,\,\ldots,\,l^{n-1}-1),$$

we find that

$$\nu\,(\zeta^{1+yl^{n-1}})=\nu\,(\zeta\rho^y)=\sigma_0+\sigma_1\rho^y+\ldots+\sigma_{l-1}\rho^{(l-1)y}.$$

Applying the theorem (6), we obtain from (5),

$$\nu^* \equiv \Sigma\sigma_h^l \equiv \Sigma\sigma_h(\zeta^l) \equiv \nu\,(\zeta^l) \pmod{l}. \tag{7}$$

† I am indebted to Mr. F. J. W. Whipple for pointing out that the following important theorem may be directly deduced; viz., that the norm of $x_0+x_1\rho+\ldots+x_{l-1}\rho^{l-1}$ is congruent to $(x_0+x_1+\ldots+x_{l-1})^{l-1} \pmod{l}$, and consequently no possible term in the expansion of the norm is absent.

It follows that the sums of the coefficients of ν^* and ν are congruent mod l, and therefore that the value of E defined in the first paper, § 12, is the same in the fields $k(\zeta)$ and $k(\zeta^l)$.

8. Let Ω be the unreduced product $\Psi_1 \dots \Psi_{g-1}$, and $e' = E^{g-1}$, and g is at present supposed prime to l; then (11) of the first paper gives

$$e'\Omega^* = \zeta^{k^* l} \Pi \nu_t^{*[g r^*_{-t}]^*} \quad (t = 0, 1, \dots, L-1), \tag{8}$$

wherein k^* is supposed to be known, and

$$e'\Omega = \zeta^k [\Pi \nu_t^{[g r_{-t}]} + \phi Z + \phi'(\zeta^{l^n} - 1)] \quad (t = 0, 1, \dots, lL-1), \tag{9}$$

where k is unknown, $Z = 0$ is the irreducible equation satisfied by ζ, and ϕ and ϕ' are rational integral functions of ζ to be so chosen as to make (9) an identity in ζ. Then putting ζ^l for ζ in (9), we see, from the first paper, § 10, that Ω becomes $\equiv \Omega^*$ (mod l); also ν_t becomes $\nu_t(\zeta^l)$, which $\equiv \nu_t^*$ (mod l), and Z becomes l. So (9) gives

$$e'\Omega^* \equiv \zeta^{kl} \Pi \nu_t^{*[g r_{-t}]} \pmod{l} \quad (t = 0, 1, \dots, lL-1). \tag{10}$$

Now $\nu^*_{t+xL} = \nu^*_t$, so the index of ν^*_t on the right is

$$\Sigma [g r_{-t-xL}] \quad (x = 0, 1, \dots, l-1).$$

9. This index may be reduced as follows.

Let $$r_{-t} = r^*_{-t} + j l^{n-1} \quad (0 \leqslant j < l),$$

and $$r_{-L} = 1 + h l^{n-1} \quad (0 < h < l).$$

Then $$r_{-t-xL} \equiv r^*_{-t} + (j + xh r_{-t}) l^{n-1} \pmod{l^n},$$

and $$r_{-t-xL} = r^*_{-t} + x' l^{n-1}, \tag{11}$$

where x' is the least positive residue of $j + xhr_{-t}$ (mod l).

Again, $$g r_{-t-xL} \equiv g r^*_{-t} + g x' l^{n-1} \pmod{l^n},$$

so $$(g r_{-t-xL}) = (g r^*_{-t})^* + x'' l^{n-1}, \tag{12}$$

where x'' is the least positive residue of $[g r^*_{-t}]^* + gx'$ (mod l). As x runs over a complete set of residues mod l, so do x' and x''. We obtain therefore, from (11),

$$\Sigma_x r_{-t-xL} = r^*_{-t} l + \tfrac{1}{2}(l-1) l^n,$$

and, from (12), $$\Sigma_x (g r_{-t-xL}) = (g r^*_{-t})^* l + \tfrac{1}{2}(l-1) l^n.$$

Therefore $$\Sigma_x [g r_{-t-xL}] = [g r^*_{-t}]^* + \tfrac{1}{2}(g-1)(l-1).$$

10. Now $\Pi \nu_t^{*\frac{1}{2}(g-1)(l-1)}$ is a power of the norm of ν^*, and is therefore $\equiv 1 \pmod{l}$. So (10) gives

$$e'\Omega^* \equiv \zeta^{kl} \Pi \nu_t^{*[gr^*_{-t}]^*} \pmod{l} \quad (t = 0, 1, \ldots, L-1);$$

and, comparing this with (8), we find that

$$\zeta^{(k-k^*)l} \equiv 1 \pmod{l}.$$

Now, writing $\lambda = 1-\zeta$, we know that $l = \eta\lambda^{(l-1)l^{n-1}}$, where η is a unit. If d is prime to l, $(\zeta^d-1)(\zeta-1)^{-1}$ is a unit, say ϵ; and so

$$\zeta^d = 1-\epsilon\lambda,$$

and
$$\zeta^{dl^x} \equiv 1-(\epsilon\lambda)^{l^x} \pmod{l}.$$

Therefore the smallest value of x such that

$$\zeta^{dl^x} \equiv 1 \pmod{l}$$

is n, and so
$$k = k^* + hl^{n-1},$$
where h is still to be found.

11. Let $\Pi = \Sigma a_v \zeta^v$ $(v = 0, 1, \ldots, l^n-1)$ be the number $\Pi \nu_t^{[gr_{-t}]}$ in its simplest (or any) form; and let $\Pi^* = \Sigma a_v^* \zeta^{vl}$ be $e'\zeta^{-k^*l}\Omega^*$ in its unreduced form. Then (9) may be written

$$e'\Omega = \zeta^{hl^{n-1}+k^*}[\Pi + \phi Z + \phi'(\zeta^{l^n}-1)], \text{ identically}; \tag{13}$$

and putting ζ^l for ζ in this, and comparing the result with (8), we get

$$l\phi(\zeta^l) + \Pi(\zeta^l) = \Pi^*,$$

and so writing
$$\phi = x_0 + x_1\zeta + \ldots + x_{l^{n-1}-1}\zeta^{l^{n-1}-1},$$
and equating coefficients,

$$lx_t + \Sigma_s a_{t+sl^{n-1}} = a_t^*. \tag{14}$$

Now
$$e'\Omega(1) \equiv (n_0/l)^{g-1} \quad \text{and} \quad \dot{\Omega}(1) \equiv 0 \pmod{l^n};$$

so differentiating the identity (13), and then putting $\zeta = 1$, we obtain

$$(hl^{n-1}+k^*)(n_0/l)^{g-1} + \dot{\Pi}(1) + l\dot{\phi}(1) \equiv 0 \pmod{l^n}, \tag{15}$$

which determines h. So beginning with the degree l, for which the units are known, we can proceed in turn to find the units of Ω for the degrees l^2, l^3, …. And in particular the unit of $X = \Psi_1 \ldots \Psi_{r-1}$ is thus determined, and then, as shewn in the first paper, § 17, ν may be made primary. The law of reciprocity enunciated in the first paper, § 20, is therefore proved.

12. I shewed in the first paper, § 16, that if ν be primary, the unit of $\Psi_1 \ldots \Psi_{g-1}$ is 1, for all values of g prime to l; so taking

$$g = 2,\ 3,\ \ldots,\ l-1,\ l+1,\ \ldots,$$

the units of Ψ_g, whenever $g \not\equiv 0$ or -1, mod l, and of $\Psi_{gl-1}\Psi_{gl}$, for all values of g, are all 1.

Conversely, is ν primary when the unit of $\Psi_1 \ldots \Psi_{g-1}$ is 1? Let

$$g \equiv r^f \pmod{l^n},$$

r being chosen so that f is a factor of $l^{n-1}(l-1)$, and let $ef = l^{n-1}(l-1)$. Then the unit of $\Psi_1 \ldots \Psi_{r^f-1}$ is 1, and so

$$\Psi_1 \ldots \Psi_{r^f-1} = e'\Pi\nu_t^{[r^f r_{-t}]};$$

putting $\nu' = \zeta^x \nu$ for ν, the index of the unit of $\Psi_1 \ldots \Psi_{r^f-1}$ becomes

$$x\Sigma r^t [r^f r_{-t}] \quad (t = 0,\ 1,\ \ldots,\ ef-1)$$

$$= x\Sigma_v \Sigma_u r^{v+uf} [r^f r_{-v-uf}] \quad (u = 0,\ 1,\ \ldots,\ e-1;\ v = 0,\ 1,\ \ldots,\ f-1)$$

$$= x(r^{ef}-1)\, l^{-n} \Sigma_v r^v r_{-v+f}$$

$$\equiv x(r^{ef}-1)\, l^{-n} f r^f \pmod{l^n}.$$

This index is prime to l if f is so; and in that case ν is uniquely determined out of the set $\zeta^x\nu$, and must therefore be primary. In particular, taking $g = 2$, ν is primary when the unit of Ψ_1 is 1, unless $2 \equiv r^{xl} \pmod{l^n}$, that is, unless $2^{l-1} \equiv 1 \pmod{l^2}$. No value of l is known for which this congruence is true, and it is known to be untrue for all values of l less than 167.†

13. Though the units of Ψ_{gl-1} and Ψ_{gl} are not needed for the law of reciprocity, they will be useful in other applications. The unit of $\Omega = \Psi_1 \ldots \Psi_{g-1}$ when $g = vl$ is a power of ζ^{l^h}, l^h being the highest power of l in g (first paper, § 15).

In this case, we get by similar reasoning to that in § 9,

$$\Sigma x [g r_{-t-xL}] = [v r^*_{-t}]^* l + \tfrac{1}{2} g(l-1),$$

and so (10) now becomes

$$e'\Omega^* \equiv \zeta^{kl} \Pi \nu_t^{*[v r^*_{-t}]^* l} \pmod{l}.$$

Let $$\Omega^*_v = \Psi^*_1 \ldots \Psi^*_{v-1},$$

† A. Cunningham, *Quart. Jour. of Math.*, Vol. XXXVII., pp. 139, 142.

then $$e'\Omega_v^* = \zeta^{k_r^* l}\,\Pi\nu_t^{*\,[vr^*_{-t}]^*},$$

and so $$\Omega^*\Omega_v^{*-l} \equiv \zeta^{(k-k_v^*)l} \pmod{l}.$$

Now the power of ζ^l, say ζ^{il}, to which $\Omega^*\Omega_r^{*-l}$ is congruent, mod l, may be supposed to be known, all the reciprocal factors for the degree l^{n-1} being known. We therefore have

$$k = i + k_v^* l + h l^{n-1},$$

and h may now be determined as before.

i may be expressed in terms of $\{l/\nu\}$ for the field $k(\zeta^{l^2})$, which I write $\{l/\nu\}^{**}$. For the argument of § 6 of the first paper remains true if l is substituted for q, and we obtain

$$F(\zeta^l)^l \equiv \{l/\nu\}^{**} F(\zeta^{l^2}) \pmod{l},$$

so, writing $\{l/\nu\}^{**} = \zeta^{jl^2}$, and changing ζ into ζ^v,

$$F(\zeta^{vl})^l \equiv \zeta^{jvl^2} F(\zeta^{vl^2}) \pmod{l}.$$

But $$\Omega^* = F(\zeta^l)^{vl} F(\zeta^{vl^2})^{-1},$$

and $$\Omega_v^* = F(\zeta^l)^v F(\zeta^{vl})^{-1},$$

so $$\Omega^*\Omega_v^{*-l} = F(\zeta^{vl})^l F(\zeta^{vl^2})^{-1} \equiv \zeta^{jvl^2} \pmod{l},$$

and $$i \equiv jvl \pmod{l^{n-1}},$$

and $$k = (jv + k_v^*)\, l + h l^{n-1}.$$

We have hitherto supposed $v > 1$; when $v = 1$, since $[r^*_{-t}]^* = 0$, (10) becomes
$$e'\Omega^* \equiv \zeta^{kl} \pmod{l}.$$

Also $$\Omega^* = F(\zeta^l)^l F(\zeta^{l^2})^{-1}$$
$$\equiv \{l/\nu\}^{**} \equiv \zeta^{jl^2} \pmod{l}.$$

So $$k = jl + h l^{n-1}.$$

14. We have so far been considering the units of Ψ_g relatively to the number ν. The whole process is evidently applicable to ψ_g when p has actual prime factors. But when the prime factors of p are not actual, one cannot speak, in the same sense as before, of the unit of ψ_g. Let $\Pi_g = \Sigma a_v \zeta^v$ be a number given as the product $\Pi p_t^{[(g+1)r_{-t}]-[gr_{-t}]}$; and let $\psi_g^* = \zeta^{k^* l}\Pi_g^*$, identically, ψ_g^* and Π_g^* being supposed to be known. We have seen in § 11 that $\Pi_g(\zeta^l)$ is invariant, mod l, in whatever form Π_g is *i.e.*, whatever multiple of Z is added to Π_g.

Comparing $\Pi_g(\zeta^l)$ with Π_g^*, the value of y such that

$$\zeta^{yl}\Pi_g(\zeta^l) \equiv \Pi_g^* \pmod{l}$$

may be determined by inspection. Now, writing Π_g in future for $\zeta^y\Pi_g$, we get, as in § 11,

$$\psi_g = \zeta^{hl^{m-1}+k^*}[\Pi_g + \phi Z + \phi'(\zeta^{l^m}-1)] \text{ identically},$$

ϕ being determined as before by (14). Finally h is found as before by differentiation.

Π_g will be capable of being distinguished from its associates

$$\zeta^x\Pi_g \quad (x = 1, 2, \ldots, l^n-1)$$

by the residues, mod l, of its coefficients; we can therefore, when

$$\psi_g = \zeta^k\Pi_g,$$

call ζ^k the unit of ψ_g relatively to Π_g.

15. In the first paper I remarked, § 4, that a test is needed to pick one out of the set $\zeta^{hl}\nu$ $(h = 0, 1, \ldots, l^{n-1}-1)$ as the standard, and, § 22, that some process for making ν primary should be found applicable directly to ν, instead of to the reciprocal factors. Such a test is as follows. Let

$$\nu = \Sigma a_v\zeta^v \qquad (v = 0, 1, \ldots, l^n-1),$$

and

$$\nu_{(k)} = \Sigma a_{vl^k}\zeta^{vl^k} \quad (v = 0, 1, \ldots, l^{n-k}-1),$$

where a_{vl^k} is the coefficient of ζ^{vl^k} in ν, so that $\nu_{(0)} = \nu$. Then, provided that $\nu_{(k)}(1)$ is prime to l for $k = 1, \ldots, n-1$, one only, say ν, of the set $\zeta^h\nu$ satisfies the n conditions

$$\dot{\nu}_{(k)}(1) \equiv 0 \pmod{l}, \ (k = 0, 1, \ldots, n-1), \tag{16}$$

where the differentiation of $\nu_{(k)}$ is with respect to ζ^{l^k}. This property holds for any form of ν, and is therefore an essential property of ν as a number.† In applying this to a given number, the conditions are to be applied in the order $k = 0, 1, \ldots, n-1$; first, h_0 is found so that $\zeta^{h_0}\nu$ satisfies the first condition, then h_1 so that $\zeta^{h_1 l+h_0}\nu$ satisfies the second, and so on. The necessity for the proviso that $\nu_{(k)}(1)$ should be prime to l arises from the fact that, if not, the process breaks down; suppose, for instance, that ν satisfies the first condition; then the second condition gives

$$h_1\nu_{(1)}(1) + \dot{\nu}_{(1)}(1) \equiv 0 \pmod{l},$$

† Cf. the first paper, § 3.

which leaves h_1 indeterminate, if $\nu_{(1)}(1) \equiv 0 \pmod{l}$. The conditions, if satisfied by ν, are satisfied also by all its conjugates.

16. The method of the last paragraph has two defects, when we attempt to use it for the purpose of ascertaining when ν is primary; firstly, it is inapplicable when $\nu_{(k)}(1) \equiv 0 \pmod{l}$, for any value of k; and secondly, the property of satisfying the conditions (16) is not, in general, invariant for multiplication, or (what is the same thing in other words) the residues mod l satisfying the conditions do not form a group. But primary numbers form a group, for if ν and ν' are primary, then $\nu\nu'$ is evidently also primary; and we cannot therefore expect to find any simple relationship between a number ν satisfying (16), and the associate $\zeta^h\nu$ which is primary.

We have shewn that the index of ζ in the unit of Ψ_g is the residue mod l^n of a rational algebraic function of the coefficients of the reciprocal factor Π_g, and of other reciprocal factors of lower degrees, which function is expressible as a function of the coefficients of ν. It follows that, when $\nu' \equiv \nu \pmod{l^n}$ and ν is primary, ν' is also primary. It is shewn in Parts III. and IV. below that in the fields of 8-th, 16-th, and 9-th roots of 1, ν' is primary whenever $\nu' \equiv \nu \pmod{l}$; and it seems probable that this is true generally, for the numbers of the set $\zeta^x\nu$ are all incongruent mod l, and one only of them is primary.

17. If ϵ is any real unit, the substitution of $\epsilon\nu$ for ν multiplies the unit of any Ψ by the norm of ϵ in the field $k(\zeta+\zeta^{-1})$, that is, by ± 1; therefore $\epsilon\nu$ is primary when ν is primary. Accordingly the group of the real units is a sub-group of the group of primary numbers; and E the group of residues mod l of the real units is a sub-group of P, the group of residues mod l of primary numbers. $\{r\}$ is also a sub-group of P, and is of order $l-1$.

P contains residues, say $\gamma, \gamma', \ldots$, none of whose powers are contained in the group $\{E, r\}$, and it seems probable that some of such residues $\gamma, \gamma', \ldots$ may be selected generating a group J, such that P is the direct product of the groups E, $\{r\}$, and J. Assuming that this is so, the group G of all semi-primary residues mod l will be $\{\zeta^l, E, r, J\}$. Consequently a method of discovering the group P is to form the group E, and to complete the analysis of G by finding residues $\gamma, \gamma', \ldots$ generating a group J, none of whose residues belong to the group $\{\zeta^l, E, r\}$; then, if $\gamma, \gamma', \ldots$ are primary, $P = \{E, r, J\}$; and if not, let

$$\gamma_0 = \zeta^x\gamma, \quad \gamma_0' = \zeta^{x'}\gamma', \quad \ldots$$

be primary, then $\gamma_0, \gamma_0', \dots$ generate a group J_0, and we shall have

$$P = \{E, r, J_0\}, \quad \text{and} \quad G = \{\zeta^{t}, E, r, J_0\}.$$

I propose to call the residues of J or J_0, as the case may be, *canonical* residues. Then every number ν may, by multiplying it by a suitable unit, be made congruent to $r^x\gamma$ mod l, where γ is one of the canonical residues. The assumption here made as to the structure of the group G is justified in the cases dealt with in Parts III. and IV. below. The use of the canonical residues in practice facilitates the calculation of the residues of the coefficients of the reciprocal factors to mod l or powers of l, and consequently the calculation of the units of the Ψ's.

PART II.

The Field of 2^n-th Roots of 1.

18. There are some differences between this and the general case, but a large part of the reasoning in the first paper and in Part I. of this paper is valid for the case $l = 2$, and need not be repeated.

There are no primitive roots mod 2^n, but every odd residue is

$$\equiv \pm 5^u \pmod{2^n}.$$

r denotes 5 or some odd power of 5.

Then $$r^{2^{n-3}} \equiv 1+2^{n-1} \quad \text{and} \quad r^{2^{n-2}} \equiv 1 \pmod{2^n}.$$

The substitution s changes ζ to ζ^r and t changes ζ to ζ^{-1}.

The prime ideal factors of p are

$$\mathfrak{p}_u = s^u\mathfrak{p} = \mathfrak{p}(\zeta^{r^u}) \quad \text{and} \quad \mathfrak{p}_u^{\dagger} = s^u t\mathfrak{p} = \mathfrak{p}(\zeta^{-r^u}) \quad (u = 0, 1, \dots, 2^{n-2}-1).$$

e, the exponent to which q appertains, is a power of 2.

$q \equiv \pm r^f \pmod{2^n}$, r being supposed chosen so that f is a power of 2.

The different residues of q mod 2^n, and the corresponding values of e and f, are as follows:—

$$\begin{array}{llll}
e = 2^{n-2}, & q \equiv \pm 1+2^2 & \pmod{2^3}, & f = 1. \\
e = 2^{n-3}, & q \equiv \pm 1+2^3 & \pmod{2^4}, & f = 2. \\
\dots & \dots & \dots & \dots \\
e = 2^{n-k}, & q \equiv \pm 1+2^k & \pmod{2^{k+1}}, & f = 2^{k-2}. \\
\dots & \dots & \dots & \dots \\
e = 2, & \begin{cases} q \equiv \pm 1+2^{n-1} \\ q \equiv -1 \end{cases} & \begin{array}{l} \pmod{2^n}, \\ \pmod{2^n}, \end{array} & \begin{array}{l} f = 2^{n-3}. \\ f = 0. \end{array} \\
e = 1, & q \equiv \quad 1 & \pmod{2^n}, & f = 0.
\end{array}$$

It appears from this table that, except when $q \equiv \pm 1 \pmod{2^n}$,

$$ef = 2^{n-2}.$$

If $q \equiv 1 \pmod 4$, the prime factors of q are

$$\mathfrak{q}_u = s^u \mathfrak{q} \quad \text{and} \quad \mathfrak{q}_u^{\dagger} = s^u t \mathfrak{q},$$

where $u = 0, 1, \ldots, 2^{n-2}-1$, when $q \equiv 1 \pmod{2^n}$, but otherwise, $u = 0, 1, \ldots, f-1$; and, if $q \equiv -1 \pmod 4$, they are

$$\mathfrak{q}_u = s^u \mathfrak{q},$$

where $u = 0, 1, \ldots, 2^{n-2}-1$, when $q \equiv -1 \pmod{2^n}$, but otherwise, $u = 0, 1, \ldots, 2f-1$.

Z denotes $\zeta^{2^{n-1}}+1$, so that $Z = 0$ is the irreducible equation satisfied by ζ.

It will be useful to adopt the convention that, when $q \equiv 1 \pmod{2^n}$ f is 2^{n-2}; this is consistent with the definition of f. With this convention, the number of prime factors of q is $2f$, except in the case

$$q \equiv -1 \pmod{2^n},$$

which is peculiar.

19. As in the first paper, § 5, we find that, when $q \equiv 1 \pmod 4$,

$$\{\nu/q\} \equiv \Pi \nu_u^{Qr_{-u}} \nu_u^{\dagger\,-Qr_{-u}} \pmod q \quad (u = 0, 1, \ldots, f-1)$$

and

$$\{\nu/q\}^q \equiv \Pi \nu_{u+f}^{Qr_{-u}} \nu_{u+f}^{\dagger\,-Qr_{-u}} \pmod q.$$

And, when $q \equiv -1 \pmod 4$, except $q \equiv -1 \pmod{2^n}$,

$$\{\nu/q\} \equiv \Pi \nu_u^{Qr_{-u}} \pmod q \quad (u = 0, 1, \ldots, 2f-1)$$

and

$$\{\nu/q\}^q \equiv \Pi \nu_{u+f}^{-Qr_{-u}} \pmod q.$$

And, when $q \equiv -1 \pmod{2^n}$,

$$\{\nu/q\} \equiv \Pi \nu_u^{Qr_{-u}} \pmod q \quad (u = 0, 1, \ldots, 2^{n-2}-1),$$

and

$$\{\nu/q\}^q \equiv \Pi \nu_u^{-Qr_{-u}} \pmod q.$$

As in the first paper, §§ 6 and 7, we have

$$\{q/\nu\}^q \equiv \Psi_1 \ldots \Psi_{q-1} \pmod q.$$

The first paper, § 8, shews that the index of ν_u in $\Psi_1 \ldots \Psi_{g-1}$ is $[g r_{-u}]$; and the index of $\nu_u^{\dagger}$ in the same expression is then $g-1-[g r_{-u}]$, since either ν_u or $\nu_u^{\dagger}$ must occur in each Ψ.

When $q \equiv 1 \pmod 4$, the proof that $\Psi_1 \dots \Psi_{q-1}$ is congruent mod q to $\{\nu/q\}^q$, so far as powers of ν and its conjugates are concerned, is similar to that in the first paper, § 9.

Secondly, when $q \equiv -1 \pmod 4 \equiv -r^f \pmod{2^n}$, we find that

$$\nu_u^q \equiv \nu_{u+}^\dagger \qquad \pmod q,$$

and so

$$\nu_u^{q^x} \equiv \nu_{u+xf} \text{ or } \nu_{u+xf}^\dagger \quad \pmod q,$$

according as x is even or odd.

In particular,

$$\nu_u^{q^e-1} \equiv 1 \quad \pmod q.$$

Since $[x]+[-x] = -1$, we have

$$\Psi_1 \dots \Psi_{q-1} = \epsilon \Pi \nu_u^{-1-[-qr_{-u}]} \nu_u^{\dagger q+[-qr_{-u}]},$$

where ϵ is a unit.

Now

$$\Pi \nu_u^{-1-[-qr_{-u}]} = \Pi_v \Pi_x \nu_{v-(2x+1)f}^{-1-[-qr_{-v+(2x+1)f}]}$$

and

$$\Pi \nu_u^{\dagger q+[-qr_{-u}]} = \Pi_v \Pi_x \nu_{v-2xf}^{\dagger q+[-qr_{-v+2xf}]},$$

where $v = 0, 1, \dots, 2f-1$ and $x = 0, 1, \dots, \frac{1}{2}e-1$.

So, expressing $\nu_{v-(2x+1)f}$ and $\nu_{v-2xf}^\dagger$ as powers of ν_{v+f}, the index of ν_{v+} in $\Psi_1 \dots \Psi_{q-1}$ is

$$\Sigma_x \{-1-[-qr_{-v+(2x+1)f}]\} q^{e-2x-2} + \Sigma_x \{q+[-qr_{-v+2xf}]\} q^{e-2x-1}$$
$$= \Sigma_x (q^{e-2x} - q^{e-2x-2}) - \Sigma_y [-qr_{-v+yf}](-q)^{e-y-1} \quad (y = 0, 1, \dots, e-1).$$

Now

$$(-qr_{-v+yf}) = r_{-v+(y+1)f};$$

so the index is q^e-1-Qr_{-v}, which proves the congruence mod q of $\Psi_1 \dots \Psi_{q-1}$ and $\{\nu/q\}^q$, so far as ν and its conjugates are concerned. Similar reasoning applies to the case of $q \equiv -1 \pmod{2^n}$.

As in the first paper, § 12, when $\nu(1) \equiv 0 \pmod 2$,

$$\Psi_g = \zeta^{2k} \Pi \nu_u^x \nu_u^{\dagger 1-x},$$

where $x = [(g+1)r_{-u}] - [gr_{-u}] = 0$ or 1. There is no need to introduce the unit -1, since it is $\zeta^{2^{n-1}}$.

And, as in the first paper, § 15, the unit of $F(\zeta)^{2^n}$ is the same as the unit of $F(-1)^2$. But $F(-1)^2 = p$, when $p \equiv 1 \pmod 4$, so the unit of $F(\zeta)^{2^n}$ is 1.

20. Corresponding to the relations (1) and (3) of § 4, *supra*, we now have

$$\psi_{-g-1} = (-1)^m \psi_g, \tag{17}$$

$$A_v^{-g-1} = A_{v+2^{n-1}m}^g. \tag{18}$$

And, in particular,

$$\psi_{2^n-1} = (-1)^{m+1}, \qquad \psi_{2^n} = -1.$$

(2) and (4) remain true when $l = 2$.

Writing the norm of ν, $N = 2^n M+1$, we have $M \equiv \Sigma m \pmod 2$, the sum being taken over all the rational prime factors of N, and so

$$\Psi_{-g-1} = (-1)^M \Psi_g.$$

We also find‡ that $\quad \dot{\psi}_g(1) \equiv (g+1)2^{n-1} m \pmod{2^n}$,

so that, when $\quad p \equiv 1 \pmod{2^{n+1}}, \quad \dot{\psi}_g(1) \equiv 0 \pmod{2^n}$,

and when $\quad p \equiv 1+2^n \pmod{2^{n+1}}, \quad \dot{\psi}_g(1) \equiv 0$ or $2^{n-1} \pmod{2^n}$,

according as g is odd or even.

And it is easily deduced that

$$\dot{\Psi}_g(1) \equiv (g+1)2^{n-1} M \pmod{2^n}, \tag{19}$$

and that, if $\quad \Omega_g = \Psi_1 \ldots \Psi_{g-1}$,

$$\dot{\Omega}_g(1) \equiv 0 \qquad \pmod{2^n},$$

when $g \equiv 1$ or $2 \pmod 4$, and

$$\dot{\Omega}_g(1) \equiv 2^{n-1} M \pmod{2^n},$$

when $g \equiv 3$ or $0 \pmod 4$.

As in (5), *supra*, we have

$$\nu_u^* = \nu_u \nu_{u+2^{n-2}} = \nu_u \nu_u(-\zeta).$$

And this specifies the power of ζ appropriate to ν, except as to a power of $\zeta^{2^{n-2}}$.

And, as in § 7, *supra*, $\nu_u = \sigma_0+\sigma_1$, $\nu_{u+2^{n-2}} = \sigma_0-\sigma_1$; so

$$\nu_u^* = \sigma_0^2-\sigma_1^2 \equiv \nu_u(\zeta^2) \pmod 2.$$

21. The reasoning of §§ 8, 9, and 10 applies when g is odd, the only difference being that here the smallest value of x such that $\zeta^{d2^x} \equiv 1 \pmod 2$ is $x = n-1$, instead of n, so that

$$k = k^* + h2^{n-2}.$$

Whether h is $\equiv 0$ or $1 \pmod 2$ may be determined, as in § 14, by comparison of the residues mod 2 of the coefficients of Π_g and Π_g^*. In

‡ Cf. H. J. S. Smith, *Report on the Theory of Numbers*, Art. 60, foot-note.

fact, using the same notation as in § 11, except that Π is reduced to its simplest form, viz., $\Sigma a_v \zeta^v$ ($v = 0, 1, \ldots, 2^{n-1}-1$), Π_g^{**} must have at least one odd coefficient; calling this a_u^{**}, since

$$a_u^* + a_{u+2^{n-2}}^* \equiv a_u^{**} \pmod 2,$$

$$a_u \not\equiv a_{u+2^{n-2}}^* \pmod 2.$$

So, if $$a_u \equiv a_u^*, \qquad a_{u+2^{n-2}} \equiv a_{u+2^{n-2}}^* \pmod 2,$$

h is even; if $$a_u \equiv a_{u+2^{n-2}}^*, \qquad a_{u+2^{n-2}} \equiv a_u^* \pmod 2,$$

h is odd. We therefore obtain

$$\Omega = \zeta^{h'2^{n-1}+k^*}[\Pi_g + \phi Z + \phi'(\zeta^{2^n}-1)]$$

identically, wherein $\Pi_g = \zeta^{2^{n-2}h}\Pi\nu_u^{[gr_{-u}]}\nu_u^{+g-1-[gr_{-u}]}$,

and h is 0 or 1, as above found. And, finally, by differentiating this identity, and then putting $\zeta = 1$, we determine h'.

22. The process of § 15 applies here with two modifications. The first is that the last of the conditions (16), corresponding to $k = n-1$, must be omitted; the other conditions (16), in the present case, fix ν except as to sign, and it is unnecessary for the present purposes to fix the sign of ν, for the unit of each Ψ is unaffected by changing the sign of ν. The other modification is that $\nu_{(k)}(1)$ is always prime to 2; for

$$\nu_{(k)}(1) \equiv \nu_{(k-1)}(1) - \dot{\nu}_{(k-1)}(1) \equiv \nu_{(k-1)}(1) \equiv 1 \pmod 2.$$

The unit of every Ψ may be made of the form ζ^{4k} by choosing $\nu = \Sigma a_v \zeta_v$ out of the set ν, $\zeta\nu$, $\zeta^2\nu$, $\zeta^3\nu$, so as to satisfy the first two of the conditions (16); these may be written

$$\left.\begin{aligned} \Sigma a_{4v+1} &\equiv \Sigma a_{4v+3} \pmod 2 \\ \Sigma a_{4v+2} &\equiv 0 \pmod 2 \end{aligned}\right\}. \tag{20}$$

And it follows from these that $\Sigma a_{4v} \equiv 1 \pmod 2$.

This property is invariant for multiplication. For, let $\mu = \Sigma b_v \zeta_v$ be another number satisfying (20), and let $\mu\nu = \Sigma c_v \zeta^v$ and $a'_v = \Sigma_u a_{4u+v}$, with similar notation for μ and $\mu\nu$. The conditions (20) then are $a'_1 \equiv a'_3$, $a'_2 \equiv 0 \pmod 2$. Then, as in the first paper, § 18,

$$c'_v \equiv \Sigma_x a'_x b'_{v-x} \pmod 2, \quad (x = 0, 1, 2, 3),$$

whence $$c'_1 \equiv c'_3, \quad c'_2 \equiv 0 \pmod 2.$$

In illustration of my assertion, in § 16, that the residues mod l

satisfying (16) do not in general form a group, I may remark that, if μ and ν, besides satisfying (20), also satisfy the third condition of (16), viz., $\Sigma a_{8v+4} \equiv 0 \pmod 2$, $\mu\nu$ does not necessarily satisfy this. A number prime to 2 and satisfying (20) will be called *semi-primary*.

The reciprocal factors ψ are semi-primary; for, writing

$$\psi_g = \Sigma A^g_u \zeta^u, \quad \text{and} \quad \psi_g(\zeta^{2^{n-2}}) = a^g_0 + a^g_1 \zeta^{2^{n-2}} + a^g_2 \zeta^{2.2^{n-2}} + a^g_3 \zeta^{3.2^{n-2}},$$

$\psi_g(\zeta^{2^{n-2}})$ being a reciprocal factor of p in the field of 4-th roots of 1, we have, as in the first paper, § 10, $\Sigma_u A^g_{4u+v} = a^g_v$.

Now, if $g \equiv 0$ or $-1 \pmod 4$,

$$a^g_0 = 2^{n-2}m - 1, \qquad a^g_1 = a^g_2 = a^g_3 = 2^{n-2}m;$$

and, if $g \equiv 1$ or $2 \pmod 4$, it is known that, when $p \equiv 1 \pmod 8$,

$$a^g_1 \equiv a^g_2 \equiv a^g_3 \equiv 0 \pmod 2.$$

Hence each Ψ also possesses this property, and therefore its unit must be ζ^{4k}.

23. In particular, the unit of $X = \Psi_1 \Psi_2 \Psi_3 \Psi_4$ is thus determined, and is of the form ζ^{4k}. The substitution of $\nu' = \zeta^{2x}\nu$ for ν in the expression for X in terms of ν (as in the first paper, § 17) causes the index of the unit to become $4k + 4hx$, where h is odd. So, by proper choice of x, ν may be made primary, *i.e.*, such that the unit of X is 1. It will be noticed that, if k be even, ν, when primary, is also semi-primary; but if k be odd, ν satisfies

$$\Sigma a_{4v+2} \equiv 1, \quad \Sigma a_{4v} \equiv 0 \pmod 2.$$

Accordingly, when $q \equiv 1 \pmod 4$,

$$\{\nu/q\}^q \equiv \{q/\nu\}^q \pmod q,$$

that is,

$$\{\nu/q\} = \{q/\nu\}.$$

Again, since

$$F(\zeta^q)\,F(\zeta^{-q}) = (-1)^m p,$$

$$\mathrm{F}(\zeta^q)\,\mathrm{F}(\zeta^{-q}) = (-1)^M N;$$

and therefore the unit of $\mathrm{F}(\zeta)^q\,\mathrm{F}(\zeta^q)^{-1}$ is $(-1)^M$ times the unit of $\mathrm{F}(\zeta)^{-q}\,\mathrm{F}(\zeta^{-q})^{-1}$; so that, if $q \equiv -r^f \pmod{2^n}$, and ν is primary, the unit of $\Psi_1 \dots \Psi_{q-1}$ is $(-1)^M$.

So, in this case, when $q \equiv -1 \pmod 4$, we obtain

$$\{\nu/q\} = (-1)^M \{q/\nu\},$$

that is

$$\{\nu/-q\} = \{-q/\nu\}.$$

With the convention that q is taken with such sign as to make $q \equiv 1 \pmod 4$, we have in each case the law—

$$\{\nu/q\} = \{q/\nu\}.$$

This is capable of the same extension as before (first paper, § 20), namely :—

If ν is any primary number of the field of 2^n-th roots of 1, and a is any odd rational number prime to ν, taken with such sign as to be $\equiv 1$ (mod 4), then

$$\{\nu/a\} = \{a/\nu\}.$$

24. Corresponding to § 12, we find that, when ν is primary, the unit of $\Psi_{2g-1}\Psi_{2g}$ is $(-1)^M$ for all values of g. And conversely, if the unit of $\Psi_1\Psi_2$ is $(-1)^M$, then ν is primary. We see that in the present case, there are no reciprocal factors whose units are always 1 when ν is primary, similar to Ψ_1, Ψ_2, ... when l is odd; but the unit of each separate Ψ_g must be determined by the process of § 13. This fact is the real reason for the difficulty referred to in § 13 of the first paper. We may here take G to be the group of all semi-primary residues mod 2, and E' to be $\{\zeta^4, E\}$, for every real unit ϵ is semi-primary. The substitution of $\epsilon\nu$ for ν leaves the unit of $\Psi_1\Psi_2$ unaltered, and so $\epsilon\nu$ is primary, when ν is primary. Canonical residues may be found as before, forming a group J, and P the group of primary residues mod 2 will then be $\{E, J\}$.

25. *The Units of $\psi_{2^{n-1}-1}$, and $\psi_{2^{n-1}}$, when* $n > 2$.—We find, from (2), that

$$\psi_{2^{n-1}-1} = \psi_{2^{n-1}-1}(\zeta^{2^{n-1}-1}) = s^{2^{n-3}}t\psi_{2^{n-1}-1};$$

so $\psi_{2^{n-1}-1}$ is a number of the field $k(\zeta+\zeta^{2^{n-1}-1})$, and we may write

$$\psi_{2^{n-1}-1} = (-1)^h\,\Pi_{2^{n-1}-1},$$

where $$\Pi_{2^{n-1}-1} = b_0+\Sigma b_v[\zeta^v+(-1)^{v+1}\zeta^{2^{n-1}-v}] \quad (v = 1, 2, \ldots, 2^{n-2}-1).$$

Applying the process of § 11, and remembering that

$$\psi^*_{2^{n-1}-1} = 2m-1+2m\Sigma\zeta^{2v} \quad (v = 1, 2, \ldots, 2^{n-1}-1),$$

we find $$2x_0+b_0 = 2m-1,$$

$$2x_v+b_v = 2m = 2x_{2^{n-1}-v}+(-1)^{v+1}b_v \quad (v = 1, \ldots, 2^{n-2}-1),$$

and $$x_{2^{n-2}} = m;$$

and so $\qquad b_0 \equiv 1 \pmod 2, \qquad b_v \equiv 0 \pmod 2.$

Accordingly $\quad \Pi_{2^{n-1}-1} + \phi Z = m + \tfrac{1}{2}(b_0 - 1) + [m - \tfrac{1}{2}(b_0+1)]\,\zeta^{2^{n-1}}$

$$+\Sigma_u\left[(m+\tfrac{1}{2}b_u)(\zeta^u+\zeta^{2^{n-1}-u})+(m-\tfrac{1}{2}b_u)(\zeta^{2^{n-1}+u}+\zeta^{2^n-u})\right] \quad (u \text{ odd})$$

$$+\Sigma_v\left[(m+\tfrac{1}{2}b_v)(\zeta^v+\zeta^{2^n-v})+(m-\tfrac{1}{2}b_v)(\zeta^{2^{n-1}+v}+\zeta^{2^{n-1}-v})\right] \quad (v \text{ even})$$

$$+m(\zeta^{2^{n-2}}+\zeta^{3\cdot 2^{n-2}}).$$

Differentiating, we obtain

$$h \equiv m + \tfrac{1}{2}(b_0+1) \pmod 2.$$

If $\Pi_{2^{n-1}-1}$ is taken with such sign as to make $b_0 \equiv 1 \pmod 4$; then the unit of $\psi_{2^{n-1}-1}$ is $(-1)^{m+1}$, and consequently the unit of $\psi_{2^{n-1}}$ is -1. We easily deduce that

$$A_{2^{n-1}}^{2^{n-1}-1} \equiv 0, \qquad A_{2^{n-1}}^{2^{n-1}} \equiv m \pmod 2.$$

Similarly, when $n = 2$, the process of § 11 gives at once that

$$\psi_1 = -(a - b\iota),$$

where $a \equiv 1 \pmod 4$, as is already known.

The fact that $A_{2^{n-1}}^{2^{n-1}} \equiv m \pmod 2$ may also be derived directly from its definition as the number of solutions of

$$R^x + R^y \equiv 1 \pmod p,$$

which satisfy $\qquad x + 2^{n-1}y \equiv 2^{n-1} \pmod{2^n},$

using reasoning which, though simpler in form, is essentially the same as that given by H. J. S. Smith‡ for the case of 8-th roots of 1.

26. $\psi_{2^{n-1}}$ and ψ_1 are the same numbers, except as regards units. For Kummer's theorem‖ shews that ψ_1 contains $\mathfrak{p}_u$, if $r_{-u} > 2^{n-1}$; and that $\psi_{2^{n-1}}$ contains $\mathfrak{p}_u$, if

$$r_{-u} + (2^{n-1}r_{-u}) > 2^n.$$

But, r_{-u} being odd, $(2^{n-1}r_{-u}) = 2^{n-1}$, so the condition becomes

$$r_{-u} > 2^{n-1}.$$

Since $\psi_{2^{n-1}}$ is unaltered by the substitution $s^{2^{n-3}}t$, if it contains $\mathfrak{p}_u$, it must also contain $\mathfrak{p}_{u+2^{n-3}}$[†].

‡ *Report on the Theory of Numbers*, Art. 121.

‖ First paper, § 8.

The results of this and the previous paragraph hold also for $\Psi_{2^{n-1}-1}$, which is $\zeta^k\Pi_1$, where
$$\Pi_1 = \Pi \nu_u \nu^{\dagger}_{u+2^{n-3}} \quad (u \text{ such that } r_{-u} > 2^{n-1}).$$
Since both $\Psi_{2^{n-1}-1}$ and Π_1 are unaltered by $s^{2^{n-3}}t$, so also is ζ^k; that is, the unit is $(-1)^k$.

27. *The Units of* $\psi_{2^{n-1}+1}$ *and* ψ_1.—Here we find, from (2), that
$$\psi_{2^{n-1}+1} = \psi_{2^{n-1}+1}(\zeta^{2^{n-1}+1}) = s^{2^{n-3}}\psi_{2^{n-1}+1},$$
and so $\psi_{2^{n-1}+1}$ is a number of the field $k(\zeta^2)$.

Now‡
$$\psi_{2g}\psi_{2g+1} = \psi_1\psi^*_g,$$
which, for $g = 2^{n-2}$, gives
$$\psi_{2^{n-1}}\psi_{2^{n-1}+1} = \psi_1\psi^*_{2^{n-2}}.$$
Now
$$\psi_{2^{n-1}} = (-1)^{\frac{1}{2}(b_0+1)}\Pi_1, \qquad \psi^*_{2^{n-2}} = (-1)^{\frac{1}{2}(b^*_0+1)}\Pi^*_1,$$
where b_0 and b^*_0 are the absolute terms in Π_1 and Π^*_1 respectively; so writing
$$\psi_1 = \theta_1\Pi_1, \qquad \psi_{2^{n-1}+1} = \theta_{2^{n-1}+1}\Pi_{2^{n-1}+1},$$
we find
$$\Pi_{2^{n-1}+1} = \Pi^*_1,$$
and
$$\theta_{2^{n-1}+1} = (-1)^{\frac{1}{2}(b_0-b^*_0)}\theta_1,$$
or, provided that $b_0 \equiv b^*_0 \pmod 4$,
$$\theta_{2^{n-1}+1} = \theta_1.$$

The coefficient of ζ^v in ψ_1 is the number of solutions of
$$R^x + R^y \equiv 1 \pmod p,$$
such that
$$x+y \equiv v \pmod{2^n}.$$
These solutions occur in pairs x, y, and y, x, except when $x = y = x'$ say, and then $2R^{x'} \equiv 1 \pmod p$. Therefore the coefficient of $\zeta^{2x'}$ in ψ_1 is odd, and the other coefficients are even. Now $\zeta^{2x'} = \{2/\mathfrak{p}\}^2$, and so, comparing ψ_1 with Π_1, we see that
$$\theta_1 = \pm\{2/\mathfrak{p}\}^2.$$

‡ First paper, § 11 (15), taking $g = 2$.

PART III.

The Fields of 8-*th and* 16-*th Roots of* 1.

28. In these fields there is only one class of ideals, the principal class, so that all ideals are actual numbers. We may therefore take ν to be a prime factor π of the rational prime p. I use throughout $r = 5$.

I give for reference the results in the field of 4-th roots of 1 which will be needed, namely, those for the case $p \equiv 1 \pmod 8$. Here we have

$$p = 8m+1, \qquad \pi = a+b\iota,$$

where
$$a \equiv 1, \quad b \equiv 0 \pmod 4, \qquad \text{and} \qquad p = a^2+b^2;$$

$$\psi_1 = \psi_2 = -(a-b\iota) = \Sigma A_\nu \zeta^\nu,$$

where
$$\begin{aligned} A_0 &= 2m-\tfrac{1}{2}(a+1), \\ A_1 &= 2m+\tfrac{1}{2}b, \\ A_2 &= 2m+\tfrac{1}{2}(a-1), \\ A_3 &= 2m-\tfrac{1}{2}b. \end{aligned}$$

29. *The Field of* 8-*th Roots of* 1.‡—I write ω for $\zeta+\zeta^3$. Here the fundamental real unit is $\epsilon = 1+\zeta+\zeta^{-1}$, and $\epsilon^2 \equiv 1 \pmod 2$, so the group E of § 17 is of order 2. Writing

$$\pi = a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3,$$

the semi-primary conditions are

$$a_1 \equiv a_3, \quad a_2 \equiv 0 \pmod 2.$$

If $a_1 \equiv 1 \pmod 2$, then $\pi \equiv \epsilon \pmod 2$, and $\epsilon\pi \equiv 1 \pmod 2$. The canonical form of π is therefore $\pi \equiv 1 \pmod 2$, that is

$$a_0 \equiv 1, \quad a_1 \equiv a_2 \equiv a_3 \equiv 0 \pmod 2.$$

The sign of π is immaterial, π and $-\pi$ being both canonical. The conjugates of π are

$$\pi_1 = \pi(-\zeta), \qquad \pi^\dagger = \pi(\zeta^{-1}), \qquad \pi_1^\dagger = \pi(-\zeta^{-1}).$$

With π in its canonical form, the equation $\pi^\bullet = \pi\pi_1$ holds good, $\pi^\bullet$ being $a+b\zeta^2$, for we find

$$\left.\begin{aligned} a &= a_0^2-a_2^2+2a_1a_3 \\ b &= -a_1^2+a_3^2+2a_0a_2 \end{aligned}\right\}. \tag{21}$$

‡ Jacobi stated the units of the reciprocal factors in this field, without proof or any indication of the method of their discovery, in "Ueber die Kreistheilung," *Crelle's Journal*, Vol. xxx., p. 166; *Collected Works*, Vol. vi., p. 254.

So $$a \equiv 1+a_2^2 \pmod 8,$$
but $$a \equiv 1+4m \pmod 8,$$
and so $$a_2 \equiv 2m \pmod 4.$$
Let $c+d\omega = \pi\pi_1^\dagger$, giving $p = c^2+2d^2$, and
$$\left.\begin{aligned} c &= a_0^2-a_1^2+a_2^2-a_3^2 \\ d &= a_0a_1-a_1a_2+a_2a_3+a_3a_0 \end{aligned}\right\}. \tag{22}$$
Whence we easily obtain
$$c \equiv 1 \pmod 4, \quad d \equiv a_1+a_3 \pmod 4,$$
and $$\tfrac{1}{4}(c-1) \equiv \tfrac{1}{4}b \equiv \tfrac{1}{2}d+m \pmod 2.$$
We know, from § 22, that the units of the reciprocal factors are ± 1; so applying the results of §§ 25–27, we see that
$$\psi_1 = (-1)^k(c-d\omega), \qquad \psi_6 = (-1)^{k+m}(c-d\omega),$$
$$\psi_3 = (-1)^{m+1}(c-d\omega), \qquad \psi_4 = -(c-d\omega),$$
$$\psi_5 = (-1)^k(a-b\zeta^2), \qquad \psi_2 = (-1)^{k+m}(a-b\zeta^2).$$

30. Applying the process of § 11 to ψ_1, in order to determine k, we have
$$2x_0+c = \tfrac{1}{2}(4m-a-1), \qquad 2x_1-d = \tfrac{1}{2}(4m+b),$$
$$2x_2 \quad = \tfrac{1}{2}(4m+a-1), \qquad 2x_3-d = \tfrac{1}{2}(4m-b).$$
And so $\phi(1) \equiv 1 \pmod 2$, $\quad \dot\phi(1) = x_1-x_3+2x_2 \equiv \tfrac{1}{2}b \pmod 4$,
whence $$k \equiv \tfrac{1}{4}b+1 \pmod 2.$$

The coefficients A_i^q of the reciprocal factors are therefore as follows:—

If $b \equiv 0 \pmod 8$,
$$A_0^1 = \tfrac{1}{4}(4m-a-2c-1), \qquad A_4^1 = \tfrac{1}{4}(4m-a+2c-1),$$
$$A_1^1 = \tfrac{1}{4}(4m+b+2d), \qquad A_5^1 = \tfrac{1}{4}(4m+b-2d),$$
$$A_2^1 = \tfrac{1}{4}(4m+a-1) \quad = \quad A_6^1,$$
$$A_3^1 = \tfrac{1}{4}(4m-b+2d), \qquad A_7^1 = \tfrac{1}{4}(4m-b-2d).$$
$$A_0^5 = \tfrac{1}{4}(4m-3a-1), \qquad A_4^5 = \tfrac{1}{4}(4m+a-1),$$
$$A_1^5 = \tfrac{1}{4}(4m+b) \quad = \quad A_5^5,$$
$$A_2^5 = \tfrac{1}{4}(4m+a+2b-1), \qquad A_6^5 = \tfrac{1}{4}(4m+a-2b-1),$$
$$A_3^5 = \tfrac{1}{4}(4m-b) \quad = \quad A_7^5.$$

And if $b \equiv 4 \pmod 8$, A_v^g has the value given above to A_{v+4}^g.

If $m \equiv 0 \pmod 2$,

$$A_0^8 = \tfrac{1}{2}(2m-c-1),$$

$$A_1^8 = A_3^8 = \tfrac{1}{2}(2m+d),$$

$$A_2^8 = A_6^8 = m,$$

$$A_4^8 = \tfrac{1}{2}(2m+c-1),$$

$$A_5^8 = A_7^8 = \tfrac{1}{2}(2m-d).$$

And if $m \equiv 1 \pmod 2$, A_v^8 has the value given above to A_{v+4}^8.

Since the unit of $\psi_1\psi_2$ is $(-1)^m$, π is primary.

And $\epsilon\pi$ is also primary, so that the primary residues, to one of which every π may be reduced by multiplication by a power of ζ, are

$$1 \quad \text{and} \quad 1+\zeta+\zeta^8 \pmod 2.$$

When π is of the second form, since $\epsilon.\epsilon(\zeta^8) = -1$, each of the units of the ψ's above stated must be changed in sign. The primary residues may also be written in the form

$$1+y_1(\lambda^2+\lambda^8) \pmod 2,$$

where $y_1 = 0$ or 1, λ being $1-\zeta$, a factor of 2.

31. *The Field of* 16-*th Roots of* 1.

Let $\omega_1 = \zeta+\zeta^7, \quad \omega_2 = \zeta^2-\zeta^6, \quad \omega_3 = \zeta^3+\zeta^5,$

and $\varpi_1 = \zeta+\zeta^{-1}, \quad \varpi_2 = \zeta^2+\zeta^{-2}, \quad \varpi_3 = \zeta^3+\zeta^{-3}.$

The fundamental real units here are

$$\epsilon_1 = 1+\varpi_1,$$

$$\epsilon_2 = 1+\varpi_1+\varpi_2,$$

$$\epsilon_3 = 1+\varpi_1+\varpi_2+\varpi_3.$$

We find that $\epsilon_2 \equiv \epsilon_1\epsilon_3 \pmod 2, \quad \epsilon_1^4 \equiv 1, \ \epsilon_3^2 \equiv 1 \pmod 2,$

so the group E is $\{\epsilon_1, \epsilon_3\}$ of order 8.

The residues of E, and the simplest corresponding units, are as follows:—

Residues of E.		Simplest Units.
1	1	
$\epsilon_1 \equiv 1+\varpi_1$	$\equiv \epsilon_1$	$= 1+\varpi_1$
$\epsilon_1^2 \equiv 1 \quad +\varpi_2$	$\equiv \epsilon^*$	$= 1 \quad +\varpi_2$
$\epsilon_1^3 \equiv 1 \quad +\varpi_2+\varpi_3$	$\equiv \epsilon_2(\zeta^3)$	$= 1 \quad -\varpi_2+\varpi_3$
$\epsilon_3 \equiv 1+\varpi_1+\varpi_2+\varpi_3$	$\equiv \epsilon_3$	$= 1+\varpi_1+\varpi_2+\varpi_3$
$\epsilon_1\epsilon_3 \equiv 1+\varpi_1+\varpi_2$	$\equiv \epsilon_2$	$= 1+\varpi_1+\varpi_2$
$\epsilon_1^2\epsilon_3 \equiv 1+\varpi_1 \quad +\varpi_3$	$\equiv -\epsilon^*\epsilon_3(\zeta^3)$	$= 1+\varpi_1 \quad +\varpi_3$
$\epsilon_1^3\epsilon_3 \equiv 1 \quad +\varpi_3$	$\equiv \epsilon_1(\zeta^3)$	$= 1 \quad +\varpi_3$

The norms of ϵ_1 and ϵ_3 in the field $k(\zeta+\zeta^{-1})$ are -1 and 1 respectively.

Writing $$\pi = \Sigma b_v \zeta^v \quad (v = 0, 1, \ldots, 7),$$

the semi-primary conditions are, the modulus being 2,

$$b_1+b_3+b_5+b_7 \equiv 0,$$
$$b_2+b_6 \equiv 0,$$
$$b_0+b_4 \equiv 1.$$

G, the group of semi-primary residues mod 2, is of order 2^5, for b_1, b_3, b_5, b_2, b_4 may each be 0 or 1. The group $E' = \{\zeta^4, E\}$ is of order 2^4. To complete the analysis of G, we need a residue of order 2, in G but not in E'. Such a residue is

$$\gamma \equiv 1+\zeta+\zeta^5, \quad \text{for} \quad \gamma^2 \equiv 1 \pmod 2;$$

so $$J = \{\gamma\}.$$

Therefore $$G = \{\zeta^4, \epsilon_1, \epsilon_3, \gamma\},$$

and the canonical residues are 1 and γ; and we take

$$\pi \equiv 1+b_1(\zeta+\zeta^5) \pmod 2,$$

where $$b_1 \equiv 0 \text{ or } 1.$$

The conjugates of π are

$$\pi_1 = \pi(\zeta^5), \quad \pi_2 = \pi(\zeta^9), \quad \pi_3 = \pi(\zeta^{13}),$$
$$\pi^\dagger = \pi(\zeta^{15}), \quad \pi_1^\dagger = \pi(\zeta^{11}), \quad \pi_2^\dagger = \pi(\zeta^7), \quad \pi_3^\dagger = \pi(\zeta^3).$$

When $b_1 \equiv 0 \pmod 2$, all the conjugates are $\equiv 1 \pmod 2$; and when $b_1 \equiv 1 \pmod 2$,

$$\pi_u \equiv 1+\zeta+\zeta^5, \quad \pi_u^\dagger \equiv 1+\zeta^3+\zeta^7 \pmod 2.$$

32. With either canonical residue, we have $\pi^* = \pi\pi_2$; in fact, this gives

$$\left.\begin{aligned} a_0 &= \quad b_0^2-b_4^2-2b_2b_6+2b_1b_7+2b_3b_5 \\ a_1 &= -b_1^2+b_5^2+2b_0b_2-2b_4b_6+2b_3b_7 \\ a_2 &= \quad b_2^2-b_6^2+2b_0b_4-2b_1b_3+2b_5b_7 \\ a_3 &= -b_3^2+b_7^2+2b_0b_6+2b_2b_4-2b_1b_5 \end{aligned}\right\}. \tag{23}$$

Then $\quad a_0 \equiv 1, \quad a_1 \equiv 0, \quad a_2 \equiv 0, \quad a_3 \equiv 2b_1 \pmod 4$;

and $\quad a \equiv 1, \quad b \equiv 4b_1, \quad c \equiv 4b_1+1 \pmod 8; \quad d \equiv 2b_1 \pmod 4$. (24)

Also, when $b \equiv 0 \pmod 8$, we find, since $p = a^2+b^2 = c^2+2d^2$,

$$a \equiv 1+8m, \quad c \equiv 1+8m+4d \pmod{32};$$

and, using the equations (21) and (22), we easily obtain

$$a_0 \equiv 1-4m+b, \quad a_2 \equiv \tfrac{1}{2}b+2d \pmod{16},$$

$$a_1+a_3 \equiv d \pmod{16};$$

and, using (23), $\quad b_4 \equiv 2m, \quad b_2+b_6 \equiv 2m+\tfrac{1}{4}b \pmod 4,$

$$b_2+b_1+b_5 \equiv \tfrac{1}{2}a_1, \quad b_6+b_3+b_7 \equiv \tfrac{1}{2}a_3 \pmod 4,$$

$$b_1+b_3+b_5+b_7 \equiv 2m+\tfrac{1}{4}b+\tfrac{1}{2}d \pmod 4.$$

Again, when $b \equiv 4 \pmod 8$,

$$a \equiv -7+8m, \quad c \equiv -11+8m \pmod{32};$$

$$a_0+a_2 \equiv 7-4m-\tfrac{1}{2}b, \quad 2a_1+a_2 \equiv -2+\tfrac{1}{2}b \pmod{16},$$

$$a_1+a_3 \equiv 4+8m+b+d \pmod{16};$$

$$b_2 \equiv \tfrac{1}{2}a_1, \quad b_6 \equiv 1+2m+\tfrac{1}{4}b+\tfrac{1}{2}a_1 \pmod 4,$$

$$b_4+b_3+b_7 \equiv 2+2m, \quad b_4+b_1+b_5 \equiv \tfrac{1}{4}b-\tfrac{1}{2}d \pmod 4.$$

In each case $\quad a \equiv 1+8m+2b \pmod{16}$. (25)

33. We have $\quad \pi\pi_2^\dagger = C_0+C_1\omega_1+C_2\omega_2+C_3\omega_3,$

where
$$\begin{aligned} C_0 &= b_0^2-b_1^2+b_2^2-b_3^2+b_4^2-b_5^2+b_6^2-b_7^2, \\ C_1 &= b_0b_1-b_1b_2+b_2b_3-b_3b_4+b_4b_5-b_5b_6+b_6b_7+b_7b_0, \\ C_2 &= b_0b_2-b_1b_3+b_2b_4-b_3b_5+b_4b_6-b_5b_7-b_6b_0+b_7b_1, \\ C_3 &= b_0b_3-b_1b_4+b_2b_5-b_3b_6+b_4b_7+b_5b_0-b_6b_1+b_7b_2, \end{aligned}$$

and so $C_0 \equiv 1+\frac{1}{2}b \pmod 4$, $C_2 \equiv 0$, $C_1 \equiv C_3 \equiv \frac{1}{2}b \pmod 2$.

Let $$\Phi_1 = (\pi\pi_2^\dagger)(\pi_3\pi_1^\dagger) = c_0+c_1\omega_1+c_2\omega_2+c_3\omega_3,$$

where
$$\begin{aligned} c_0 &= C_0^2-2C_2^2, \\ c_1 &= C_0C_1+C_0C_3-2C_2C_3, \\ c_2 &= C_1^2-C_3^2+2C_1C_3, \\ c_3 &= -C_0C_1+C_0C_3-2C_1C_2, \end{aligned}$$

and so $c_0 \equiv 1 \pmod 8$, $c_2 \equiv \frac{1}{2}b \pmod 4$, $c_1 \equiv c_3 \equiv 0 \pmod 2$. (26)

Let $$\Phi_2 = (\pi\pi_2)(\pi_3\pi_1^\dagger) = \Sigma d_v\zeta^v \quad (v = 0, 1, \ldots, 7),$$

where
$$\begin{aligned} d_0 &= a_0C_0+(-a_1+a_3)\,C_2, \\ d_1 &= (a_2+a_3)\,C_1+(a_0-a_1)\,C_3, \\ d_2 &= a_1C_0-(a_0+a_2)\,C_2, \\ d_3 &= (-a_0+a_3)\,C_1+(a_1-a_2)\,C_3, \\ d_4 &= a_2C_0-(a_1+a_3)\,C_2, \\ d_5 &= -(a_0+a_1)\,C_1+(a_2-a_3)\,C_3, \\ d_6 &= a_3C_0+(a_0-a_2)\,C_2, \\ d_7 &= -(a_1+a_2)\,C_1+(a_0+a_3)\,C_3, \end{aligned}$$

and so
$$\left.\begin{array}{c} d_0 \equiv 1+\frac{1}{2}b, \quad d_4 \equiv 0 \pmod 4 \\ d_2 \equiv d_6 \equiv 0 \pmod 2, \quad d_2+d_6 \equiv \frac{1}{2}b \pmod 4 \\ d_1 \equiv d_3 \equiv d_5 \equiv d_7 \equiv \frac{1}{4}b \pmod 2, \quad d_1 \equiv d_7, \quad d_3 \equiv d_5 \pmod 4 \\ d_1+d_7 \equiv d_3+d_5 \equiv \frac{1}{2}b \pmod 4, \quad \text{and} \quad \Sigma d_v \equiv 1 \pmod 4 \end{array}\right\}. \quad (27)$$

34. The reciprocal factors of p are—

$$\begin{aligned} \psi_1 &= \zeta^{4k_1}t\Phi_1 &&= \zeta^{4k_1}(c_0-c_1\omega_1+c_2\omega_2-c_3\omega_3), \\ \psi_3 &= \zeta^{4k_3}s^3\Phi_2 &&= \zeta^{4k_3}(d_0+d_5\zeta-d_2\zeta^2-d_7\zeta^3+d_4\zeta^4-d_1\zeta^5-d_6\zeta^6+d_3\zeta^7), \\ \psi_5 &= \zeta^{4k_5}st\Phi_2 &&= \zeta^{4k_5}(d_0+d_3\zeta+d_6\zeta^2-d_1\zeta^3-d_4\zeta^4-d_7\zeta^5+d_2\zeta^6+d_5\zeta^7), \\ \psi_7 &= (-1)^{m+1}t\Phi_1 &&= (-1)^{m+1}(c_0-c_1\omega_1+c_2\omega_2-c_3\omega_3), \\ \psi_9 &= \zeta^{4k_1}(c-d\zeta^2-d\zeta^6), \\ \psi_{11} &= \zeta^{4k_{11}}t\Phi_2 &&= \zeta^{4k_{11}}(d_0-d_7\zeta-d_6\zeta^2-d_5\zeta^3-d_4\zeta^4-d_3\zeta^5-d_2\zeta^6-d_1\zeta^7), \\ \psi_{13} &= \zeta^{4k_{13}}t\Phi_2, \end{aligned}$$

and $$\psi_{2g} = (-1)^m\psi_{15-2g}.$$

We find, from (2), that

$$\psi_{13} = \psi_5(\zeta^{13}) \quad \text{and} \quad \psi_{11} = \psi_3(\zeta^{11}),$$

whence $\qquad k_{13} \equiv k_5, \quad k_{11} \equiv -k_3 \pmod 4,$

and the relation used in § 27, $\psi_{2g}\psi_{2g+1} = \psi_1\psi_g^{*}$, gives,

$$\left.\begin{array}{ll}\text{when } g=1, & k_1-k_3-k_5 \equiv 2m+\tfrac{1}{2}b+2 \\ \text{and when } g=2, & k_1+k_3-k_5 \equiv 2m+\tfrac{1}{2}b+2\end{array}\right\} \pmod 4; \tag{28}$$

therefore $\qquad k_3 \equiv 0, \quad k_1 \equiv k_5 \pmod 2.$

35. To determine k_1, we find, from § 27, that $4k_1 = b+8h$; and so,

$$\begin{aligned} 2x_0+c_0 &= \tfrac{1}{4}(8m-a-2c-1), & 2x_4 &= \tfrac{1}{4}(8m-a+2c-1), \\ 2x_1-c_1 &= \tfrac{1}{4}(8m+b+2d), & 2x_5-c_3 &= \tfrac{1}{4}(8m+b-2d), \\ 2x_2+c_2 &= \tfrac{1}{4}(8m+a-1), & 2x_6-c_2 &= \tfrac{1}{4}(8m+a-1), \\ 2x_3-c_3 &= \tfrac{1}{4}(8m-b+2d), & 2x_7-c_1 &= \tfrac{1}{4}(8m-b-2d). \end{aligned}$$

Then $\qquad 2\dot{\phi}(1) \equiv -b \pmod{16}, \qquad \phi(1) \equiv 1 \pmod 2,$

and $\qquad \dot{\Pi}_1(1) \equiv 2b \pmod{16};$

and so $\qquad 4k_1 \equiv b+8 \pmod{16},$

that is, the unit of ψ_1 is $\qquad \zeta^{b+8} = -\zeta^b,$

Next, since $k_3 \equiv 0 \pmod 2$, so that the unit of ψ_3 is ζ^{8h}, we have

$$\begin{aligned} 2x_0+d_0 &= \tfrac{1}{2}(4m-c-1), & 2x_4+d_4 &= \tfrac{1}{2}(4m+c-1), \\ 2x_1+d_5 &= \tfrac{1}{2}(4m+d), & 2x_5-d_1 &= \tfrac{1}{2}(4m-d), \\ 2x_2-d_2 &= 2m, & 2x_6-d_6 &= 2m, \\ 2x_3-d_7 &= \tfrac{1}{2}(4m+d), & 2x_7+d_3 &= \tfrac{1}{2}(4m-d). \end{aligned}$$

Then $\qquad 2\dot{\phi}(1) \equiv 8m+d_1+d_3-d_5-d_7+2d_2-2d_6 \pmod{16},$

$\qquad \dot{\Pi}_3(1) \equiv \quad -d_1-d_3+d_5+d_7-2d_2+2d_6 \pmod{16},$

and $\qquad \phi(1) \equiv 1 \pmod 2;$

and so $\qquad 4k_3 \equiv 8m+8 \pmod{16},$

that is, the unit of ψ_3 is $(-1)^{m+1}$.

Then, by (28), $\qquad 4k_5 \equiv -b+8 \pmod{16}.$

36. The units are therefore as follows, θ_g being the unit of ψ_g,

$$\begin{array}{ll} \theta_1 = \theta_9 = \zeta^{b+8}, & \theta_2 = \theta_{10} = \zeta^{-b+8m+8}, \\ \theta_3 = \theta_7 = \theta_{11} = (-1)^{m+1}, & \theta_4 = \theta_8 = \theta_{12} = -1, \\ \theta_5 = \theta_{13} = \zeta^{-b+8}, & \theta_6 = \theta_{14} = \zeta^{b+8m+8}; \end{array}$$

and so, since $\theta_1\theta_2 = (-1)^m$, ν is primary, if it is in either of the canonical forms.

When $b \equiv 0 \pmod 8$, corresponding to the canonical residue 1 (mod 2), primary numbers are those whose residues mod 2 are those of the group E. And when $b \equiv 4 \pmod 8$, corresponding to the canonical residue γ, primary numbers are those whose residues belong to the set γE, which are as follows:—

$$\begin{array}{rl} \gamma \equiv & 1 + \zeta + \zeta^5 \\ \epsilon_1^2\gamma \equiv & 1+\zeta^2+\zeta^6+\zeta+\zeta^5 \\ \epsilon_1^2\epsilon_3\gamma \equiv & 1+\zeta^3+\zeta^7 \\ \epsilon_3\gamma \equiv & 1+\zeta^2+\zeta^6+\zeta^3+\zeta^7 \\ \epsilon_1^3\gamma \equiv & \zeta^4+\zeta+\zeta^3 \\ \epsilon_1^3\epsilon_3\gamma \equiv & \zeta^4+\zeta^2+\zeta^6+\zeta+\zeta^3 \\ \epsilon_1\epsilon_3\gamma \equiv & \zeta^4+\zeta^5+\zeta^7 \\ \epsilon_1\gamma \equiv & \zeta^4+\zeta^2+\zeta^6+\zeta^5+\zeta^7 \end{array}$$

Any residue of G may also be written in the form

$$1+y_2(\lambda^2+\lambda^3)+y_4\lambda^4+y_5\lambda^5+y_6\lambda^6+y_7\lambda^7 \pmod 2,$$

where each $y \equiv 0$ or 1. It might be expected that some simple relation between the y's would hold for primary residues; and, having expressed all the primary residues in this form, I find that the following congruences hold:—

when $y_2 \equiv 0$, then $\qquad y_4+y_5+y_6+y_7 \equiv 0 \pmod 2$;

and when $y_2 \equiv 1$, then $\qquad y_4+y_6+y_7 \equiv 1 \pmod 2$;

so that, in each case, the condition that a residue is primary is

$$(y_2+1)(y_5+1)+y_4+y_6+y_7 \equiv 1 \pmod 2.$$

PART IV.

The Field of 9th Roots of 1.

37. The notation used here for the field of 3rd roots of 1 is: ρ is a 3rd root of 1, $p = 9m+1$, $\pi^* = a+a'\rho$, in the semi-primary form, and taken with such sign that

$$a \equiv 1, \quad a' \equiv 0 \pmod 3.$$

Then
$$p = a^2 - aa' + a'^2 = \tfrac{1}{4}(A^2+3a'^2),$$

where
$$A = 2a - a' \equiv -1 \pmod 3.$$

$$\psi_1 = -(a+a'\rho^2) = \Sigma A_v \rho^v,$$

where
$$A_0 = 3m + \tfrac{1}{3}(-2a+a'-1),$$

$$A_1 = 3m + \tfrac{1}{3}(a+a'-1),$$

$$A_2 = 3m + \tfrac{1}{3}(a-2a'-1).$$

We have
$$A^2 \equiv 4p \equiv 4+9m \pmod{27},$$

that is, and
$$\left.\begin{aligned} A &\equiv 2+9m \pmod{27} \\ a+a' &\equiv 1 \pmod 9 \end{aligned}\right\}, \tag{29}$$

38. The fundamental real units in the field of 9th roots are

$$\epsilon_0 = -\zeta - \zeta^{-1}, \quad \epsilon_1 = -\zeta^2 - \zeta^{-2}, \quad \epsilon_2 = -\zeta^4 - \zeta^{-4},$$

where
$$\epsilon_0 \epsilon_1 \epsilon_2 = 1, \quad \text{and} \quad \epsilon_0 + \epsilon_1 + \epsilon_2 = 0.$$

The units are taken with such sign as to make the sum of the coefficients of each unit $\equiv 1 \pmod 3$.

The group E is $\{\epsilon_0, \epsilon_1\}$, where

$$\epsilon_0^3 \equiv \epsilon_1^3 \equiv 1 \pmod 3.$$

Every residue of E may be expressed in the form $x+y\epsilon_0+z\epsilon_1$, where $x+y+z \equiv 1 \pmod 3$, and also in the form $1+\Sigma y_v \lambda^v$ $(v = 2, \ldots, 5)$; the

residues and the simplest corresponding units are :—

Residues.			Simplest Units.
$1 \equiv$	1	$\equiv 1$	1
$\epsilon_0^2\epsilon_1 \equiv$	$1-\epsilon_0+\epsilon_1$	$\equiv 1 \qquad -\lambda^4+\lambda^5$	$\epsilon_0^2\epsilon_1$
$\epsilon_0\epsilon_1^2 \equiv$	$1+\epsilon_0-\epsilon_1$	$\equiv 1 \qquad +\lambda^4-\lambda^5$	$\epsilon_0\epsilon_1^2$
$\epsilon_0 \equiv$	ϵ_0	$\equiv 1-\lambda^2-\lambda^3-\lambda^4-\lambda^5$	ϵ_0
$\epsilon_1 \equiv$	ϵ_1	$\equiv 1-\lambda^2-\lambda^3+\lambda^4$	ϵ_1
$\epsilon_0^2\epsilon_1^2 \equiv$	$-\epsilon_0-\epsilon_1$	$\equiv 1-\lambda^2-\lambda^3 \qquad +\lambda^5$	ϵ_2
$\epsilon_0^2 \equiv$	$-1 \qquad -\epsilon_1$	$\equiv 1+\lambda^2+\lambda^3-\lambda^4$	$\epsilon_1\epsilon_2$
$\epsilon_0\epsilon_1 \equiv$	$-1-\epsilon_0$	$\equiv 1+\lambda^2+\lambda^3+\lambda^4+\lambda^5$	$\epsilon_0\epsilon_1$
$\epsilon_1^2 \equiv$	$-1+\epsilon_0+\epsilon_1$	$\equiv 1+\lambda^2+\lambda^3 \qquad -\lambda^5$	$\epsilon_2\epsilon_0$

These simplest units are

$$\epsilon_0^2\epsilon_1 = -2-\epsilon_0+\epsilon_1, \quad \epsilon_0\epsilon_1^2 = 1+\epsilon_0-\epsilon_1, \quad \epsilon_0\epsilon_1 = -1-\epsilon_0,$$

$$\epsilon_1\epsilon_2 = -1-\epsilon_1, \quad \epsilon_2\epsilon_0 = -1-\epsilon_2.$$

The primitive root mod 9 used is 2, so that s is the substitution changing ζ to ζ^2.

$$\pi = b_0+b_3\zeta^3+b_1\zeta+b_2\zeta^2+b_7\zeta^7+b_8\zeta^8,$$

is a prime factor of p, the terms in ζ^4, ζ^5, and ζ^6, being disposed of by the identity

$$\zeta^6+\zeta^3+1 = 0.$$

π being taken with such sign that $\pi(1) \equiv 1 \pmod 3$, and being also semi-primary, we have

$$b_0+b_3-1 \equiv b_1+b_7 \equiv b_2+b_8 \pmod 3.$$

G, the group of residues mod 3 satisfying these congruences is of order 3^4, and $E' = \{\zeta^3, E\}$ is a sub-group of order 3^3; the residue

$$\gamma \equiv 1+\zeta-\zeta^7 \pmod 3$$

is of order 3; and therefore $G = \{\zeta^3, \epsilon_0, \epsilon_1, \gamma\}$, and there are three canonical residues 1, γ, and

$$\gamma^2 \equiv 1-\zeta+\zeta^7 \pmod 3.$$

We may therefore take

$$\pi \equiv 1+b_1(\zeta-\zeta^7) \pmod 3,$$

where

$$b_1 \equiv 0, \ \pm 1 \pmod 3.$$

39. The conjugates of π are

$$\pi_1 = \pi(\zeta^2), \quad \pi_2 = \pi(\zeta^4), \quad \pi_3 = \pi(\zeta^8), \quad \pi_4 = \pi(\zeta^7), \quad \text{and} \quad \pi_5 = \pi(\zeta^5).$$

Applying (5) and (7) to the present case, we have

$$\pi^* = \pi\pi_2\pi_4 \equiv \pi(\zeta^3) \pmod 3,$$

and, writing $\sigma_0 = b_0+b_3\zeta^3$, $\sigma_1 = b_1\zeta+b_7\zeta^7$, $\sigma_2 = b_2\zeta^2+b_8\zeta^8$,

$$\begin{aligned}\pi^* &= \sigma_0^3+\sigma_1^3+\sigma_2^3-3\sigma_0\sigma_1\sigma_2 \\ &\equiv 1+\sigma_1^3 \equiv 1+3b_1-3b_1\rho \pmod 9,\end{aligned}$$

and so

$$b_1 \equiv -\tfrac{1}{3}a' \pmod 3. \tag{30}$$

Similarly, taking the residue of π^* (mod 27), we find

$$b_0+b_3 \equiv 1-3m+a' \pmod 9;$$

and, when $b_1 \equiv 0 \pmod 3$, $b_0 \equiv 1-3m-\tfrac{1}{3}a'$, $b_3 \equiv \tfrac{1}{3}a' \pmod 9$;

when $b_1 \equiv 1 \pmod 3$, $b_2+b_8 \equiv -1-\tfrac{1}{3}a'+b_3 \pmod 9$;

and, when $b_1 \equiv -1 \pmod 3$, $b_2+b_8 \equiv -1+\tfrac{1}{3}a'-b_3 \pmod 9$.

40. Let $\phi = \pi\pi_1\pi_2 = c_0+c_3\zeta^3+c_1\zeta+c_2\zeta^2+c_7\zeta^7+c_8\zeta^8$;

then, since $\pi\pi_1\pi_2 \equiv 1+b_1(-\zeta-\zeta^2+\zeta^7+\zeta^8) \pmod 3$,

$$c_0-1 \equiv c_3 \equiv 0, \quad c_1 \equiv c_2 \equiv -b_1, \quad c_7 \equiv c_8 \equiv b_1 \pmod 3. \tag{31}$$

And, taking the residue of ϕ mod 9, we find

$$c_0+c_3-1 \equiv c_1+c_2+c_7+c_8 \equiv -3b_1^2 \pmod 9;$$

and, when $b_1 \equiv 0 \pmod 3$,

$$c_0 \equiv 1-\tfrac{1}{3}a', \quad c_3 \equiv \tfrac{1}{3}a' \pmod 9,$$

$$c_1 \equiv -c_8 \equiv -b_2+b_7, \quad c_2 \equiv -c_7 \equiv b_1+b_2-b_7-b_8 \pmod 9.$$

41. Writing $\theta_g = \zeta^{3k_g}$ for the unit of ψ_g, we find, from (1) and (2), that

$$\theta_1 = \theta_4 = \theta_7, \quad \theta_2 = \theta_6, \quad \theta_3 = \theta_5 = \theta_2^{-1};$$

the reciprocal factors therefore are

$$\begin{aligned}\psi_1 = \psi_7 &= -\theta_1 s\phi = -\zeta^{3k_1}(c_0+c_3\zeta^3+c_1\zeta^2+c_2\zeta^4+c_7\zeta^5+c_8\zeta^7), \\ \psi_2 = \psi_6 &= -\theta_2 s^2\phi = -\zeta^{3k_2}(c_0+c_3\zeta^3+c_7\zeta+c_1\zeta^4+c_8\zeta^5+c_2\zeta^8), \\ \psi_3 = \psi_5 &= -\theta_2^{-1}s\phi, \\ \psi_4 &= -\theta_1 s^3\phi = -\zeta^{3k_1}(c_0+c_3\zeta^6+c_8\zeta+c_7\zeta^2+c_2\zeta^7+c_1\zeta^8).\end{aligned}$$

Then, applying the process of § 11 to ψ_1,

$$3x_0+c_0+c_3 = -3m-\tfrac{1}{3}(-2a+a'-1),$$
$$3x_1+c_2+c_8 = -3m-\tfrac{1}{3}(a+a'-1),$$
$$3x_2+c_1+c_7 = -3m-\tfrac{1}{3}(a-2a'-1);$$

and so
$$k_1 \equiv \tfrac{1}{3}a'+c_3-c_1-c_2+c_7+c_8 \pmod 3$$
$$\equiv \tfrac{1}{3}a'+b_1 \equiv 0 \pmod 3.$$

The canonical forms are therefore primary. And we find for ψ_2,

$$3x_0+c_0+c_3 = -3m+1,$$
$$3x_1+c_1+c_7 = -3m,$$
$$3x_2+c_2+c_8 = -3m,$$

and so
$$k_2 \equiv -c_3-c_1+c_8 \equiv \tfrac{1}{3}a' \pmod 3.$$

Therefore
$$\theta_1 = 1, \qquad \theta_2 = \rho^{\frac{1}{3}a'}.$$

42. When $a' \equiv 0 \pmod 9$, the corresponding canonical residue being 1 (mod 3), primary numbers are those whose residues mod 3 are those of the group E.

When $a' \equiv -3 \pmod 9$, the corresponding canonical residue being γ, primary numbers are those whose residues mod 3 belong to the set $\gamma\{E\}$, namely—

$$\begin{aligned}
\gamma &\equiv 1 +\zeta -\zeta^7 \equiv 1 -\lambda^3+\lambda^4\\
\epsilon_0^2\epsilon_1\gamma &\equiv 1 -\zeta-\zeta^2+\zeta^7+\zeta^8 \equiv 1 -\lambda^3 +\lambda^5\\
\epsilon_0\epsilon_1^2\gamma &\equiv 1 +\zeta^2 -\zeta^8 \equiv 1 -\lambda^3-\lambda^4-\lambda^5\\
\epsilon_0\gamma &\equiv 1-\zeta^3-\zeta-\zeta^2 \equiv 1-\lambda^2+\lambda^3\\
\epsilon_1\gamma &\equiv 1-\zeta^3 +\zeta^2-\zeta^7+\zeta^8 \equiv 1-\lambda^2+\lambda^3-\lambda^4+\lambda^5\\
\epsilon_0^2\epsilon_1^2\gamma &\equiv 1-\zeta^3+\zeta +\zeta^7-\zeta^8 \equiv 1-\lambda^2+\lambda^3+\lambda^4-\lambda^5\\
\epsilon_0^2\gamma &\equiv 1+\zeta^3-\zeta-\zeta^2-\zeta^7-\zeta^8 \equiv 1+\lambda^2 -\lambda^5\\
\epsilon_0\epsilon_1\gamma &\equiv 1+\zeta^6 +\zeta^2+\zeta^7 \equiv 1+\lambda^2 -\lambda^4\\
\epsilon_1^2\gamma &\equiv 1+\zeta^3+\zeta +\zeta^8 \equiv 1+\lambda^2 +\lambda^4+\lambda^5
\end{aligned}$$

And, when $a' \equiv 3 \pmod 9$, the corresponding canonical residue being γ^2,

primary numbers are those whose residues mod 3 belong to the set $\gamma^2\{E\}$, namely—

$$\begin{array}{rl}
\gamma^2 \equiv & 1 - \zeta + \zeta^7 \equiv 1 + \lambda^3 - \lambda^4 \\
\epsilon_0^2\epsilon_1\gamma^2 \equiv & 1 - \zeta^2 + \zeta^8 \equiv 1 + \lambda^3 + \lambda^4 + \lambda^5 \\
\epsilon_0\epsilon_1^2\gamma^2 \equiv & 1 + \zeta + \zeta^2 - \zeta^7 - \zeta^8 \equiv 1 + \lambda^3 - \lambda^5 \\
\epsilon_0\gamma^2 \equiv & -1 + \zeta^3 - \zeta + \zeta^2 + \zeta^8 \equiv 1 - \lambda^2 + \lambda^4 + \lambda^5 \\
\epsilon_1\gamma^2 \equiv & -1 + \zeta^3 - \zeta^7 - \zeta^8 \equiv 1 - \lambda^2 - \lambda^5 \\
\epsilon_0^2\epsilon_1^2\gamma^2 \equiv & -1 + \zeta^3 + \zeta - \zeta^2 + \zeta^7 \equiv 1 - \lambda^2 - \lambda^4 \\
\epsilon_0^2\gamma^2 \equiv & -\zeta^3 + \zeta + \zeta^8 \equiv 1 + \lambda^2 - \lambda^3 + \lambda^4 + \lambda^5 \\
\epsilon_0\epsilon_1\gamma^2 \equiv & -\zeta^3 - \zeta - \zeta^2 - \zeta^7 - \zeta^8 \equiv 1 + \lambda^2 - \lambda^3 - \lambda^5 \\
\epsilon_1^2\gamma^2 \equiv & -\zeta^3 + \zeta^2 + \zeta^7 \equiv 1 + \lambda^2 - \lambda^3 - \lambda^4
\end{array}$$

It is evident that if π is primary, so also are its conjugates. It appears that $1+y_2\lambda^2+y_3\lambda^3+y_4\lambda^4+y_5\lambda^5$ is a primary residue in the following cases:—

when $y_2 \equiv 0$, and $y_4+y_5 \equiv -y_3$ (mod 3),

when $y_2 \equiv 1$, and $y_4+y_5 \equiv -1$ (mod 3),

and when $y_2 \equiv -1$, and $y_4+y_5 \equiv y_3-1$ (mod 3).

These three cases may be expressed in one congruence, viz.,

$$y_2^2-y_2y_3+y_3+y_4+y_5 \equiv 0 \pmod 3.$$

OSCILLATING SUCCESSIONS OF CONTINUOUS FUNCTIONS

By W. H. YOUNG, Sc.D., F.R.S.

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1. The theory of series which neither converge nor diverge to a definite limit has been little studied. It is clear none the less that from the point of view of theory of functions such series have as real a claim to consideration as the more usual ones; there is, moreover, the important application to the theory of derivates.

The investigator is, however, met on the threshold by the difficulty that the methods which were fruitful in the more simple case do not at once apply. On the one hand, the nature of the sum function, or rather of the two functions which now correspond to it, has not been elucidated, and on the other, the usual definition of uniform convergence by means of the remainder function does not lend itself easily to generalisation. This last difficulty is removed by the formulation given by myself in a recent paper presented to this Society, a formulation which depends on the introduction of what I call the peak and chasm functions. Moreover, the application of the method of monotone sequences leads readily to the required information as to the nature of the upper and lower functions, which, in the general case, replace the sum-function.

Except in the fundamental theorem which concerns the nature of the functions defined by the extreme limits at every point, I confine my attention to series of continuous functions. In the case of the fundamental theorem, however, this restriction is unnecessarily narrow. The nature of the upper function is, in fact, found to be the same, in general, whether the generating functions are continuous, or only lower semi-continuous. In the same way the lower function has the same nature whether the generating functions are continuous, or only upper semi-continuous.

The fundamental theorem in question is as follows:—

The upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous, except possibly at a set of points of the first category. This is, moreover, true not only with respect to the continuum, but with respect to every perfect

set. In particular, the upper and lower derivates have these properties respectively.

This fundamental theorem includes as a particular case Baire's theorem when the upper and lower functions coincide. In a paper in the *Messenger of Mathematics** I shewed that, in the case of a single variable, this theorem of Baire's still held when there was continuity on one side only at each point. In the present paper I shew that the corresponding property is possessed by the upper and lower functions, viz., that their nature is the same when the lower and upper semi-continuity of the generating functions is on one side only.

Turning to the nature of the convergence and divergence, we have now, in general, what is usually called oscillatory divergence existing at every point. It at once follows from the fundamental theorem that the points at which the measure of this divergence, or, as I prefer to say, the measure of the oscillation, is greater than k form an ordinary inner limiting set.

The consideration of the peak and chasm functions leads, on the other hand, to the splitting up of the concept of uniform convergence and divergence into two components, which I call *uniform oscillation above and below*. I shew that the points of uniform oscillation above and below have each the same distribution as have in the simple case the points of uniform convergence, viz., that they fill up the continuum except possibly for a set of the first category.†

Further, all the results as to the distinction of right and left obtained in connexion with ordinary uniform convergence and divergence still hold. In other words, non-uniform oscillation above and below have each on the right and left the same measure except possibly at a countable set of points.

Closely connected with the main subject of the paper is the consideration of the distinction of right and left in the case of derivates. Here we are concerned in general with two distinct sequences, that leading to a right-hand and that leading to a left-hand derivate. The result obtained is that whatever be the nature of the function and of its derivates, bounded or unbounded, the derivates are the same on the right and left, except at a set of points of the first category.

* W. H. Young, "A New Proof of a Theorem of Baire's," *Messenger of Math.*, New Series, August, 1907.

† From this it follows in particular, by the characteristic property of a set of the first category, that the well known theorems concerning the mode of convergence of a series of continuous functions and the nature of its limiting function still hold *when oscillating divergence is allowed at a set of the first category*, viz., the points of uniform convergence and divergence still fill up the continuum, except for a set of the first category, and the limiting functions are all pointwise discontinuous.

2. Let $f_1, f_2, \ldots$ be a sequence of functions, not necessarily having a definite limit at each point. At each point we shall then have a highest possible limit, and a lowest possible limit, and perhaps intermediate limits. The function $\bar{f}$ whose value at each point is the highest possible limit is called *the upper function*, and the function $\underline{f}$ whose value at each point is the lowest possible limit is called *the lower function*. With this explanation we may write shortly,

$$\bar{f}(x) = \text{highest} \operatorname*{Lt}_{n=\infty} f_n(x),$$

$$\underline{f}(x) = \text{lowest} \operatorname*{Lt}_{n=\infty} f_n(x).$$

We now define the *left- and right-hand peak and chasm functions* as follows:—

We take an interval (P, Q) with P as right-hand end-point, and denote the upper bound of $f_n(x)$ for points x internal to this interval by $M_{n, Q}$. The highest possible limit of $M_{n, Q}$, as n increases indefinitely, we denote by M_Q.

Now, if Q_1 and Q_2 are two positions of Q, of which Q_2 lies between P and Q_1, it follows from the definitions that

$$M_{n, Q_2} \leqslant M_{n, Q_1},$$

Hence, also, $$M_{Q_2} \leqslant M_{Q_1}.$$

If therefore we make Q move up to P as limit, the quantities M_Q will form a monotone decreasing sequence, and will therefore have a definite limit not greater than any of them; this limit we take as the value $\pi_L(P)$ of *the left-hand peak function* at P.

Similarly, working on the right, we get *the right-hand peak function* $\pi_R(P)$. The function $\pi(P)$ whose value at each point is that one of π_L and π_R which is not less than the other, is the peak function *par excellence*.

Similarly, interchanging "upper" and "lower" we get *the chasm functions* χ_L, χ_R, and χ.

3. THEOREM 1.—*If f denote either the upper or the lower function,*

$$\chi_L(P) \leqslant \psi_L(P) \leqslant \phi_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, if x be any point inside the interval (P, Q) on the right of P, and $M_{n, Q}$ denote the upper bound of $f_n(x)$ in this interval,

$$f_n(x) \leqslant M_{n, Q}.$$

Making n increase indefinitely, $f(x)$, being one of the limits on the left, cannot lie above the highest limit M_Q on the right, that is,

$$f(x) \leqslant M_Q.$$

Now, letting x describe a sequence having P as limit, any limit which we may obtain on the left is less than or equal to M_Q, so that

$$\phi_L(P) \leqslant M_Q.$$

Since this is true for all positions of Q, it is true when Q moves up to P, so that

$$\phi_L(P) \leqslant \pi_L(P).$$

Similarly

$$\psi_L(P) \geqslant \chi_L(P),$$

which proves the theorem.

THEOREM 2.—*If the functions f_n are continuous at P,*

$$\chi_L(P) \leqslant f(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, since $f_n(x)$ is continuous at P, it has the definite limit $f_n(P)$, as x approaches P, so that

$$f_n(P) \leqslant M_{n,Q}.$$

Since this is true for all values of n, $f(P)$ cannot be higher than the highest limit of the right-hand side, that is,

$$f(P) \leqslant M_Q.$$

Since this is true for all positions of Q,

$$f(P) \leqslant \pi_L(P).$$

Similarly the other inequality holds, which proves the theorem.

From these theorems it follows that if the peak and chasm functions are equal at P, the upper and lower functions agree and are both continuous at P, supposing the f_n's to be continuous functions at P.*

Again, it follows that *at a point where the peak function is equal to the upper function, the latter is upper semi-continuous, while at a point where the chasm function is equal to the lower function the latter is lower semi-continuous.*

* If the f_n's are not continuous, the same holds with a countably infinite set of possible exceptions, by the results of my paper on "The Distinction of Right and Left at Points of Discontinuity," *Quart. Jour. of Math.*, 1907.

4. Let $f_1, f_2, \ldots$ be a sequence of lower (upper) semi-continuous functions. Let $v_{1,2}$ denote the function* which at each point has the value of the greater of f_1 and f_2, or is equal to both, if they are equal. Then $v_{1,2}$ is a lower (upper) semi-continuous function.†

Let $v_{1,n}$ be the function which at each point has the value of the greater of f_n and v_{n-1}, or is equal to both. Then it follows, for each value of n, that $v_{1,n}$ is a lower (upper) semi-continuous function. Also

$$v_{1,2} \leqslant v_{1,3} \leqslant v_{1,4} \leqslant \ldots$$

is a monotone ascending sequence.

Thus, if the original functions $f_1, f_2, \ldots$ were lower semi-continuous, the limit v_1 of the last sequence is a lower semi-continuous function. This function v_1 is such that at each point its value is the upper bound of $f_1, f_2, \ldots$ at that point.

Similarly we define $v_2, v_3, \ldots, v_n$ being got from $f_n, f_{n+1}, \ldots$ as v_1 was from $f_1, f_2, \ldots$.

Then $$v_1 \geqslant v_2 \geqslant v_3 \geqslant \ldots$$

is a monotone descending sequence of functions, which, if the f_n's were lower semi-continuous, are lower semi-continuous. The limit of this sequence has at each point its value equal to the highest limit approached by $f_1, f_2, \ldots$, and is accordingly the upper function $\bar{f}$ of the original sequence.

Similarly, if the original functions were upper semi-continuous, we get the lower function $\underline{f}$ represented as the limit of a monotone ascending sequence of upper semi-continuous functions.

Thus we have the following theorem :—

THEOREM 3.—*The upper (lower) function of a sequence of lower (upper)*

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* Lebesgue, in his *Intégration*, p. 121, uses a similar device in the case of a sequence of measurable functions to shew that the upper and lower functions are measurable.

† For, if f_1 and f_2 are both lower semi-continuous, and A_k any number less than $f_k(x')$, we can find an interval throughout which

$$f_1(x) > A_1, \quad f_2(x) > A_2;$$

and therefore $v_{1,2}$ is greater than the greater of A_1 or A_2, that is, greater than any number less than its value at x', and is therefore lower semi-continuous. Similarly, if f_1 and f_2 are both upper semi-continuous, and A_k any number greater than $f_k(x')$, we can find an interval throughout which

$$f_1 < A_1, \quad f_2(x) < A_2;$$

and therefore $v_{1,2}$ is less than the greater of A_1 and A_2, that is, less than any number greater than its value at x', and is therefore upper semi-continuous.

semi-continuous functions is the limit of a monotone descending (ascending) sequence of lower (upper) semi-continuous functions.

COR. 1.—*The upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous with respect to any perfect set, except at a set of points of the first category with respect to that set.*

For all the common points of continuity of the generating semi-continuous functions yield points of upper (lower) semi-continuity of the limiting function. This is, moreover, true whether we refer to the continuum or to any other perfect set.

COR. 2.—*The points at which the upper function of a sequence of lower semi-continuous functions is $+\infty$, or is $>k$, form an ordinary inner limiting set.*

For they are the points

$$v_1 = +\infty \text{ or } > k, \quad v_2 = +\infty \text{ or } > k, \quad \dots,$$

that is, the inner limiting set of a sequence of ordinary inner limiting sets.*

Similarly,—

The points at which the lower function of a sequence of upper semi-continuous functions is $-\infty$, or $<k$, form an ordinary inner limiting set.

Hence also, *the points at which the upper (lower) function of a sequence of lower (upper) semi-continuous functions $\leqslant k$ ($\geqslant k$) form an ordinary outer limiting set.*

It may be noticed that in the case when the functions $f_1, f_2, \dots$ are not only lower semi-continuous but never $-\infty$, each of the functions v_n has a finite value at each point, and therefore, being lower semi-continuous, is bounded below. Hence the lower integral of v_n (which is its generalised, or Lebesgue, integral) is the upper limit of the lower summations.

A similar result holds when $f_1, f_2, \dots$ are upper semi-continuous, and never $+\infty$, for the generating upper semi-continuous functions of the lower function.

5. Applying the results of the preceding article to the case when the

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* Young, *Theory of Sets of Points* (Cambridge University Press, 1906), p. 72, Theorem 38*a*.

original functions $f_1, f_2, \ldots$ are continuous functions, we have the following:—

The upper function is upper semi-continuous, and the lower function lower semi-continuous with respect to any perfect set, except possibly at a set of the first category with respect to that set.

The upper function is the limit of a monotone descending sequence of lower semi-continuous functions, and the lower function the limit of a monotone ascending sequence of upper semi-continuous functions.

The points at which the difference of the upper and lower functions is greater than k form an ordinary inner limiting set. Or, as we may say, the points at which the "measure of the oscillation" is greater than k form an ordinary inner limiting set.

Any function which is the limit of a sequence of continuous functions can be expressed as the limit of a decreasing sequence of lower semi-continuous functions, and also as the limit of an ascending sequence of upper semi-continuous functions. Such a function is therefore pointwise discontinuous with respect to every perfect set.

This last result is Baire's theorem, which is here proved in another new way. Assuming the converse, which has also been proved by Baire, it shews that any function which is pointwise discontinuous with respect to every perfect set can be expressed in each of these two modes, and gives a criterion, which may sometimes be convenient, for determining whether a function belongs to Baire's first class.

6. Although we have throughout worked with a discontinuous parameter n, which approaches the value infinity along a countable set of values, the whole discussion might equally well have been based upon a sequence depending on a continuous parameter h, which approaches the value 0, say.

Since the right-hand upper and lower derivates $f^+(x)$ and $f_+(x)$ of a continuous function $f(x)$ are the upper and lower functions of a sequence of continuous functions

$$\frac{f(x+h)-f(x)}{h},$$

where h is a continuous positive variable which approaches the value 0, or (Hobson's *Functions of a Real Variable*, p. 552) of a sequence

$$\frac{f(x+h_n)-f(x)}{h_n},$$

all that has been said about upper and lower functions applies to derivates.

7. THEOREM 4.—*There is no distinction of right and left with respect to derivates, except possibly at a set of the first category.*

For, since in every closed interval the right-hand upper derivate f^+ and the left-hand upper derivate f^- have the same upper bound, it follows that at each point they have the same associated upper limiting function of Baire, say ϕ_B.

But, except at points of a set of the first category, both the upper derivates are upper semi-continuous, and therefore, both being equal to ϕ_B, they are equal to one another, which proves the theorem as far as the upper derivates are concerned.

Similarly it follows for the lower derivates, and therefore for the upper and lower derivates simultaneously, since the sum of two sets of the first category is a set of the first category.

This result may be compared with that of Lebesgue* that a differential coefficient exists in the case of a large class of functions, and in particular functions with bounded derivates, except at a set of content zero.

The above theorem is true without any limitations, and is not, even in Lebesgue's case, *included* in his result. It is easy, in fact, to construct a set of the first category whose content is that of the continuum, and whose complementary set is therefore of content zero without being of the first category.†

Combining the two results in the case of the functions considered by Lebesgue we have the result that there is no distinction of right and left with regard to derivates, except possibly at a set of the first category of content zero.

It should be noted that we cannot obtain any information with regard to the identity of upper and lower derivates by our method of procedure, still less prove that they agree except at a set of content zero. It is easy, in fact, to construct two bounded functions which have all the properties of the derivates utilised above, and which do not agree at any point of an interval.

Ex.—Let $f_1(x) = 2^{-q}$ at all the rational points with denominator 2^{-q},
and $\qquad\qquad = 1$ elsewhere.

Let $f_2(x) = 1-3^{-q}$ at all the rational points with denominator 3^{-q},
and $\qquad\qquad = 0$ elsewhere.

* Lebesgue, *Intégration*, pp. 123 *seq.* See, however, Hobson, *Functions of a Real Variable* (Cambridge University Press, 1907), p. 556.

† W. H. Young, "On the Construction of a Pointwise Discontinuous Function all of whose Continuities are Infinities and which has a Generalised Integral," *Quarterly Journal*, February, 1908.

Both functions are bounded, f_1 is everywhere greater than f_2, both functions have the upper bound 1 and the lower bound 0 in every interval, and, while f_1 is upper semi-continuous except at a set of the first category (here countable), f_2 is lower semi-continuous except at a set of the first category (countable). The Lebesgue integrals are 1 and 0 respectively, integrating from 0 to 1.

8. On account of its importance we give an alternative proof of the result just obtained, a proof moreover independent of the fact that the derivates have the same ϕ_B and ψ_B.

Alternative proof of the theorem :—

There is no distinction of right and left with respect to derivates except at points of a set of the first category.

Let h_1, h_2, ... be a sequence of positive quantities, monotone and decreasing with zero as limit, and such that the upper function

$$f(x) = \text{highest} \underset{n=+\infty}{\text{Lt}} f_n(x), \tag{1}$$

where

$$f_n(x) = \frac{F(x+h_n)-F(x)}{h_n} \tag{2}$$

is the upper right-hand derivate F^+ of the function $F(x)$.*

Let corresponding dashed letter apply to the left-hand upper derivate, so that

$$f'(x) = \text{highest} \underset{n=+\infty}{\text{Lt}} f'_n(x), \tag{3}$$

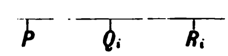
is the upper left-hand derivate F^-, where

$$f'_n(x) = \frac{F(x-h'_n)-F(x)}{-h'_n}. \tag{4}$$

It follows that, if $g_n(x)$ denote the function got by suppressing the dashes in (4), and $g(x)$ be the upper function of the series g_1, g_2, ...,

$$g(x) \leqslant f'(x) \leqslant F^-. \tag{5}$$

Now, let P be any point, and Q_i and R_i the points to the right of P, such that



$$PQ_i = Q_iR_i = h_i. \tag{6}$$

Let $x_{1,n}$ be the point of the interval (P, Q_1) where the continuous function $f_n(x)$ attains its upper bound M_{n,Q_1}.

Then the point $y_{1,n}$ lies in the interval (P, R_1), if

$$y_{1,n} = x_{1,n}+h_n, \tag{7}$$

* Hobson, *Theory of Functions of a Real Variable*, p. 552.

since $h_1 > h_n$, for all values of n. Also

$$g_n(y_{1,n}) = \frac{F(y_{1,n}-h_n)-F(y_{1,n})}{-h_n} = f_n(x_{1,n}) = M_{n,\,Q_1},$$

whence it follows that the upper bound, say G_{n,R_1}, of g_n in the interval (P, R_1) is not less than M_{n,Q_1}, *i.e.*,

$$M_{n,\,Q_1} \leqslant G_{n,\,R_1}. \tag{8}$$

Since this is true for all values of n, it is true of the highest limits M_{Q_1} and G_{R_1} approached by the two sides of (8), that is,

$$M_{Q_1} \leqslant G_{R_1}. \tag{9}$$

Similarly, since $h_i > h_{i+r}$, we have for all values of $n \geqslant i$,

$$M_{n,\,Q_i} \leqslant G_{n,\,R_i}, \tag{8'}$$

whence $$M_{Q_i} \leqslant G_{R_i}, \tag{9'}$$

for all values of i.

Now, by the definition of the right-hand peak function $\pi_R(P)$ of the sequence $f_1, f_2, \ldots$, it is the limit of the quantities M_{Q_i}, since the points Q_i form a sequence having P as limiting point on the left.

Similarly, since the points R_i form a sequence having P as limit on the left, the peak function of the sequence $g_1, g_2, \ldots$ is the limit of G_{R_i}. Hence, by (9′),

$$\pi_R(P) \leqslant \text{the right-hand peak function of the } g_i\text{'s}.$$

But the peak function is equal to the upper function, except at a set of the first category, hence

$$f(x) \leqslant g(x),$$

except at a set of the first category, *a fortiori*, by (5),

$$f(x) \leqslant f'(x),$$

except at a set of the first category, that is,

$$F^+ \leqslant F^-, \tag{10}$$

except at a set of the first category.

Similarly, $$F^- \leqslant F^+, \tag{11}$$

except at a set of the first category.

From (10) and (11) it follows that

$$F^+ = F^-,$$

except at a set of the first category.

x 2

Similarly, the right- and left-hand lower derivates are equal except at a set of the first category. Thus, finally, *the two upper derivates are equal, and the two lower derivates are equal except at a set of the first category.*

9. THEOREM 5.—*If the upper and the lower function coincide at one of the points where the upper function is upper semi-continuous, and the lower function lower semi-continuous, both functions are continuous there.*

For the ψ of the upper function is not less than that of the lower function, and is therefore not less than the value of the lower function, since the lower function is lower semi-continuous. Since the upper function has the same value, this shews that

$$\bar{\psi} \geqslant \bar{f}.$$

But, since the upper function is upper semi-continuous at the point,

$$\bar{\phi} \leqslant \bar{f},$$

since, for any function, $\psi \leqslant \phi$,

this proves that $\bar{\psi} = \bar{f} = \bar{\phi}$,

so that the upper function, and similarly the lower function, is continuous.

COR. 1.—*If the points at which the upper and lower functions do not coincide form a set of the first category, the points at which the limiting functions are discontinuous form a set of the first category.*

In particular, if the points known by Lebesgue's theorem to be of zero content, at which the differential coefficient of a function of the class specially considered by him does not exist, form a set of the first category, the points at which the derivates are discontinuous form a set of the first category.

We surmise that, even in the cases considered by Lebesgue, the points at which the differential coefficient does not exist will not, in general, form a set of the first category.

We may state Cor. 1 a little differently as follows :—

Unless the points at which a definite limit exists form a set of the first category only, there is certainly a set of the second category at which all the limiting functions are continuous.

COR. 2.—*At any point at which the series of non-negative continuous functions*

$$u_1 - u_2 + u_3 - \ldots,$$

where $u_1 \geqslant u_2 \geqslant u_3 \geqslant \ldots$ *has a definite limit, the upper and lower functions, and, of course, therefore, all intermediate limiting functions, are continuous.*

In fact, the upper and lower functions are respectively upper and lower semi-continuous throughout the whole interval, the upper function being obtained as the limit of the sum of the first $(2n+1)$ terms, and the lower function as the sum of the first $2n$ terms, when n increases without limit.

10. *Uniform Oscillation.*—When the functions f_n are continuous functions, I shewed that uniform convergence or divergence at a point P might be characterised by the equality of the peak and chasm functions. In this case both are equal to the limiting function, which is moreover continuous at P. It was then shewn that such points of uniform convergence or divergence always exist, and indeed that they form the complementary set of a set of the first category only.

Our theorems shew that, in the more general case, the peak and chasm functions cannot coincide without the upper and lower functions also coinciding. Such points may not exist at all. The preceding theorems, however, suggest a generalisation of the notion of uniform convergence or divergence which subsequent investigations further justify.

DEF.—*At a point where the peak function is equal to the upper function the sequence is said to oscillate uniformly above.*

At a point where the chasm function is equal to the lower function the sequence is said to oscillate uniformly below.

At a point where both these occur, the sequence is said to oscillate uniformly.

The last result of Article 3 may now be re-stated in the following form:—

At a point where a sequence of continuous functions oscillates uniformly above (below), the upper (lower) function is upper (lower) semi-continuous.

11. The theorems proved for the peak and chasm functions in § 12 of my paper on "Convergence and Divergence of a Series of Continuous Functions, ..." are independent of the existence or non-existence of a definite limiting function; it is therefore unnecessary to reproduce the proofs. The enunciations are as follows:—

THEOREM 6.—*Any limit approached by* $\pi(x)$, $\pi_L(x)$, *or* $\pi_R(x)$ *as* x

approaches a point P as limit on the right $\leqslant \pi_L(P)$, and, as x approaches P as limit on the left $\leqslant \pi_R(P)$.

COR.—*π_L is upper semi-continuous on the left and π_R on the right, while π is an upper semi-continuous function. As such* π_L and π_R, as well as π, are at most pointwise discontinuous.*

THEOREM 7.—*At every point of continuity of π,*

$$\pi_L = \pi_R = \pi,$$

and both π_L and π_R are continuous.

THEOREM 8.—*The only points at which both π_L and π_R are continuous are the points of continuity of π.*

THEOREM 9.—*The points, if any, at which π_L differs from π_R are countable.*

Similar results, interchanging the signs $>$ and $<$, hold, of course, for the chasm functions.

THEOREM 10.—*At any point where the peak and chasm functions are equal both are continuous.*

For as x approaches P as limit on the right, by Theorem 6,

$$\chi_L(P) \leqslant \text{Lt}\, \chi_L(x) \leqslant \text{Lt}\, \pi_L(x) \leqslant \pi_L(P),$$

at such a point as is contemplated, therefore, the sign of equality must hold throughout. The left-hand peak and chasm functions are therefore continuous on the left. Similarly we can prove the result on the right.

12. The following theorem, which in its form of proof exactly corresponds to that given in my paper quoted in § 11, proving that the points at which both the peak and chasm functions are continuous are points of uniform convergence or divergence, shews that points of uniform oscillation (above and below) always occur, and that their distribution is precisely that of the points of uniform convergence and divergence in the more special case.

THEOREM 11.—*At every point where the peak function is continuous and the upper function upper semi-continuous, the peak function is equal to the upper function (that is, there is uniform oscillation above).*

For, if possible, let P be a point at which the peak function is con-

* W. H. Young, "Note on Left- and Right-Handed Semi-Continuous Functions," *Quart. Jour. of Math.*, 1908.

tinuous and the upper function upper semi-continuous, but these two functions are not equal. Then, by Theorem 2, we have

$$f(P) < \pi(P).$$

By the sense of this relation $\pi(P)$ cannot be $-\infty$, nor $f(P)$ be $+\infty$; therefore we can find two numbers α and β, such that

$$f(P) < \beta, \quad \alpha < \pi(P), \tag{2}$$

while at the same time

$$\beta < \alpha. \tag{3}$$

Since P is a point of continuity of the peak function, we can find a whole interval (A, B) containing P as internal point, at every internal point of which

$$\alpha < \pi(x). \tag{4}$$

From the definition of the peak function it now follows from (2) that we can find a point Q in (A, B), such that

$$\alpha < M_Q;$$

and therefore we can find an integer n_1, greater than some assigned integer, such that

$$\alpha < M_{n_1, Q}.$$

Since $M_{n_1, Q}$ is the upper bound of the values of $f_{n_1}(x)$ in the interval (P, Q), there is certainly a point of this interval where $f_{n_1}(x) > \alpha$. Hence f_{n_1} being continuous, there is a whole interval (A_1, B_1), internal to (A, B) at every point of which

$$\alpha < f_{n_1}(x),$$

while, of course, the relation (4) still holds.

By the same reasoning we shew that there is an interval (A_2, B_2) inside (A_1, B_1), such that at every point of it,

$$\alpha < f_{n_2}(x),$$

n_2 being a certain integer greater than n_1.

Proceeding thus we get a series of intervals (A, B), (A_1, B_1), (A_2, B_2), ... each lying inside the preceding, and a corresponding series of increasing integers, $n_1, n_2, \ldots, n_r, \ldots$ such that at every point of (A_r, B_r),

$$\alpha < f_{n_r}(x).$$

These intervals have at least one common internal point x, at which the preceding inequality holds for all integers r, so that the upper function f there is certainly greater than or equal to α; we have therefore found a point x of (A, B) at which

$$\alpha \leqslant f(x).$$

Since this is true for every smaller interval (A, B) containing the point P as internal point, it follows that

$$\alpha \leqslant \phi(P),$$

while, by (2) and (3), $\qquad f(P) < \beta < \alpha < \phi(P),$

which is impossible, since at the point P the upper function f is upper semi-continuous. This, therefore, proves that at P, the peak function being continuous and the upper function upper semi-continuous, these two functions must be equal.

COR.—*The points, if any, where the peak function differs from the upper function (that is, the points of non-uniform oscillation above) form a set of the first category.*

For they belong to the set consisting of the discontinuities of the peak function and the points at which the upper function is not upper semi-continuous. But the discontinuities of the peak function form a set of the first category, and so do the points at which the upper function is not upper semi-continuous. Since the set consisting of all the points of two sets of the first category is a set of the first category, this proves the corollary.

Similarly we have the alternative theorem and corollary :—

THEOREM 11′.—*At every point where the chasm function is continuous and the lower function lower semi-continuous, the chasm function is equal to the lower function (that is, there is uniform oscillation below).*

COR. 1.—*The points, if any, where the chasm function differs from the lower function (that is, the points of non-uniform oscillation below) form a set of the first category.*

COR. 2.—*With the possible exception of the points of a set of the first category, the peak function is equal to the upper function and the chasm function to the lower function (that is, the oscillation, both above and below, is uniform).*

13. In the case when a definite limiting function exists, as already mentioned, it was shewn that the definition of a point of uniform convergence or divergence of a sequence of continuous functions as a point where the peak and chasm functions were equal, was concomitant to the old $R_n(x)$-definition and its extension to the case when infinite values are allowed.

If we seek a corresponding formulation of uniform oscillation, it is found that discrepancies occur, except in the case when the upper and lower functions coincide at the point in question. These discrepancies, which arise when infinite values are allowed, occur none the less when only finite values are permitted. In fact it may be shewn* that, at a point of uniform oscillation where the upper and lower functions are both finite, we can find an interval d, corresponding to any assigned positive quantity e, containing P, and such that for all points x within it,

$$|f(x)-f_n(x)| \leqslant k+e,$$

f denoting not only the upper but also the lower function, for all values of $n \geqslant m$, where m is an integer and k a quantity, both independent of x, which can be determined. The converse of this theorem, however, seems only to hold when k is zero.

The $R_n(x)$-definition, indeed, does not lead directly to generalisation when the upper and lower functions are distinct. The very plausible generalisation of the inequality

$$|R_n(x)| < e,$$

in the form
$$\underline{f}(x)-e \leqslant f_n(x) \leqslant \bar{f}(x)+e,$$

is found to lead to a point at which the upper function is lower and the lower function upper semi-continuous. Such points, by the results we have already obtained, will rarely occur at all, so that any theory based on their existence will be of very limited application. All this points to advantages in the new definition of uniform convergence by means of the peak and chasm functions.

14. The concept of uniform convergence at a point, however, is one which may be extended to the case where a definite limit exists at one or more, but not at all points.

DEF.—*The sequence of functions $f_1(x)$, $f_2(x)$, ... is said to converge uniformly to a definite limit at the point P if, given any positive quantity e, an interval d can be described, having P as internal point, so that, for all points x within this interval d,*

$$|\bar{f}(x)-f_n(x)| < e,$$

and also
$$|\underline{f}(x)-f_n(x)| < e$$

* As in the proof of Theorem 3, pp. 37 of my paper already quoted.

($\bar{f}$ *and* $\underline{f}$ *being respectively the upper and lower functions*), *for all values of* $n \geqslant m$, *where* m *is an integer independent of* x.

Similarly, we define the expressions right-handed and left-handed uniform convergence at P: in this case the interval d will have P as end-point.

This definition may also be adapted to give "uniform divergence to an infinite limit at the point P." We merely have to replace the two above inequalities by the single inequality

$$f_n(x) > A,$$

or

$$f_n(x) < -A,$$

according as the infinite limit is positive or negative, A being any positive quantity.

The reasoning employed in the proofs of Theorems 3 and 4 of my paper on "Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," *Proc. London Math. Soc.*, 1907, may then be transferred almost verbatim to the case in point. We only have to exercise ordinary care wherever the limiting function f occurs, to modify the wording so as to refer both to the upper and to the lower functions instead of to the single limiting function. The result is then as follows:—

THEOREM 12.—*If the* f_n*'s are continuous functions, and* P *a point at which the left-hand peak and chasm functions are equal, the sequence converges or diverges uniformly on the left at* P.

Conversely, if the sequence converges, or diverges, uniformly on the left at P, *the left-hand peak and chasm functions are equal at* P.

(Similar results hold, of course, on the right.)

It may be emphasized that, in general, there are no points of uniform convergence. When such do occur, they are special cases of points of uniform oscillation which, as we saw, always do occur. Wherever at a point of uniform oscillation we have a single limiting value, the point is one of uniform convergence.

15. We shall now require the following theorem about monotone sequences:—*

THEOREM 13.—*If* $f_1 \geqslant f_2 \geqslant \dots$ *is a monotone decreasing sequence of functions, whose limit is* f, *then the chasm function is the associated*

* This is a generalisation of the theorem in the paper quoted, "On Monotone Sequences of Continuous Functions."

lower limiting function of f, that is

$$\chi = \psi,$$

and at any common point of continuity of $f_1, f_2, \ldots$, the peak function is the limiting function f,

$$\pi = f.$$

This follows from the Theorem of the Bounds, viz.,* the lower bounds as well as the upper bounds form a monotone decreasing sequence; the limit of the lower bounds is the lower bound of the limit, while the limit of the upper bounds is only $\geqslant$ the upper bound of the limit.

Now, if P be any point, and Q a near point on the right, and, as usual, $L_{n,Q}$ denote the lower bound of f_n in the interval (P, Q), and $M_{n,Q}$ the upper bound, we have, by the theorem of the bounds,

$$L_{1,Q} \geqslant L_{2,Q} \geqslant \ldots \geqslant L_Q \text{ as limit,}$$

where L_Q denotes the lower bound of f in the same interval.

Now the chasm function is defined as the limit as Q moves up to P of the limit of $L_{n,Q}$, as n increases indefinitely, that is, Lt L_Q. But, by the definition of the associated lower limiting function of f, the limit of L_Q is χ, thus

$$\chi(P) = \psi(P);$$

this proves the first part of the theorem.

If A denote any number greater than $f(P)$, when $f(P)$ is not $+\infty$, we can, since $f(P)$ is the limit of $f_n(P)$, find an integer m such that for all integers $n \geqslant m$,

$$f_n(P) < A.$$

Since $f_n(x)$ is continuous at P, we can find an interval (P, Q) to the right of P, such that

$$M_{m,Q} < A,$$

But

$$M_{m,Q} \geqslant M_{m+1,Q} \geqslant \ldots,$$

so that, for all integers $n \geqslant m$,

$$M_{n,Q} < A.$$

Proceeding to the limit with n,

$$M_Q \leqslant A,$$

and letting Q move up to P, we have, in the limit,

$$\pi_R(P) \leqslant A.$$

* W. H. Young, "On Functions defined by Monotone Sequences and their Upper and Lower Bounds," *Messenger of Mathematics*, New Series, February, 1908.

Since A is any number greater than $f(P)$, it follows that

$$\pi_R(P) \leqslant f(P). \tag{1}$$

But since $f_n(x)$ is continuous at P, and for all values x in the open interval (P, Q),

$$f_n(x) \leqslant M_{n, Q};$$

therefore

$$f_n(P) \leqslant M_{n, Q}.$$

Since this is true for all values of n,

$$f(P) \leqslant M_Q.$$

Since this is true for all positions of Q as it moves up to P,

$$f(P) \leqslant \pi_R(P), \tag{2}$$

whence, by (1),

$$f(P) = \pi_R(P),$$

when $f(P)$ is not $+\infty$. But when $f(P) = +\infty$, the same result follows at once from (2), which proves the theorem.

16. Applying the preceding theorem to the monotone descending sequence of lower semi-continuous functions $v_1 \geqslant v_2 \geqslant \dots$ whose limit is the upper function $\bar{f}$, we see that at every point except at the points of a set of the first category, where one of the functions v_n at least is discontinuous, the upper function $\bar{f}$ is itself the peak function of the v_n's.

Now, by the definition of the v_n's, it is evident that the upper limit of $v_n \geqslant$ that of $f_n, f_{n+1}, f_{n+2}, \dots$ in any interval (P, Q), and is therefore $\geqslant$ the M_Q of the f_n's. Hence the M_Q of the v_n's $\geqslant$ that of the f_n's, so that the peak function of the v_n's $\geqslant$ that of the f_n's.

Hence, *since, by Theorem* 2, *$\bar{f}$ is never greater than π at all the common points of continuity of the v_n's, the upper function $\bar{f} =$ the peak function of the f_n's, that is,*

$$\bar{f} = \pi,$$

except at a set of the first category.

Similarly, *the lower function is the chasm function except at a set of the first category*, that is,

$$\underline{f} = \chi.$$

Hence, *when the upper and lower functions agree, except at a set of the first category, the peak and chasm functions agree, except at a set of the first category.* In particular, *if there is a definite limiting function at every point, the peak and chasm functions agree except at a set of the first category, that is, there is uniform convergence and divergence, except at a set of the first category.*

17. We have seen that in general the points at which the upper and lower functions are respectively not upper and not lower semi-continuous form a set of the first category, as do also the points at which the oscillation is non-uniform. In the following examples both these sets of points are only countable.

Ex. 1.—Let $f_n(x)$ for each value of n be a monotone increasing function of x. Then, if Q lie on the right of P,

$$M_{n,Q} = f_n(Q), \qquad L_{n,Q} = f_n(P).$$

Hence it easily follows that—

(1) The right-hand peak function is the associated upper limiting function of the upper function, *i.e.*,

$$\pi_R(P) = \bar{\phi}_R(P).$$

(2) The left-hand peak function is the upper function, *i.e.*,

$$\pi_L(P) = \bar{f}(P).$$

Similarly,

(3) The right-hand chasm function is the lower function, *i.e.*,

$$\chi_R(P) = \underline{f}(P).$$

(4) The left-hand chasm function is the lower associated limiting function of the lower function, *i.e.*,

$$\chi_L(P) = \underline{\psi}_L(P).$$

We can at once deduce that the upper (lower) function is upper (lower) semi-continuous except at a countable set of points, and that with the same exceptions the oscillation above (below) is uniform.

Again, if the upper (lower) function is upper (lower) semi-continuous throughout the interval considered, the oscillation above (below) is uniform throughout.

Further, when a limiting function exists throughout an interval it is continuous except at a countable set of points, and the convergence or divergence is uniform, except at a countable set of points.

Ex. 2.—Now let $f_n(x)$ for each value of n be a continuous function with finite total fluctuation, and suppose that in every interval this fluctuation has a definite limit when n is infinite. With the notation

of Lebesgue* we may write

$$f_n(x) = f_n(a) + P_n(x) - N_n(x),$$

where $P_n(x)$ and $N_n(x)$ are both monotone increasing functions. Also,

$$V_n(x) = P_n(x) + N_n(x),$$

where $V_n(x)$ is also monotone increasing, and, since it is the total fluctuation of $f_n(x)$ in the interval (a, x), has a definite limit, say $V(x)$.

By Ex. 1, $V(x)$ is continuous and the convergence of $V_n(x)$ to $V(x)$ is uniform, except at a countable set of points.

Now assume further that at a, that is, at some one point, $f_n(x)$ has a definite limit. Then the above equations shew that

$$\bar{f}(x) = f(a) - V(x) + 2\bar{P}(x),$$

and a similar equation for the lower function. This proves that here also the upper and lower functions are respectively upper and lower semi-continuous and the oscillation is uniform, except at a countable set of points.

18. So far the work has in general applied to functions of any number of variables. In the special case when there is only one independent variable, we can work with continuity and semi-continuity on one side only. I have already made this extension in the case of one theorem of Baire's.† We now prove that our main result remains true if the generating functions $f_1, f_2, \ldots$ are continuous on one side only.

THEOREM 14.—*If $f_1, f_2, \ldots$ be continuous on the right, and F the upper function of the sequence, F is upper semi-continuous excepting only at a set of the first category.*

Let $v_{1,1}$ be the function which at every point is equal to both f_1 and f_2, or to the greater of these. Then it is easily proved that $v_{1,1}$ is also continuous on the right.‡

Similarly, if $v_{1,2}$ be defined from $v_{1,1}$ and f_3, and each of the functions

* *Intégration*, pp. 52 *seq.*

† *Loc. cit.*, p. 299, footnote *.

‡ For, if $f_1 = f_2 = v$, the only limit which can be approached by v is the only limit which can be approached by f_1 or f_2, viz., the common value, so that v is continuous at the point. If, however, $f_1 = v$, and $f_2 < v - 2e$, there will be a whole interval to the right throughout which $f_1 > v - e$, and $f_2 < v - e$, whence $f_1 = v$ at every point, so that v, like f_1, is continuous on the right.

$v_{1,n}$ from $v_{1,n-1}$ and f_{n+1}, the functions

$$v_{1,1} \leqslant v_{1,2} \leqslant v_{1,3} \leqslant \dots$$

are all continuous on the right, and form a monotone increasing sequence. Their limit v_1 is therefore lower semi-continuous on the right, and has at each point P the highest possible limiting value of the quantities $f_1(P)$, $f_2(P)$, ..., or one of these values, if it is greater than all such limiting values.

Let $v_2, v_3, \dots, v_n, \dots$ be defined in like manner, omitting in turn the first, the first two, ..., the first $(n-1)$, ..., of the functions f_r. Then, evidently,

$$v_1 \geqslant v_2 \geqslant \dots$$

is a monotone decreasing sequence of functions, each of which is lower semi-continuous on the right, and has at each point the highest possible limiting value of the quantities $f_1(P)$, $f_2(P)$, ..., and is therefore none other than the upper function F.

Since* a function which is lower semi-continuous on the right is continuous with respect to every perfect set, excepting only at a set of the first category with respect to that set, the points at which one at least of the functions $v_1, v_2, \dots$ is discontinuous form a set of the first category. At any point not belonging to this set all these functions are continuous, and therefore the upper function F is upper semi-continuous. This proves the theorem.

19. We shall not attempt to deduce any of the obvious consequences of the theorems above given. As one example we may, however, note the following application.

Let
$$f(x) = \operatorname*{Lt}_{n=\infty} f_n(x),$$

where $f_n(x)$ is for every value of x a continuous function with a continuous differential coefficient $f'_n(x)$. Then

$$\frac{f(x+h)-f(x)}{h} = \operatorname*{Lt}_{n=\infty} \frac{f_n(x+h)-f_n(x)}{h} = \operatorname*{Lt}_{n=\infty} f'_n(x+\theta h),$$

where θ is > 0 and < 1.

Now, consider the set of functions of which $f'_n(x)$ is a type, and let $M_{n,Q}$, M_Q, $\Pi(x)$, ... refer to this set of functions, the rest of the notation being the same as in the previous articles.

Then, evidently,
$$f'_n(x+\theta h) \leqslant M_{n,Q};$$

* *Loc. cit.*, p. 310, footnote.

and therefore $\overline{\underset{\substack{\text{upper} \\ h=0}}{\mathrm{Lt}}} \underset{n=\infty}{\mathrm{Lt}} f_n'(x+\theta h) \leqslant \overline{\underset{\substack{\text{upper} \\ h=0}}{\mathrm{Lt}}} M_Q \leqslant \Pi_R(x)$;

therefore the upper derivate $f^+(x) \leqslant \Pi_R(x)$;

similarly the lower derivate $f_+(x) \geqslant X_R(x)$,

with similar results on the left.

It at once follows that, except at a set of the first category,

$$f^+(x) \leqslant \text{the upper function of the set of } f_n'(x),$$

$$f_+(x) \geqslant \text{the lower function of this set},$$

and further that if the oscillation is uniform above and below throughout an interval, then these and the corresponding inequalities on the left hold throughout. Further, if even at an isolated point at which the oscillation is uniform above and below, we have convergence, or divergence to a definite infinite limit, for the series of differential coefficients, then at that point term by term differentiation is allowable.

NOTE ON A SOLUBLE DYNAMICAL PROBLEM

By L. J. ROGERS.

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A DYNAMICAL system whose equation of energy in n independent coordinates $x_1, x_2, \ldots, x_n$ is

$$Y_1\dot{x}_1^2+Y_2\dot{x}_2^2+\ldots=\frac{P_1}{Y_1}+\frac{P_2}{Y_2}+\ldots, \tag{1}$$

where $Y_1, Y_2, \ldots, P_1, P_2$ are functions of $x_1, x_2, \ldots$, is satisfied by the equations

$$Y_1\dot{x}_1=\sqrt{P_1}, \quad Y_2\dot{x}_2=\sqrt{P_2}, \quad \ldots, \tag{2}$$

provided $P_1, P_2, \ldots$ are respectively functions of $x_1, x_2, \ldots$ only. For, in Lagrange's equation,

$$2\frac{d}{dt}Y_1\dot{x}_1-\dot{x}_1^2\frac{\partial Y_1}{\partial x_1}-\dot{x}_2^2\frac{\partial Y_2}{\partial x_1}-\ldots=\frac{\partial}{\partial x_1}\frac{P_1}{Y_1}+P_2\frac{\partial}{\partial x_1}\frac{1}{Y_2}+\ldots,$$

we have

$$2\frac{d}{dt}Y_1\dot{x}_1=2\frac{d}{dt}\sqrt{P_1}=\frac{\dot{P}_1}{\sqrt{P_1}}=\frac{\dot{x}_1}{\sqrt{P_1}}\frac{\partial P_1}{\partial x_1}=\frac{1}{Y_1}\frac{\partial P_1}{\partial x_1},$$

and

$$\dot{x}_r^2\frac{\partial Y_r}{\partial x_1}=-P_r\frac{\partial}{\partial x_1}\frac{1}{Y_r} \quad (r=1, 2, \ldots, n),$$

so that the equation is easily seen to be satisfied.

If $Y_r=(X_r-X_1)(X_r-X_2)\ldots(X_r-X_{r-1})(X_r-X_{r+1})\ldots(X_r-X_n)$, (3)

where $X_1, X_2, \ldots$ are respectively functions of $x_1, x_2, \ldots$ only, the solution of the dynamical problem is general and complete. For, if

$$P_r=F_r+a_0+a_1X_r+a_2X_r^2+\ldots+a_{n-2}X_r^{n-2}+CX_r^{n-1},$$

the equation of energy becomes

$$Y_1\dot{x}_1^2+\ldots=C+\frac{F_1}{Y_1}+\frac{F_2}{Y_2}+\ldots,$$

in which the constants $a_0, a_1, \ldots, a_{n-2}$ do not occur.

Moreover, equations (2) may be written

$$\left.\begin{aligned}
\frac{dx_1}{\sqrt{P_1}}+\frac{dx_2}{\sqrt{P_2}}+\ldots &= 0\\
\frac{X_1\,dx_1}{\sqrt{P_1}}+\frac{X_2\,dx_2}{\sqrt{P_2}}+\ldots &= 0\\
\ldots\quad\ldots\quad\ldots&\quad\ldots\\
\frac{X_1^{r-2}\,dx_1}{\sqrt{P_1}}+\frac{X_2^{r-2}\,dx_2}{\sqrt{P_2}}+\ldots &= 0\\
\frac{X_1^{r-1}\,dx_1}{\sqrt{P_1}}+\frac{X_2^{r-1}\,dx_2}{\sqrt{P_2}}+\ldots &= dt.
\end{aligned}\right\} \tag{4}$$

The solution is therefore general, since we have $2n$ arbitrary constants, viz., $a_0, a_1, \ldots, a_{n-2}, C$, and the n constants obtained by integrating (4). It is also complete, as equations (4) are directly integrable.

If $x_1, x_2, \ldots$ are the generalized elliptic coordinates of a particle moving in n-fold space, the *vis viva* of the particle can be reduced to a form corresponding to that in (1), subject to condition (3). If the functions F are all zero, we shall get the conditions for a straight line.

Hence the equations of a straight line in n-fold elliptic coordinates are the algebraic solutions of the system (4) of Abelian differential equations.

We may, moreover, extend the generality of the solution by supposing that the *vis viva* contains terms such as $L\dot{\phi}^2$, in addition to those assumed above, L being a function of $x_1, x_2, \ldots, x_n$ only, of a form such that

$$\frac{1}{L}=\frac{1}{\phi_1(X_1)\,Y_1}+\frac{1}{\phi_2(X_2)\,Y_2}+\ldots+\frac{1}{\phi_n(X_n)\,Y_n}.$$

In such a case ϕ is an ignorable coordinate, leading to an integral of the type $L\dot{\phi}=h$, and as such is subject to the ordinary laws which modify the form of the energy-equation in which ignorable coordinates exist. When, however, as in this case, the ignorable coordinate occurs in one term only, through the square of its time-flux, in the kinetic energy, it is worth noticing that we may eliminate the variables entirely before applying the other Lagrangian equation, provided the corresponding term be placed in the force-function.

For instance, if $T+\frac{1}{2}L\dot{\phi}^2=U$ be the equation of energy where T, L, U are independent of ϕ, and θ any coordinate of the system, the true Lagrangian equation for θ is

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}}-\frac{\partial T}{\partial\theta}-\tfrac{1}{2}\dot{\phi}^2\,\frac{\partial L}{\partial\theta}=\frac{\partial U}{\partial\theta},$$

i.e.,
$$\frac{d}{dt}\frac{\partial T}{\partial \dot\theta}-\frac{\partial T}{\partial \theta}+\tfrac{1}{2}h^2\frac{\partial}{\partial \theta}\frac{1}{L}=\frac{\partial U}{\partial \theta},$$

which is the same as if the equation had been derived from an energy-equation
$$T=U-\tfrac{1}{2}\frac{h^2}{L}.$$

If, then, L has the form assumed above, the dynamical system has now a complete solution for all variables except ϕ.

The last, however, is given by
$$\frac{\dot\phi}{h}=\frac{1}{L}=\frac{1}{\phi_1(X_1)\,Y_1}+\frac{1}{\phi_2(X_2)\,Y_2}+\dots$$
$$=\frac{\dot x_1}{\phi_1(X_1)\sqrt{P_1}}+\frac{\dot x_2}{\phi_2(X_2)\sqrt{P_2}}+\dots,$$

so that
$$\frac{d\phi}{h}=\frac{dx_1}{\phi_1(X_1)\sqrt{P_1}}+\frac{dx_2}{\phi_2(X_2)\sqrt{P_2}}+\dots,$$

where now
$$P_r=F_r+a_0+a_1X_r+\dots+CX_r^{n-1}-\tfrac{1}{2}\frac{h^2}{\phi(X_r)}.$$

As an instance of the introduction of extra terms such as $\frac{1}{2}L\dot\phi^2$ into the kinetic energy, we may shew how Euler's problem of the motion of a particle attracted by Newtonian forces towards two fixed centres of force may be *fully* solved, *i.e.*, when the path of the particle is not confined to a plane.

If the force-centres lie at the foci of the ellipse
$$\frac{x^2}{a}+\frac{y^2}{b}=1 \quad (a>b),$$

and
$$x^2=\frac{(a+\lambda)(a+\mu)}{a-b},\quad y^2=\frac{(b+\lambda)(b+\mu)}{b-a},$$

then the focal distances are $\sqrt{a+\lambda}\pm\sqrt{a+\mu}$.

Taking the coordinates of the particle as x, y in the plane containing the particle and the force-centres, and ϕ the angle this plane makes with a fixed plane with which it is coincident at some epoch, the energy-equation is
$$\tfrac{1}{2}(\dot x^2+\dot y^2+y^2\dot\phi^2)=C+\frac{\kappa_1}{r_1}+\frac{\kappa_2}{r_2},$$

where r_1, r_2 are the distances from the centres of force.

In the elliptic coordinates this becomes

$$\tfrac{1}{8}(\lambda-\mu)\frac{\dot{\lambda}^2}{(a+\lambda)(b+\lambda)}+\tfrac{1}{8}(\mu-\lambda)\frac{\dot{\mu}^2}{(a+\mu)(b+\mu)}-\tfrac{1}{2}\frac{(b+\lambda)(b+\mu)}{a-b}\dot{\phi}^2$$

$$=C+\frac{(\kappa_1+\kappa_2)\sqrt{a+\lambda}}{\lambda-\mu}+\frac{(\kappa_2-\kappa_1)\sqrt{a+\mu}}{\mu-\lambda}.$$

Here L, the coefficient of $\dot{\phi}^2$ satisfies the relation

$$\frac{1}{L}=-(a-b)\left\{\frac{1}{(b+\lambda)(\lambda-\mu)}+\frac{1}{(b+\mu)(\mu-\lambda)}\right\}.$$

so that a complete solution of the problem is obtained according to the method above indicated.*

When $n=3$, and X, Y, Z are elliptic coordinates, the *vis viva* takes the form

$$\tfrac{1}{4}(X-Y)(X-Z)\dot{x}^2+\tfrac{1}{4}(Y-Z)(Y-X)\dot{y}^2+\tfrac{1}{4}(Z-X)(Z-Y)\dot{z}^2,$$

where $\left(\dfrac{dX}{dx}\right)^2=X^3+pX^2+qX+r,\quad \left(\dfrac{dY}{dy}\right)^2=Y^3+pY^2+qY+r,$

and $$\left(\frac{dZ}{dz}\right)^2=Z^3+pZ^2+qZ+r.$$

Taking the equation of energy to be

$$(X-Y)(X-Z)\dot{x}^2+(Y-Z)(Y-X)\dot{y}^2+(Z-X)(Z-Y)\dot{z}^2$$

$$=C+\frac{P}{(X-Y)(X-Z)}+\frac{Q}{(Y-Z)(Y-X)}+\frac{R}{(Z-X)(Z-Y)},$$

we have seen that the dynamical system is completely soluble if P, Q, R are respectively functions of X, Y, Z.

These functions can be put into a simple general form when subjected to the condition that the forces are due to gravitation only.

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* When the attractions are zero the path is a straight line whose equations are algebraic in λ, μ and circular functions of ϕ. There are three equations for the solution, involving elliptic integrals. The equation involving λ and μ, leads to the addition equation of elliptic integrals of the first kind; while the equations giving t and ϕ lead to the fundamental properties of those of the second and third kind respectively. There is probably no analytical process which yields these results so concisely.

For it is known that if the square of the velocity of a particle be denoted by

$$\frac{\dot{x}^2}{h_1^2}+\frac{\dot{y}^2}{h_2^2}+\frac{\dot{z}^2}{h_3^2},$$

where x, y, z are orthogonal coordinates, then Laplace's operation is equivalent to

$$h_1h_2h_3\left(\frac{\partial}{\partial x}\frac{h_1}{h_2h_3}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{h_2}{h_3h_1}\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\frac{h_3}{h_1h_2}\frac{\partial}{\partial z}\right),$$

i.e.,

$$(Y-Z)\frac{\partial^2}{\partial x^2}+(Z-X)\frac{\partial^2}{\partial y^2}+(X-Y)\frac{\partial^2}{\partial z^2}, \tag{5}$$

so that this operation must annihilate the function

$$C+\frac{P}{(X-Y)(X-Z)}+\frac{Q}{(Y-Z)(Y-X)}+\frac{R}{(Z-X)(Z-Y)}.$$

It will be found convenient to write X_1 for $\dfrac{dX}{dx}$, and X_2 for

$$\frac{d^2X}{dx^2}, = \tfrac{1}{2}(3X^2+2pX+q),$$

with corresponding meanings for Y_1, Z_1, Y_2, Z_2; and to use Ω for the operation (5), which, of course, only differs from Laplace's ∇^2 by a factor here unnecessary.

Now it can readily be shown that

$$\Omega\frac{X_1}{(X-Y)(X-Z)}$$

$$=\frac{\partial^2}{\partial x^2}\left(\frac{X_1}{X-Y}-\frac{X_1}{X-Z}\right)-X_1\frac{\partial^2}{\partial y^2}\frac{1}{X-Y}+X_1\frac{\partial^2}{\partial z^2}\frac{1}{X-Z}$$

$$=\frac{\partial}{\partial x}\left\{\frac{X_2}{X-Y}-\frac{X_2}{X-Z}-\frac{X_1^2}{(X-Y)^2}+\frac{X_1^2}{(X-Z)^2}\right\}$$

$$-X_1\frac{\partial}{\partial y}\frac{Y_1}{(X-Y)^2}+X_1\frac{\partial}{\partial z}\frac{Z_1}{(X-Z)^2}$$

$$=\frac{\partial}{\partial x}\left\{\frac{Y_2}{X-Y}-\frac{Z_2}{X-Z}-\frac{2Y_2}{X-Y}-\frac{Y_1^2}{(X-Y)^2}+\frac{2Z_2}{X-Z}+\frac{Z_1^2}{(X-Z)^2}\right\}$$

$$-X_1\frac{\partial}{\partial y}\frac{Y_1}{(X-Y)^2}+X_1\frac{\partial}{\partial z}\frac{Z_1}{(X-Z)^2}$$

$$=0, \tag{6}$$

so that it will be better to substitute UX_1 for P, since the coefficient of U in ΩUX_1 will then be zero.

Thus
$$\Omega \frac{UX_1}{(X-Y)(X-Z)}$$
$$= \frac{\partial^2}{\partial x^2} UX_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) + \ldots$$
$$= X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial^2 U}{\partial x^2} + 2\frac{\partial U}{\partial x} \frac{\partial}{\partial x} X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right),$$

which, if we put $\dfrac{\partial U}{\partial x} = \dfrac{L}{X_1^2}$, becomes

$$X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial}{\partial x} \frac{L}{X_1^2} + 2\frac{L}{X_1^2} \frac{\partial}{\partial x} X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right)$$
$$= \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial L}{\partial X} - 2L \left\{\frac{1}{(X-Y)^2} - \frac{1}{(X-Z)^2}\right\}. \quad (7)$$

Hence, if M and N be functions respectively of y and z derived from Q and R, just as L is derived from P, we see that the final condition required is that the sum of the three expressions of which (7) is the type must be equal to zero.

The form of the relation being purely algebraic in X, Y, Z, we are naturally led to test for what value of n the assumptions $L = X^n$, $M = Y^n$, $N = Z^n$ lead to the satisfying of the identity. Such values are easily seen to be $n = 0, 1, 2, 3, 4$ and no other.

Finally, we see that the necessary value of P, *i.e.*, $X_1 \displaystyle\int \frac{L}{X_1^3}\, dX$, is

$$(X^3+pX^2+qX+r)^{\frac{1}{2}} \int \frac{AX^4+BX^3+CX^2+DX+E}{(X^3+pX^2+qX+r)^{\frac{3}{2}}}\, dX,$$

while Q and R are the same functions of Y and Z respectively. In consequence of (6) it is evident that the lower limit of these integrals may be taken as arbitrary.

THE RELATION BETWEEN THE CONVERGENCE OF SERIES AND OF INTEGRALS

By T. J. I'A. BROMWICH.

[Read and Received April 30th, 1908.]

IT has long been known that, when $f(x)$ is a positive function which steadily decreases as x tends to infinity, the series $\sum^{\infty} f(n)$ and the integral $\int^{\infty} f(x)\,dx$ converge or diverge together: and in case of divergence, the difference

$$\int_{\mu}^{\nu} f(x)\,dx - \sum_{\mu}^{\nu} f(n)$$

tends to a definite limit as ν tends to infinity.*

But, when the series contains terms of both signs (although steadily decreasing in numerical value), it is not easy to make a corresponding statement with reference to the relation between the convergence of the series and of the integral.†

It is known, of course, that, when $f(x)$ decreases steadily to zero as a limit, the two series and the two integrals

$$\sum^{\infty} f(n) \sin na, \quad \int^{\infty} f(x) \sin ax\,dx, \quad \sum^{\infty} f(n) \cos na, \quad \int^{\infty} f(x) \cos ax\,dx$$

are all convergent.‡ And these results can be easily extended to cases in which the periodic factors are of the forms

$$\sin ax\, P(\sin^2 ax), \qquad \cos ax\, P(\sin^2 ax),$$

where P is a *polynomial*. But, if the polynomial P is replaced by

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* The main part of the theorem goes back to Maclaurin: for a proof of the latter part, see my *Infinite Series*, Art. 11. When the monotonic condition is removed from $f(x)$, the theorem is no longer true; for examples, see p. 423 of my book.

† Of course, in a large number of interesting cases, the terms decrease fast enough to ensure *absolute* convergence. This case is covered, from the practical point of view, by Maclaurin's rule, and we shall suppose that absolute convergence is excluded from the cases discussed here.

‡ See, for example, *Infinite Series*, Arts. 19, 20, 169.

a general continuous function, the cosine-series is known to diverge for certain values of a, although the integral is always convergent; however, with the same values of a the sine-series is convergent.*

If the periodic factor in the series is of the form

$$\sin\phi(x) \quad \text{or} \quad \cos\phi(x),$$

where $\phi(x)$ tends to infinity *more rapidly than* x, it is practically certain that the convergence of the integral gives no information with regard to the nature of the series.

Thus, for instance, the integrals

$$\int^{\infty} f(x)\sin(ax^p)\,dx, \qquad \int^{\infty} f(x)\cos(ax^p)\,dx,$$

where p is positive, will converge if $f(x)$ steadily decreases to any finite limit (not necessarily zero) † : but, on the other hand, the two series

$$\Sigma f(n)\sin(an^p), \qquad \Sigma f(n)\cos(an^p)$$

have only been considered for rational values of a/π and integral values of p; they are then known to diverge [even if $f(x)$ tends to zero] unless a certain condition is satisfied; and this condition is certainly broken even in the simplest case (when p is 2) except for special values of a.‡

The object of the following note is to prove that [with certain restrictions on the functions, stated in (a)–(δ) on p. 329], *when $\phi(x)$ tends steadily to infinity, but* MORE SLOWLY THAN *x, the behaviour of the integrals*

$$\int^{\infty} f(x)\sin\phi(x)\,dx, \qquad \int^{\infty} f(x)\cos\phi(x)\,dx$$

entirely settles the character of the series

$$\overset{\infty}{\Sigma} f(n)\sin\phi(n), \qquad \overset{\infty}{\Sigma} f(n)\cos\phi(n).$$

This theorem is then applied, in § 2, to extend (and simplify the proofs of) certain known theorems, the simplest of which is that if

$$A_\nu + iB_\nu = \sum_1^\nu \frac{1}{n^{1+ai}} \quad (a \text{ real}),$$

then $A_\nu + iB_\nu - i/a\nu^{ai}$ tends to a definite limit, so that both A_ν and B_ν have a range of oscillation $2/a$ as ν tends to infinity.

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* See Bromwich and Hardy, *Quarterly Journal*, Vol. XXXIX., May, 1908, pp. 232, 236, 240, and also below, p. 338.

† See my *Infinite Series*, Art. 169 and Ex. 8, p. 468.

‡ See below, § 3, p. 338; and Genocchi, *Atti di Torino*, t. X., 1875, p. 991.

1. *Proof of the Theorem.*

Let us write $F(x) = f(x)\sin\phi(x)$, where we suppose that

$$\left.\begin{array}{ll}(\alpha) & f(x) \textit{ tends steadily to zero*} \\ (\beta) & \phi(x) \textit{ tends steadily to infinity} \\ (\gamma) & \phi'(x) \textit{ tends steadily to zero}\end{array}\right\} \textit{as } x\to\infty.$$

From these conditions it follows that

$$f'(x) \leqslant 0, \qquad \phi'(x) \geqslant 0. \tag{1}$$

Now, if we write $\quad X_n = \int_n^{n+1} F(x)\,dx - F(n),$

we have the equations

$$X_n = \int_n^{n+1}\{F(x)-F(n)\}\,dx = \int_0^1\{F(n+t)-F(n)\}\,dt, \tag{2}$$

$$F(n+t)-F(n) = \int_0^t F'(n+v)\,dv. \tag{3}$$

But $\qquad F'(x) = f'(x)\sin\phi(x)+f(x)\,\phi'(x)\cos\phi(x)$;

and so $\qquad |F'(n+v)| \leqslant |f'(n+v)|+|f(n+v)|\,.\,|\phi'(n+v)|.$

Now, from conditions (α), (β), (γ) and from the inequalities (1), we see that

$$|f'(n+v)| = -f'(n+v),$$

and $\qquad |f(n+v)|\,.\,|\phi'(n+v)| \leqslant \; f(n)\,\phi'(n).$

It follows that $\qquad |F'(n+v)| \leqslant f(n)\,\phi'(n)-f'(n+v).$

Thus, making use of the last inequality in the equation (3), we find that

$$\begin{aligned}|F(n+t)-F(n)| &\leqslant tf(n)\,\phi'(n)+f(n)-f(n+t)\\ &\leqslant f(n)\,\phi'(n)+f(n)-f(n+1),\end{aligned} \tag{3'}$$

provided that t belongs to the interval (0, 1). Hence, combining (3′) and (2), we see that

$$|X_n| \leqslant f(n)\,\phi'(n)+f(n)-f(n+1). \tag{4}$$

We now introduce the condition that

(δ) *the integral* $\int^\infty f(x)\,\phi'(x)\,dx$ *is convergent.*

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* As remarked above (foot-note, p. 327), we suppose absolute convergence excluded, so that $\int^\infty f(x)\,dx$ is divergent. When this integral is convergent, the discussion given here is quite superfluous.

Consequently, by Maclaurin's theorem quoted in the introduction to this paper (p. 327), we see that the series

(5) $$\sum^{\infty} f(n)\,\phi'(n)$$

is also convergent because $f(x)\,\phi'(x)$ tends steadily to zero, in virtue of conditions (α), (γ). Further,

$$\sum_{\mu}^{\nu} \{f(n)-f(n+1)\} = f(\mu)-f(\nu+1),$$

and this tends to the limit $f(\mu)$ as ν tends to infinity; thus the series

(6) $$\sum^{\infty} \{f(n)-f(n+1)\}$$

is also convergent.

It follows from (4), (5), and (6) that the series $\Sigma|X_n|$ converges: and consequently the series ΣX_n is absolutely convergent. But

$$\sum_{\mu}^{\nu} X_n = \int_{\mu}^{\nu} F(x)\,dx - \sum_{\mu}^{\nu} F(n) + \int_{\nu}^{\nu+1} F(x)\,dx$$

and $$\left|\int_{\nu}^{\nu+1} F(x)\,dx\right| < \int_{\nu}^{\nu+1} f(x)\,dx < f(\nu),$$

so that $$\lim_{\nu\to\infty} \int_{\nu}^{\nu+1} F(x)\,dx = 0.$$

Thus

(7) $$\lim_{\nu\to\infty} \left\{\int_{\mu}^{\nu} F(x)\,dx - \sum_{\mu}^{\nu} F(n)\right\} = \sum_{\mu}^{\infty} X_n;$$

and accordingly, since the series on the right has been proved to converge, the limit on the left is also definite.

It follows at once that, *if* $f(x)$ *and* $\phi(x)$ *are subject to the conditions* (α), (β), (γ), *and* (δ), *the series*

$$\sum^{\infty} f(n)\sin\phi(n)$$

and the integral $$\int^{\infty} f(x)\sin\phi(x)\,dx$$

converge, diverge, or oscillate together.

Further, the equation (7) shews that, in case both oscillate, the amplitude of oscillation is the *same* for the series as for the integral; and, in case of divergence, the limit on the left of (7) is still finite.

Under the same conditions (α)–(δ), *the same results apply to the series*

$$\sum^{\infty} f(n)\cos\phi(n)$$

and the integral $$\int^{\infty} f(x) \cos \phi(x)\, dx\,;$$

and, consequently, the same conclusions apply also to the series

$$\sum^{\infty} f(n) \exp \{\pm i\phi(n)\}$$

and the integrals $$\int^{\infty} f(x) \exp \{\pm i\phi(x)\}\, dx.$$

We note as a special case that the conditions (α)–(δ) are certainly satisfied by the function

$$\{M(x)\}^{-\kappa} \quad (\kappa = \beta + i\gamma,\ \beta > 0),$$

provided that $M(x)$ tends steadily to infinity in such a way that $M'(x)/M(x)$ tends steadily to zero. For we have then to take

$$f(x)e^{\pm i\phi(x)} = \{M(x)\}^{-\kappa}, \text{ or } f(x) = \{M(x)\}^{-\beta}, \text{ and } \phi(x) = |\gamma| \log \{M(x)\},$$

from which it is evident that the first three conditions are satisfied; as to the fourth condition, we must examine the integral

$$\int^{\infty} f(x)\, \phi'(x)\, dx = \int^{\infty} \frac{|\gamma|\, M'(x)}{\{M(x)\}^{(1+\beta)}}\, dx = |\gamma| \int^{\infty} \frac{dy}{y^{1+\beta}},$$

which is clearly convergent, so that all the four conditions are satisfied.

Again, if the two functions

$$F(x) = f(x)\, e^{\pm i\phi(x)}, \qquad G(x) = g(x)\, e^{\pm i\psi(x)}$$

satisfy the prescribed conditions, their product $F(x)\, G(x)$ will also satisfy the conditions, provided that*

(β') $\phi(x) - \psi(x)$ tends steadily to infinity,

(γ') $\phi'(x) - \psi'(x)$ tends steadily to zero.

For then the product $f(x)\, g(x)$ will obviously tend steadily to zero, and each of the integrals

$$\int^{\infty} f(x)\, g(x)\, \phi'(x)\, dx, \qquad \int^{\infty} f(x)\, g(x)\, \psi'(x)\, dx$$

is convergent, because $\lim f(x) = 0$, $\lim g(x) = 0$.

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* These additional conditions are superfluous when the signs in the two exponential functions are the same.

2. *Examination of certain Special Series.*

1. Consider first the simple case*

$$\Sigma \frac{1}{n^{1+ai}} \quad (a \text{ real}).$$

Here we can take $M(x) = x$, $\beta = 1$, in the result at the end of § 1; then $M(x)$ tends steadily to infinity, while $M'(x)/M(x)$ tends steadily to zero.

Consequently, the behaviour of the series is determined by that of the integral

$$\int_1^\nu \frac{dx}{x^{1+ai}} = \frac{1}{ai}\left\{1 - \frac{1}{\nu^{ai}}\right\}.$$

Now, as $\nu \to \infty$, this integral oscillates, the amplitude (both for real and for imaginary parts) being $2/a$; and so the same is true of the series. The theorem of § 1 can also be applied to cases such as

$$M(x) = \log x, \quad \log(\log x), \quad \ldots,$$

but the range of oscillation for the corresponding integrals and series is infinite.

2. Secondly, let us consider the type of series which is obtained by introducing a complex index in the general logarithmic series: that is, we consider the series

$$\Sigma \frac{1}{n \,.\, l_1 n \,.\, l_2 n \,\ldots\, l_{k-1} n \,(l_k n)^{1+ai}},$$

where† $\qquad l_k x = |\log(l_{k-1} x)|, \qquad l_1 x = \log x.$

Here we can take $\qquad \phi(x) = a\, l_{k+1} x, \qquad f(x) = \phi'(x)/a$

or‡ $\qquad f(x) = 1/\{x \,.\, l_1 x \,.\, l_2 x \ldots l_k x\}.$

Then we can find a constant K, so that

$$f(x)\,\phi'(x) < K/x^2,$$

and so condition (δ) of § 1 is satisfied and the other conditions are

* This series can be discussed by Weierstrass's rule depending on the quotient of two consecutive terms in the series (see my *Infinite Series*, p. 204). The particular case of the rule which is needed here is, however, rather troublesome to establish; and it would be almost impossible to use a similar method in the other cases given below.

† Of course, after a certain stage, the logarithms are all positive, and the sign of the modulus may then be omitted; it is often simpler to suppose that the earlier terms are left out from the series so as to avoid this complication.

‡ Because, using accents for differential coefficients, we have

$$l'_{k+1} = \frac{l'_k}{l_k}, \quad l'_k = \frac{l'_{k-1}}{l_{k-1}}, \quad \ldots, \quad l'_1 = \frac{1}{x}.$$

evidently satisfied. Thus we have to consider the integral

$$\int_\mu^\nu f(x)\, e^{-i\phi(x)}\, dx = \frac{1}{ia}\,(e^{-i\phi(\mu)} - e^{-i\phi(\nu)}),$$

which again oscillates (as $\nu \to \infty$) with an amplitude $2/a$; and so the series has the same range of oscillation.

3. It follows without further proof that, *if* $\psi(x)$ *is any function tending steadily to zero* (*as* $x \to \infty$) *the series*

$$\Sigma \frac{\psi(n)}{n^{1+ai}}, \qquad \Sigma \frac{\psi(n)}{n \,.\, l_1 n \,.\, l_2 n \,\ldots\, l_{k-1} n \,.\, (l_k n)^{1+ai}}$$

are both convergent, in virtue of Dirichlet's test of convergence* and the results obtained in (1), (2) above.

Thus, as a simple example, we may note that the series

$$\Sigma \frac{1}{n^{1+ai} (\log n)^\beta}$$

is convergent if β is positive.

We can generalise these results still further by supposing that $\psi(x)$ *is complex, but tends to zero as* x *tends to infinity in such a way that*

$$\int^\infty |\psi'(x)|\, dx$$

is convergent. For then we have the inequality †

$$\left| \int_{X_1}^{X_2} F(x)\, \psi(x)\, dx \right| < HV \quad (X_2 > X_1),$$

where

$$V = \int_{X_1}^\infty |\psi'(x)|\, dx,$$

and H is the upper limit to the integral

$$\left| \int_{X_1}^{X} F(x)\, dx \right| \quad (X_1 \leqslant X \leqslant X_2).$$

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* See my *Infinite Series*, Art. 20.

† See *Proc. London Math. Soc.*, Vol. 6, 1907, p. 65; it is perhaps worth while to remark that there the integral is proved to be less than HV', where

$$V' = \int_{X_1}^{X_2} |\psi'(x)|\, dx + |\psi(X_2)|.$$

But

$$\psi(X_2) = -\int_{X_2}^\infty \psi'(x)\, dx;$$

and so

$$|\psi(X_2)| \leqslant \int_{X_2}^\infty |\psi'(x)|\, dx.$$

Thus

$$V' \leqslant \int_{X_1}^\infty |\psi'(x)|\, dx,$$

and so the value of V given in the text is larger than V'.

Thus, since $H \not< 2/a$ [see (1), (2) above], we can choose X_1 so that

$$\left|\int_{X_1}^{X_2} F(x)\,\psi(x)\,dx\right| < \epsilon \quad (\text{if } X_2 > X_1),$$

because, by proper choice of X_1, the integral V can be made as small as we please. Consequently the integral

$$\int^{\infty} F(x)\,\psi(x)\,dx$$

is convergent: and so the same is true of the series

$$\sum^{\infty} F(n)\,\psi(n),$$

provided that the conditions of § 1 are satisfied by the function $F(x)\,\psi(x)$.

As an illustration of the last result, we may take

$$F(x) = x^{-(1+ai)}, \quad \psi(x) = (\log x)^{-\kappa} \quad (\kappa = \beta + i\gamma,\ \beta > 0).$$

For then
$$|\psi'(x)| = \frac{|\kappa|}{x(\log x)^{1+\beta}},$$

and so
$$\int^{\infty} |\psi'(x)|\,dx = \int^{\infty} \frac{|\kappa|\,dx}{x(\log x)^{1+\beta}},$$

which converges when β is positive, because the indefinite integral is

$$\frac{|\kappa|}{\beta}(\log x)^{-\beta}.$$

Further, as was pointed out in (1) above, the conditions of § 1 are satisfied by the functions $F(x)$, $\psi(x)$; and so the conditions are satisfied also by their product, since the function in the exponential is here

$$a \log x + \gamma \log(\log x).$$

Thus we see that the series

$$\Sigma \frac{1}{n^{1+ai}(\log n)^{\beta+\gamma i}} \quad (\beta > 0)$$

is convergent; and the same method can be easily extended to more complicated cases. In this way it can be proved that the series*

$$\Sigma \frac{(l_2 n)^{\kappa_2}(l_3 n)^{\kappa_3}\ldots(l_k n)^{\kappa_k}}{n^{1+ai}(l_1 n)^{\beta+\gamma i}} \quad (\beta > 0)$$

is convergent, whatever the indices $\kappa_2, \kappa_3, \ldots, \kappa_k$ may be (real or complex).

* These examples were suggested to me by Mr. Hardy.

4. As a last example, we shall find an asymptotic formula for the series

$$\sum_1^\nu \frac{(\log n)^\kappa}{n^{1+ai}} \quad (\kappa=\beta+i\gamma,\ \beta>0),$$

which has been discussed in the special case $\kappa=1$, by Mertens.*

Here we have to put

$$f(x)=\frac{(\log x)^\beta}{x}, \quad \phi(x)=a\log x-\gamma\log(\log x),$$

and so we find
$$f'(x)=-\frac{(\log x)^\beta}{x^2}\left(1-\frac{\beta}{\log x}\right),$$

$$\phi'(x)=\frac{1}{x}\left(a-\frac{\gamma}{\log x}\right).$$

Thus the first three conditions of § 1 will be satisfied, as soon as $\log x$ is greater than both β and γ/a; further, the integral (δ) of § 1 will converge, provided that

$$\int^\infty \frac{(\log x)^\beta}{x^2}\,dx$$

is convergent; but this reduces to the known integral

$$\int^\infty \xi^\beta e^{-\xi}\,d\xi \quad (\text{if } \xi=\log x),$$

and so all the conditions of § 1 are satisfied here.

Thus the asymptotic formulæ for the series

$$\sum_1^\nu \frac{(\log n)^\kappa}{n^{1+ai}}$$

is given by the asymptotic formula for the integral

$$\int_1^\nu \frac{(\log x)^\kappa}{x^{1+ai}}\,dx=\int_0^{\log\nu}\xi^\kappa e^{-ai\xi}\,d\xi.$$

Thus, on integrating by parts, we get the formula

$$\begin{aligned}\int_1^\nu \frac{(\log x)^\kappa}{x^{1+ai}}\,dx=-\frac{1}{\nu^{ai}}\Big[\frac{(\log\nu)^\kappa}{ai}+\kappa\frac{(\log\nu)^{\kappa-1}}{(ai)^2}+\kappa(\kappa-1)\frac{(\log\nu)^{\kappa-2}}{(ai)^3}+\dots\\ +\kappa(\kappa-1)\dots(\kappa-m+1)\frac{(\log\nu)^{\kappa-m}}{(ai)^{m+1}}\Big]\\ +\int_0^{\log\nu}\frac{\kappa(\kappa-1)\dots(\kappa-m)}{(ai)^{m+1}}\xi^{\kappa-m-1}e^{-ai\xi}\,d\xi,\end{aligned}$$

provided that $\beta-m$ is positive.

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* *Göttingen Nachrichten*, 1887, p. 266; the method adopted by Mertens is to differentiate the first series of p. 332 with respect to a.

When β is an integer, the formula just obtained is only true if $m = \beta - 1$; to extend it to $m = \beta$, we divide the remainder integral into two, from 0 to 1 and from 1 to $\log \nu$. The second of these integrals may be again integrated by parts, and so we obtain the term given by $m = \beta$ together with a constant; the new remainder integral is then proportional to

$$\int_1^{\log \nu} \xi^{i\gamma - 1} e^{-ai\xi} \, d\xi,$$

which is easily seen to converge to a definite value as ν tends to infinity.* Consequently, if we write $m = \beta$ in the expression in square brackets at the foot of p. 335, the difference between this formula and the sum of the series will tend to a definite limit as ν tends to infinity.

When β is not an integer, we take m as the integer next less than β, and the remainder integral can then be proved to converge (as $\nu \to \infty$) by a method similar to that used in the last case.

Summing up, we have now the result that

$$\sum_1^{\nu} \frac{(\log n)^{\kappa}}{n^{1+ai}} \sim -\frac{1}{\nu^{ai}} \left[\frac{(\log \nu)^{\kappa}}{ai} + \kappa \frac{(\log \nu)^{\kappa-1}}{(ai)^2} + \kappa(\kappa-1) \frac{(\log \nu)^{\kappa-2}}{(ai)^3} + \dots \right.$$
$$\left. + \kappa(\kappa-1) \dots (\kappa - m + 1) \frac{(\log \nu)^{\kappa-m}}{(ai)^{m+1}} \right],$$

where m is the integral part of β. Thus in particular we have Mertens's result

$$\sum_1^{\nu} \frac{\log n}{n^{1+ai}} \sim -\frac{1}{\nu^{ai}} \left\{ \frac{\log \nu}{ai} + \frac{1}{(ai)^2} \right\}.$$

These asymptotic equations imply that the difference between the expressions on the two sides of the symbol $\sim$ has a finite limit as ν tends to infinity.

It would be easy to multiply examples of this type by introducing more logarithms, but enough has been said to indicate the scope of the method.

8. *A Different Test for Series which contain Periodic Factors.*

Suppose that we wish to discuss the series

$$\Sigma f(n)\, v(n),$$

* In fact, the real part of the integral can be written in the form

$$\text{const.} + \int_c^{\log \nu} \left\{ \left(a - \frac{\gamma}{\xi} \right) \cos (a\xi - \gamma \log \xi) \right\} \frac{d\xi}{a\xi - \gamma} \quad (ac > \gamma),$$

to which we can apply Dirichlet's test for convergence (*Infinite Series*, Art. 169). Similarly the imaginary part of the integral can be proved to converge.

where $f(x)$ has the same properties as in § 1, but $v(n)$ has the period ω, so that

$$v(n+\omega) = v(n).$$

Then the necessary and sufficient condition for convergence is that

$$\sum_1^{\omega} v(n) = 0.$$

For, suppose that

$$\Omega = \sum_1^{\omega} v(n),$$

then

$$v(\omega) = \Omega - v(1) - v(2) - \ldots - v(\omega - 1).$$

Then, since $v(r\omega + s) = v(s)$, we find that

$$\sum_1^{\lambda\omega} f(n)\, v(n) = \Omega \sum_{r=1}^{\lambda} f(r\omega) + \sum_{s=1}^{\omega-1} v(s)\, S,$$

where $S = f(s) - f(\omega) + f(s+\omega) - f(2\omega) + \ldots + f[s + (\lambda - 1)\omega] - f(\lambda\omega)$.

Now, in virtue of the decreasing character of $f(x)$, the sum S has a definite limit $\phi(s)$ less than $f(s)$, as λ tends to infinity, and so

$$\lim_{\lambda \to \infty} \sum_{s=1}^{\omega-1} v(s)\, S = \sum_{s=1}^{\omega-1} v(s)\, \phi(s).$$

But

$$\sum_{r=1}^{\lambda} f(r\omega) > \int_1^{\lambda} f(\omega\xi)\, d\xi = \int_1^{\omega\lambda} f(x)\, dx,$$

and so this sum tends to infinity with λ, since $\int^{\infty} f(x)\, dx$ is divergent.

It follows that the sum $\sum_1^{\lambda\omega} f(n)\, v(n)$

also tends to infinity with λ unless Ω is zero; in the latter case, the sum has the finite limit

$$\sum_{s=1}^{\omega-1} v(s)\, \phi(s).$$

Now, if $0 < \mu < \omega$, we have

$$\left| \sum_{\lambda\omega+1}^{\lambda\omega+\mu} f(n)\, v(n) \right| < V\mu f(\lambda\omega),$$

where V denotes the largest of the values $|v(1)|, |v(2)|, \ldots, |v(\omega)|$. Consequently

$$\lim_{\lambda \to \infty} \sum_{\lambda\omega+1}^{\lambda\omega+\mu} f(n)\, v(n) = 0,$$

and so the behaviour of the general series

$$\sum^{\infty} f(n)\, v(n)$$

is the same as that of $\lim_{\lambda\to\infty} \sum_1^{\lambda} f(n)\, v(n)$.

Thus the series diverges unless Ω is zero, and converges if Ω is zero.

As a simple example, we can infer the convergence of the series

$$\Sigma f(n) \sin nh\, \Theta(\sin^2 nh),$$

where $h = p\pi/q$ (p, q being positive integers), and Θ is any continuous function; because here $\omega = 2q$ if p is odd, or $\omega = q$ if p is even, and in either case $\Omega = 0$.

But the series $\quad \Sigma f(n) \cos nh\, \Theta(\sin^2 nh)$

can be made to diverge by adjustment of Θ if p is even and q is odd.*

Similarly the series $\quad \Sigma(-1)^{[nh]} f(n)$,

where $[\xi]$ denotes the integral part of ξ, will diverge if $h = p/q$, where p is even and q is odd,† because again $\omega = q$ and so $\Omega = 1$.

The applications to series of the type

$$\Sigma f(n) \sin (n^s h), \qquad \Sigma f(n) \cos (n^s h),$$

where s is a positive integer, are equally obvious. Thus, for example, the series

$$\Sigma f(n) \sin \left(n^2 \frac{2\pi}{q}\right), \quad \Sigma f(n) \cos \left(n^2 \frac{2\pi}{q}\right)$$

can converge only if

$$\sum_0^{q-1} \sin \left(n^2 \frac{2\pi}{q}\right) = 0, \qquad \sum_0^{q-1} \cos \left(n^2 \frac{2\pi}{q}\right) = 0,$$

respectively.

Thus, when q is of the form $4k+1$ the sine-series converges, but the cosine-series diverges; but if q is of the form $4k+3$, the cosine-series converges, while the sine-series is divergent; if q is of the form $4k$, both series diverge, but if q is of the form $4k+2$, both converge. These results follow from the values found by Gauss for $\Sigma \exp(2\pi i n^2/q)$ in his investigations on quadratic residues.‡

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* For instance, if we take $p = 2$, $q = 3$, the value of Ω is easily seen to be

$$\Theta(0) - \Theta(\tfrac{3}{4}),$$

which, of course, may have any value.

† Bromwich and Hardy, *l.c.*, p. 240.

‡ *Werke*, Bd. II., p. 9 (§ 19).

ON A FORMULA FOR THE SUM OF A FINITE NUMBER OF TERMS OF THE HYPERGEOMETRIC SERIES WHEN THE FOURTH ELEMENT IS UNITY

(*Second Communication.*)

By M. J. M. HILL.

[Received March 10th, 1908.—Read March 12th, 1908.]

Abstract.

IN a paper communicated to the Society and printed in the *Proceedings*, Ser. 2, Vol. 5, pp. 335–341, it was shown that when the real part of $\gamma-\alpha-\beta$ is negative, then, in general, the sum of s terms of the series

$$1+\frac{\alpha\beta}{1\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\;\gamma(\gamma+1)}+\dots$$

was asymptotic with $\dfrac{\Pi(\gamma-1)\,s^{\alpha+\beta-\gamma}}{(\alpha+\beta-\gamma)\,\Pi(\alpha-1)\,\Pi(\beta-1)}$, *l.c.*, p. 339.

The proof given did not, however, apply to the special case in which $\gamma-\alpha-\beta$ is a negative integer, and it did not apply when $\gamma-\alpha-\beta$ is equal to zero.

The object of the present communication is to show that the formula given above does hold when $\gamma-\alpha-\beta$ is a negative integer, but that, when $\gamma-\alpha-\beta$ is equal to zero, then the sum of s terms is asymptotic with

$$\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\log_e s.$$

This last result was obtained in the first instance by taking the expression given in Art. 4 of the former paper, putting $\gamma=\alpha+\beta+\epsilon$, and equating the terms independent of ϵ. The difficulty of obtaining a thoroughly satisfactory proof by this method led me to build up an independent proof.

The method adopted has a point of interest.

Calling the terms of the original series $T_1+T_2+\dots+T_s$, certain factors

$U_1, U_2, \ldots, U_s$ are obtained, such that $U_n T_n$ can be put into the form $V_n - V_{n-1}$. From this it follows that

$$U_1 T_1 + U_2 T_2 + \ldots + U_s T_s = V_s - V_0.$$

The factors $U_1, U_2, \ldots, U_s$ depend upon an integer r, in such a way that when r is increased to infinity, these factors all tend to unity.

To sum the series $T_1 + \ldots + T_s$, all that remains is to make r infinitely great in $V_{s+1} - V_0$, and then determine the simplest expression with which $V_{s+1} - V_0$ is asymptotic. The result is as given above.

Thus the only discontinuity in the formula takes place at the value of γ which separates those series which are convergent from those which are divergent.

1. With the notation used in the former paper, and also the following,

$$t_{r,s} = \frac{1}{\alpha+r-1} + \frac{1}{\beta+r-1} - \frac{1}{r} - \frac{1}{\alpha+\beta+s+r},$$

$$u_{r,s} = 1 + \frac{\alpha\beta}{\alpha+\beta+s+1} + \frac{\alpha_2\beta_2}{2!\,(\alpha+\beta+s+1)_2} + \ldots + \frac{\alpha_{r-1}\beta_{r-1}}{(r-1)!\,(\alpha+\beta+s+1)_{r-1}},$$

$$w_{r,s} = t_{1,s}u_{1,s} + t_{2,s}u_{2,s} + \ldots + t_{r,s}u_{r,s},$$

it will be proved that

$$\frac{\alpha_s\beta_s}{s!\,(\alpha+\beta)_s}\left(1 - \frac{\alpha_r\beta_r}{r!\,(\alpha+\beta+s)_r} - \frac{\alpha_r\beta_r}{(r-1)!\,(\alpha+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{\alpha+\beta+s+r}\right)$$
$$= \frac{\alpha_{s+1}\beta_{s+1}}{s!\,(\alpha+\beta)_{s+1}} w_{r,s} - \frac{\alpha_s\beta_s}{(s-1)!\,(\alpha+\beta)_s} w_{r,s-1}. \quad \text{(I.)}$$

Dividing out by $\dfrac{\alpha_s\beta_s}{(s-1)!\,(\alpha+\beta)_s}$, it is necessary to prove

$$\frac{1}{s}\left(1 - \frac{\alpha_r\beta_r}{r!\,(\alpha+\beta+s)_r} - \frac{\alpha_r\beta_r}{(r-1)!\,(\alpha+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{\alpha+\beta+s+r}\right)$$
$$= \frac{(\alpha+s)(\beta+s)}{s(\alpha+\beta+s)} w_{r,s} - w_{r,s-1}$$
$$= \frac{(\alpha+s)(\beta+s)}{s(\alpha+\beta+s)} \sum_{v=1}^{r} t_{v,s}u_{v,s} - \sum_{v=1}^{r} t_{v,s-1}u_{v,s-1}$$
$$= \sum_{v=1}^{r} t_{v,s}\left(\frac{(\alpha+s)(\beta+s)}{s(\alpha+\beta+s)} u_{v,s} - u_{v,s-1}\right) + \sum_{v=1}^{r} u_{v,s-1}(t_{v,s} - t_{v,s-1}).$$

2. ...ng the demonstration of equation (I.)

ind...

...parate proof is required.

...nd $w_{r,s-1}$ disappear, and it might be

...ed reduces to

$$\ldots \frac{a_r\beta_r}{\ldots 1)!\,(a+\beta)_{r+1}} = \frac{a\beta}{a+\beta}\,w_{r,0}. \qquad \text{(II.)}$$

3. ...

$$\frac{1}{s} \ldots u_{2,0}+\ldots+t_{r,0}\,u_{r,0}.$$

$$\ldots \frac{}{\ldots+1)_2}+\ldots+\frac{a_{r-1}\beta_{r-1}}{(a+\beta+1)_{r-1}\,(r-1)!}$$

$$= \ldots$$

...paper. Also

$$\frac{\ldots 1)_{r-1}\,(\beta+1)_{r-1}}{\ldots !\,(a+\beta+1)_{r-1}}$$

$$= \frac{\ldots 1)_{r-1}}{\ldots 1)_{r-1}}$$

$$\frac{\ldots 1)_{r-2}\,(\beta+1)_{r-2}}{\ldots 1)!\,(a+\beta+1)_{r-1}}$$

$$\frac{\ldots 1)_{r-1}}{\ldots}$$

$$\frac{\ldots_{-2}\,(\beta+1)_{r-2}}{\ldots a+\beta+1)_{r-2}}$$

$$\frac{\ldots\beta+1)_{r-1}}{\ldots+1)_{r-1}}.$$

...m the first and second

$$\ldots \frac{}{\ldots\beta+1};$$

Now
$$t_{v,s}\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}\left[\frac{a+\beta+2v-2}{(a+v-1)(\beta+v-1)}-\frac{a+\beta+s+2v}{v(a+\beta+s+v)}\right]$$
$$+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_{v-1}\beta_{v-1}}{s.(v-1)!\,(a+\beta+s)_v}[(a+\beta+s+v-1)+(v-1)-s]$$
$$-\frac{a_v\beta_v}{s.v!\,(a+\beta+s)_{v+1}}[(a+\beta+s+v)+v]+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_{v-1}\beta_{v-1}}{s.(v-1)!\,(a+\beta+s)_{v-1}}+\frac{a_{v-1}\beta_{v-1}}{s.(v-2)!\,(a+\beta+s)_v}$$
$$-\frac{a_v\beta_v}{s.v!\,(a+\beta+s)_v}-\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_{v+1}}.$$

The third and fourth of the above terms are obtainable from the first and second by increasing v by unity.

When $v=1$, the formula is
$$t_{1,s}\frac{a\beta}{s(a+\beta+s)}+\frac{1}{a+\beta+s}=\frac{1}{s}-\frac{a\beta}{s(a+\beta+s)}-\frac{a\beta}{s(a+\beta+s)_2};$$

therefore
$$\sum_{v=1}^{r}\left(t_{v,s}\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}\right)$$
$$=\frac{1}{s}-\frac{a_r\beta_r}{s.r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{s.(r-1)!\,(a+\beta+s)_{r+1}}.$$

Hence the equation to be proved is
$$\frac{1}{s}\left(1-\frac{a_r\beta_r}{r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}-\frac{s\,u_{r,s-1}}{a+\beta+s+r}\right)$$
$$=\frac{1}{s}-\frac{a_r\beta_r}{s.r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{s.(r-1)!\,(a+\beta+s)_{r+1}}-\frac{u_{r,s-1}}{a+\beta+s+r},$$

which is correct.

5. There is no difficulty in applying the demonstration of equation (I.) to the case $s = 1$.

But for the first term $s = 0$ a separate proof is required.

In this case the terms $u_{r,s-1}$ and $w_{r,s-1}$ disappear, and it might be expected that the identity to be proved reduces to —

$$1-\frac{a_r\beta_r}{r!\,(a+\beta)_r}-\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}}=\frac{a\beta}{a+\beta}\,w_{r,0}. \qquad \text{(II.)}$$

To prove that this is so

$$w_{r,0}=t_{1,0}u_{1,0}+t_{2,0}u_{2,0}+\ldots+t_{r,0}u_{r,0}.$$

Now $$u_{r,0}=1+\frac{a\beta}{a+\beta+1}+\frac{a_2\beta_2}{2!\,(a+\beta+1)_2}+\ldots+\frac{a_{r-1}\beta_{r-1}}{(a+\beta+1)_{r-1}\,(r-1)!}$$

$$=\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_{r-1}},$$

by the result in Art. 5, (a), of the former paper. Also

$$t_{r,0}u_{r,0}=\frac{a+\beta+2(r-1)}{(a+r-1)(\beta+r-1)}\,\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_{r-1}}$$

$$-\frac{a+\beta+2r}{r(a+\beta+r)}\,\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_{r-1}}$$

$$=\left[(r-1)+(a+\beta+r-1)\right]\frac{(a+1)_{r-2}\,(\beta+1)_{r-2}}{(r-1)!\,(a+\beta+1)_{r-1}}$$

$$-\left[r+(a+\beta+r)\right]\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{r!\,(a+\beta+1)_r}$$

$$=\frac{(a+1)_{r-2}\,(\beta+1)_{r-2}}{(r-2)!\,(a+\beta+1)_{r-1}}+\frac{(a+1)_{r-2}\,(\beta+1)_{r-2}}{(r-1)!\,(a+\beta+1)_{r-2}}$$

$$-\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_r}-\frac{(a+1)_{r-1}\,(\beta+1)_{r-1}}{r!\,(a+\beta+1)_{r-1}}.$$

The third and fourth of the above terms come from the first and second by increasing r by unity. Also

$$t_{1,0}u_{1,0}=\frac{1}{a}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{a+\beta+1};$$

therefore

$$w_{r,0} = t_{1,0}u_{1,0}+t_{2,0}u_{2,0}+\ldots+t_{r,0}u_{r,0}$$

$$= \frac{1}{a}+\frac{1}{\beta}-\frac{(a+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_r}-\frac{(a+1)_{r-1}(\beta+1)_{r-1}}{r!\,(a+\beta+1)_{r-1}};$$

therefore $$\frac{a\beta}{a+\beta}w_{r,0} = 1-\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}}-\frac{a_r\beta_r}{r!\,(a+\beta)_r},$$

which is the equation (II.) to be proved.

6. Hence, by means of equations (I.) and (II.) together, it follows that

$$1-\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}}-\frac{a_r\beta_r}{r!\,(a+\beta)_r}$$

$$+\sum_{s=1}^{s}\frac{a_s\beta_s}{s!\,(a+\beta)_s}\left(1-\frac{a_r\beta_r}{r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}-\frac{s\,u_{r,s-1}}{a+\beta+s+r}\right)$$

$$=\frac{a\beta}{a+\beta}w_{r,0}+\sum_{s=1}^{s}\left(\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}}w_{r,s}-\frac{a_s\beta_s}{(s-1)!\,(a+\beta)_s}w_{r,s-1}\right)$$

$$=\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}}w_{r,s}. \tag{III.}$$

In this result it is necessary to make r infinite.

Now $$\frac{a_r\beta_r}{r!\,(a+\beta)_r}=\frac{\Pi(r,\,a+\beta-1)}{r.\Pi(r,\,a-1)\,\Pi(r,\,\beta-1)},$$

and therefore tends to zero as r tends to infinity. Similarly

$$\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}},\quad \frac{a_r\beta_r}{r!\,(a+\beta+s)_r},\quad \frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}$$

all tend to zero as r tends to infinity.

Also $u_{r,s-1}$ is a convergent series and therefore finite, and therefore $\frac{s\,u_{r,s-1}}{a+\beta+s+r}$ tends to zero as r tends to infinity. Hence, when r tends to infinity, the left-hand side of equation (III.) tends to

$$1+\sum_{s=1}^{s}\frac{a_s\beta_s}{s!\,(a+\beta)_s}.$$

Next $$\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}} = \frac{\Pi(s+1,\, a+\beta-1)}{\Pi(s+1,\, a-1)\,\Pi(s+1,\beta-1)},$$

and hence when s is large it will be asymptotic with

$$\frac{\Pi(a+\beta-1)}{\Pi(a-1)\,\Pi(\beta-1)}.$$

It remains to examine $w_{r,s}$.

$$w_{r,s} = t_{1,s}u_{1,s} + t_{2,s}u_{2,s} + \ldots + t_{r,s}u_{r,s}$$

$$= t_{1,s} + t_{2,s} + \ldots + t_{r,s} + t_{1,s}(u_{1,s}-1) + t_{2,s}(u_{2,s}-1) + \ldots + t_{r,s}(u_{r,s}-1).$$

Now

$$u_{r,s}-1 = \frac{a\beta}{a+\beta+s+1} + \frac{a_2\beta_2}{2!\,(a+\beta+s+1)_2} + \ldots + \frac{a_{r-1}\beta_{r-1}}{(r-1)!\,(a+\beta+s+1)_{r-1}};$$

therefore

$$\frac{u_{r,s}-1}{\left(\dfrac{a\beta}{a+\beta+s+1}\right)} = 1 + \frac{(a+1)(\beta+1)}{2!\,(a+\beta+s+2)} + \ldots + \frac{(a+1)_{r-2}(\beta+1)_{r-2}}{(r-1)!\,(a+\beta+s+2)_{r-2}}. \quad \text{(IV.)}$$

Now when s is large the right-hand side of (IV.) tends to the limit 1.

Suppose that the right-hand side of (IV.) lies between $1-\epsilon$ and $1+\eta$, where ϵ, η are two functions of s which tend to zero as s increases.

It follows that

$$t_{1,s}(u_{1,s}-1) + t_{2,s}(u_{2,s}-1) + \ldots + t_{r,s}(u_{r,s}-1)$$

lies between $$\frac{a\beta}{a+\beta+s+1}(t_{1,s}+t_{2,s}+\ldots+t_{r,s})(1-\epsilon)$$

and $$\frac{a\beta}{a+\beta+s+1}(t_{1,s}+t_{2,s}+\ldots+t_{r,s})(1+\eta).$$

Now $$t_{1,s}+t_{2,s}+\ldots+t_{r,s}$$

$$= \left(\frac{1}{a}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{a+\beta+s+1}\right)$$

$$+\left(\frac{1}{a+1}+\frac{1}{\beta+1}-\frac{1}{2}-\frac{1}{a+\beta+s+2}\right)$$

$$+\ldots$$

$$+\left(\frac{1}{a+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{a+\beta+s+r}\right)$$

$$= \left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{\alpha+\beta-1}\right)+\frac{1}{\alpha+\beta-1}-\frac{1}{\alpha+\beta+s+1}$$

$$+\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}-\frac{1}{2}-\frac{1}{\alpha+\beta}\right)+\frac{1}{\alpha+\beta}-\frac{1}{\alpha+\beta+s+2}$$

$$+\dots$$

$$+\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)+\frac{1}{\alpha+\beta+r-2}-\frac{1}{\alpha+\beta+s+r}$$

$$= \sum_{r=1}^{r}\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)$$

$$+\left(\frac{1}{\alpha+\beta-1}+\frac{1}{\alpha+\beta}+\dots+\frac{1}{\alpha+\beta+s}\right)$$

$$-\left(\frac{1}{\alpha+\beta+r-1}+\frac{1}{\alpha+\beta+r}+\dots+\frac{1}{\alpha+\beta+s+r}\right).$$

Now $$\sum_{r=1}^{r}\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)$$

$$= -\sum_{r=1}^{r}\frac{(\alpha-1)(\beta-1)[\alpha+\beta+2(r-1)]}{r(\alpha+r-1)(\beta+r-1)(\alpha+\beta+r-2)},$$

and is therefore a convergent series with a finite sum, when r is infinite.

Next

$$\frac{1}{\alpha+\beta+r-1}+\frac{1}{\alpha+\beta+r}+\dots+\frac{1}{\alpha+\beta+s+r}<\frac{s+2}{\alpha+\beta+r-1},$$

and therefore tends to zero as r tends to infinity.

Next $$\frac{1}{\alpha+\beta-1}+\frac{1}{\alpha+\beta}+\dots+\frac{1}{\alpha+\beta+s}$$

is asymptotic with $\log_e s$. Hence

$$t_{1,s}+t_{2,s}+\dots+t_{r,s}$$

is asymptotic with $\log_e s$, whilst

$$\frac{\alpha\beta}{\alpha+\beta+s+1}(t_{1,s}+\dots+t_{r,s})$$

tends to the value $$\frac{\alpha\beta}{\alpha+\beta+s+1}\log_e s;$$

and is therefore very small when s is large; therefore

$$t_{1,s}(u_{1,s}-1)+\ldots+t_{r,s}(u_{r,s}-1)$$

is very small when r is infinite and s is large; therefore $w_{r,s}$ is asymptotic with

$$t_{1,s}+t_{2,s}+\ldots+t_{r,s};$$

and therefore with $\log_e s$. Hence

$$\frac{\alpha_{s+1}\beta_{s+1}}{s!\,(\alpha+\beta)_{s+1}}\,w_{r,s}$$

is asymptotic with $$\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\log_e s.$$

Hence $$1+\sum_{s=1}^{s}\frac{\alpha_s\beta_s}{s!\,(\alpha+\beta)_s}$$

is asymptotic with $$\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\log_e s.$$

7. Consider now the case where $\gamma=\alpha+\beta-n$.

Putting $$\gamma=\alpha+\beta-n,\qquad t=n-1,$$

in equations (V.) and (VI.), p. 337, of the former paper, it follows that

$$G(\alpha,\beta,\alpha+\beta-n,s)=\frac{(\alpha-n)_n(\beta-n)_n}{(-n)_n(\alpha+\beta-n)_n}G(\alpha,\beta,\alpha+\beta,s)$$
$$+\frac{\alpha_{s+1}\beta_{s+1}}{n\,.\,s!\,(\alpha+\beta-n)_{s+1}}f(\alpha,\beta,\alpha+\beta-n,s,n-1)$$

where $$f(\alpha,\beta,\alpha+\beta-n,s,n-1)$$

$$=1+\frac{(\alpha-n)(\beta-n)}{(-n+1)(\alpha+\beta-n+s+1)}+\frac{(\alpha-n)_2(\beta-n)_2}{(-n+1)_2(\alpha+\beta-n+s+1)_2}+\ldots$$
$$+\frac{(\alpha-n)_{n-1}(\beta-n)_{n-1}}{(-n+1)_{n-1}(\alpha+\beta-n+s+1)_{n-1}}.$$

Now $$\frac{\alpha_{s+1}\beta_{s+1}}{n\,.\,s!\,(\alpha+\beta-n)_{s+1}}=\frac{\Pi(s+1,\alpha+\beta-n-1)}{\Pi(s+1,\alpha-1)\,\Pi(s+1,\beta-1)}\,\frac{(s+1)^n}{n};$$

and is therefore, for a large s, asymptotic with

$$\frac{\Pi(\alpha+\beta-n-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\,\frac{(s+1)^n}{n}.$$

Also $f(\alpha,\beta,\alpha+\beta-n,s,n-1)$ tends to unity as s increases.

Further, $G(\alpha, \beta, \alpha+\beta, s)$ has been shown to be asymptotic with a finite multiple of $\log_e s$, which is very small compared with $(s+1)^n$, when n is a positive integer and s a large positive integer.

Consequently $G(\alpha, \beta, \alpha+\beta-n, s)$ is asymptotic with

$$\frac{\Pi(\alpha+\beta-n-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\,\frac{(s+1)^n}{n},$$

which is the value assumed by the expression

$$\frac{\Pi(\gamma-1)\,(s+1)^{\alpha+\beta-\gamma}}{(\alpha+\beta-\gamma)\,\Pi(\alpha-1)\,\Pi(\beta-1)}$$

given in Art. 4, on p. 339 of the former paper, when $\gamma=\alpha+\beta-n$.

ON A GENERAL CONVERGENCE THEOREM, AND THE THEORY OF THE REPRESENTATION OF A FUNCTION BY SERIES OF NORMAL FUNCTIONS

By E. W. HOBSON.

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THE theory of integral equations, as recently developed by Hilbert and by Schmidt, has resulted in a certain unification of the theory of the series of normal functions which represent prescribed functions in an interval. By this means the validity of such representation has, however, in the first instance, been established only for the case of a function which, together with its first and second differential coefficients, is continuous in the whole interval of representation, and which satisfies at the ends of the interval the same conditions that are imposed upon the normal functions themselves. An extension of the theory to the case of functions of a less restricted type has recently been given by Kneser.* A method of development of the theory of series of normal functions, on foundations laid by Schwarz and Poincaré has been given in detailed investigations by Stekloff and others, but involves a restriction upon the type of the functions represented by the series, of a similar character to that in the theory of integral equations.

It seems desirable to obtain sufficient conditions for the convergence of the series at a particular point, and for the uniform convergence of the series in any interval contained in the whole interval of representation, comparable in generality with the known sufficient conditions applicable in the case of Fourier's series. In the present communication a fundamental convergence theorem is established, which, when applied to the case of series of Sturm-Liouville functions,† suffices to shew that the question whether the series corresponding to a given function converges, or not, at a particular point, depends only upon the nature of the function in

* *Math. Annalen*, Vol. LXIII.

† The convergence of the series of Sturm-Liouville functions has been investigated for the case of a function of limited total fluctuation by Kneser, *Math. Annalen*, Vols. LVIII., LX. The case of the series representing analytical functions has been treated by A. C. Dixon, by the method of residues, *Proceedings*, Ser. 2, Vol. 3.

an arbitrarily small neighbourhood of the point, whilst the nature of the function throughout the whole interval of representation is restricted only by the postulation that it shall possess a Lebesgue integral in that interval; the function being therefore not necessarily limited in the interval. The convergence theorem is further employed to shew that, subject to the same condition as regards the nature of the function, the question whether the series converges uniformly, or not, in an interval in which the function is continuous, and which is contained in the given interval of representation, depends only on the nature of the function in an interval which encloses the interval of continuity in its interior, exceeding it in length only by an arbitrarily small amount.

It is further shewn that a sufficient restriction upon the nature of the function in an arbitrarily small neighbourhood of a particular point, to ensure the convergence of the series, is that the function shall be of limited total fluctuation (*à variation bornée*) in that neighbourhood. It is shewn that a similar restriction is sufficient in the case of uniform convergence in an interval.

When the end-points of the interval of representation are singular points of the linear differential equation satisfied by the normal functions, as, for example, in the case of Legendre's or Bessel's functions, the fundamental theorem is not applicable to the whole interval of representation. In this case, neighbourhoods of the end-points must be excluded and the theorem applied to the remainder of the interval; the parts of the series depending on the excluded neighbourhoods of the end-points requiring separate consideration. As an instance of such series, the case of the series of Legendre's functions is treated in detail.

A few other applications of the fundamental convergence theorem are given, to the proof of the validity, under very general conditions, of known modes of representation of functions by means of definite integrals and by series.

A GENERAL CONVERGENCE THEOREM.

1. The following convergence theorem will be first established :—

Let $f(x')$ be a function which has a Lebesgue integral in the interval (α, β) of the variable x', whether the function be limited or unlimited in that interval. Let $F(x', x, n)$ be a function defined for all values of x' in (α, β), and for values of x belonging to a certain set of points G contained in (α, β), and for positive values of n. Let $F(x', x, n)$ satisfy the following conditions :—(1) that $|F(x', x, n)|$ does not exceed a definite

positive number $\bar{F}$, *for all values of* x' *and* x *such that* $|x'-x|$ *is not less than a fixed positive number* μ $(< b-a)$, *where* x' *belongs to* (a, β), *and* x *to* G, *and for all values of* n; (2) *that* $\int_{a_1}^{\beta_1} F(x', x, n)\, dx'$ *exists as a Lebesgue integral for all values of* a_1, β_1 *such that* $a \leqslant a_1 < \beta_1 \leqslant \beta$, *for each value of* x *belonging to* G *but not interior to the interval* $(a_1-\mu, \beta_1+\mu)$, *and that it is less in absolute value, for each value of* n, *than a positive number* A_n, *independent of* a_1, β_1 *and* x; (3) *that*

$$\lim_{n=\infty} A_n = 0.$$

Then $\int_a^{x-\mu} f(x') F(x', x, n)\, dx'$ *converges, as* n *is indefinitely increased, uniformly to zero, for all values of* x *belonging to* G, *and in the interval* $(a+\mu, \beta)$; *also* $\int_{x+\mu}^{\beta} f(x') F(x', x, n)\, dx'$ *converges uniformly to zero as* n *is indefinitely increased, for all values of* x *belonging to* G *and in the interval* $(a, \beta-\mu)$. *The positive number* n *may be either a variable capable of having all positive values, or it may be restricted to have the values in a sequence with no upper limit, as, for example, the sequence of positive integers.*

In particular, the set G *may consist of the whole interval* (a, β), *in which case the integrals converge uniformly to zero as* n *is increased indefinitely for all values of* x *in the intervals* $(a+\mu, \beta)$, $(a, \beta-\mu)$ *respectively; or* G *may consist of the points of an interval* $(a+\lambda, \beta-\lambda)$ *contained in* (a, β), *in which case the integrals converge uniformly in the intervals* $(a+\lambda, \beta-\lambda)$, *if* $\lambda \geqslant \mu$, *or in the intervals* $(a+\mu, \beta)$, $(a, \beta-\mu)$ *respectively, if* $\lambda \leqslant \mu$.

In proving this theorem, it will be sufficient to consider the first of the two integrals only. Let it be first assumed that $f(x')$ is limited in the given interval (a, β); and let U, L denote its upper and lower limits respectively in that interval.

We may divide the interval (L, U) into portions

$$(c_0, c_1),\ (c_1, c_2),\ \ldots,\ (c_{p-1}, c_p),$$

where $c_0 = L$, $c_p = U$, and such that $c_q - c_{q-1}$ is less than an arbitrarily chosen positive number η, for all the values $1, 2, 3, \ldots, p$, of q.

Let that set of points in (a, β) for which $c_q \leqslant f(x') < c_{q+1}$ be denoted by E_q; and for any fixed value of x $(\geqslant \mu)$ and belonging to G, let e_q be that part of E_q which is in the interval $(a, x-\mu)$.

Let a function $f_1(x')$ be defined by the following rule:—For those values of x' for which $c_q \leqslant f(x') < c_{q+1}$, let $f_1(x') = c_q$, for each value of q; and for $f(x') = c_p$, let $f_1(x') = c_p$.

We have now

$$\left| \int_a^{x-\mu} f(x') F(x', x, n)\, dx' - \int_a^{x-\mu} f_1(x') F(x', x, n)\, dx' \right| < \eta \bar{F} (\beta - a);$$

where x is any point of the set G in the interval $(a+\mu, \beta)$, and for all values of n. Also we have

$$\int_a^{x-\mu} f_1(x') F(x', x, n)\, dx' = \sum_{q=0}^{q=p} c_q \int_{(e_q)} F(x', x, n)\, dx'.$$

Let the set of points E_q be enclosed in the interiors of a set of non-overlapping intervals H_q, such that $m(H_q) - m(E_q) = \zeta$; where $m(H_q)$, $m(E_q)$ denote the measures of H_q and of E_q, and ζ is an arbitrarily chosen positive number sufficiently small. A finite, or infinite, set H_q of intervals can always be so chosen that this condition is satisfied. If h_q denote that part of H_q which is in the interval $(a, x-\mu)$, it can be seen that

$$m(h_q) - m(e_q) \leqslant \zeta.$$

For, if possible, let $\qquad m(h_q) - m(e_q) = \zeta + \gamma,$

where γ is a positive number. Let the set e_q be enclosed in the interiors of non-overlapping intervals of a set l_q, all in the interval $(a, x-\mu)$, such that $m(l_q) < m(e_q) + \gamma$; and let $\bar{H}_q$ denote that set of non-overlapping intervals which consists of the set l_q together with that part of H_q which is not in the interval $(a, x-\mu)$. Observing that $m(l_q) < m(h_q) - \zeta$, we have then

$$m(\bar{H}_q) = m(H_q) - m(h_q) + m(l_q) < m(H_q) - \zeta < m(E_q);$$

and this is impossible, since E_q cannot be enclosed in intervals of a set of smaller measure than $m(E_q)$. Since therefore no such positive number γ can exist, we have $m(h_q) - m(e_q) \leqslant \zeta$. We have now

$$\left| \int_{(e_q)} F(x', x, n)\, dx' - \int_{(h_q)} F(x', x, n)\, dx' \right| < \zeta \bar{F}.$$

Let the intervals of the set H_q have lengths $\gamma_1, \gamma_2, \gamma_3, \ldots$ in descending order of magnitude. We may choose r_q so that

$$m(H_q) - (\gamma_1 + \gamma_2 + \ldots + \gamma_{r_q}) < \zeta.$$

Of the intervals $\gamma_1, \gamma_2, \ldots, \gamma_{r_q}, \ldots$, let $\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_t}, \ldots$ fall wholly or partly in the interval $(a, x-\mu)$; one of these intervals may be only partially in $(a, x-\mu)$. Let s_t be the greatest of the numbers $s_1, s_2, \ldots, s_t, \ldots$ which does not exceed r_q; we have then

$$\gamma_{s_{t+1}} + \gamma_{s_{t+2}} + \ldots < \zeta, \quad \text{and} \quad m(h_q) - (\gamma_{s_1} + \gamma_{s_2} + \ldots + \gamma_{s_t}) < \zeta,$$

as may be seen by applying the same argument that has been employed above to shew that
$$m(h_q)-m(e_q) \leqslant \zeta.$$
Here, in case the point $x-\mu$ is interior to an interval γ_{s_r}, we take only that part of γ_{s_r} which is in the interval $(a, x-\mu)$.

We have now $m(h_q)-m(D_q) < \zeta$, where D_q is the finite set of intervals $\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_t}$; the number t of intervals of this set D_q does not exceed the number r_q, which is independent of x.

We have now
$$\left| \int_{(h_q)} F(x', x, n)\, dx' - \int_{(D_q)} F(x', x, n)\, dx' \right| < \zeta \bar{F}.$$
Also
$$\left| \int_{(D_q)} F(x', x, n)\, dx' \right| < tA_n < r_q A_n.$$
Combining the inequalities which have been shewn to hold, we find that
$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \eta \bar{F}(\beta-a) + \sum_{q=0}^{q=p} c_q (2\zeta \bar{F} + r_q A_n).$$

Now, let ϵ be an arbitrarily fixed positive number; we can then fix η so that $\eta \bar{F}(\beta-a) < \frac{1}{3}\epsilon$; then the numbers c_q for $q = 0, 1, 2, \ldots, p$ can be fixed. We can then choose ζ so that
$$2\zeta \bar{F} \sum_{q=0}^{q=p} c_q < \tfrac{1}{3}\epsilon.$$
The numbers r_q depend only on ζ and q, being independent of x, and thus $\sum_{q=0}^{q=p} r_q c_q$ is fixed; we can choose n_1 so that
$$A_n \sum_{q=0}^{q=p} r_q c_q < \tfrac{1}{3}\epsilon,$$
provided
$$n \geqslant n_1.$$

We now have
$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \epsilon, \quad \text{for } n \geqslant n_1,$$
and for all values of x belonging to G, and in the interval $(a+\mu, \beta)$. The uniform convergence of the integral to zero has accordingly been established.

Next, let $f(x')$ be no longer limited in (a, β). A positive number N can be so determined that
$$\int |f(x')|\, dx' < \tfrac{1}{3}\epsilon / \bar{F},$$

when the integral is taken over that set of points K_N in (a, β) for each of which $|f(x')| > N$. If k_N be that part of K_N which lies in $(a, x-\mu)$, for any fixed value of x in $(a+\mu, \beta)$, we have

$$\int_{k_N} |f(x')|\, dx' < \tfrac{1}{2}\epsilon/\bar{F}.$$

Let the function $f_2(x')$ be defined by the rule that $f_2(x') = f(x')$, when $|f(x')| \leqslant N$, and $f_2(x') = 0$, when $|f(x')| > N$. Thus $f_2(x')$ vanishes at all points of K_N, and it is a limited summable function. We have now

$$\int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' = \int_{(k_N)} f(x')\, F(x', x, n)\, dx' + \int_a^{x-\mu} f_2(x')\, F(x', x, n)\, dx'.$$

A value n_1 of n can be so determined that

$$\left| \int_a^{x-\mu} f_2(x')\, F(x', x, n)\, dx' \right| < \tfrac{1}{2}\epsilon, \quad \text{for } n \geqslant n_1,$$

and for all values of x belonging to G, and in the interval $(a+\mu, \beta)$. Also

$$\left| \int_{k_N} f(x')\, F(x', x, n)\, dx' \right| < \tfrac{1}{2}\epsilon,$$

for all the values of x and of n. Therefore we have

$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \epsilon, \quad \text{for } n \geqslant n_1,$$

and for all the values of x. The theorem has now been completely established. A special case of this theorem, in a somewhat different form, was given in my paper* "On the Uniform Convergence of Fourier's Series," and was there applied to the theory of Dirichlet's integral.

2. It may be remarked that the above proof is applicable to establish the following somewhat more general theorem:—

If the functions $f(x')$, $F(x', x, n)$ *satisfy the conditions before stated,* $\int_{a_1}^{\beta_1} f(x')\, F(x', x, n)\, dx'$ *converges to zero, as* n *is indefinitely increased, uniformly for all values of* a_1, β_1 *and* x, *which are such that* $a \leqslant a_1 < \beta_1 \leqslant \beta$, *and such that* x *belongs to the set* G, *and is not interior to the interval* $(a_1-\mu, \beta_1+\mu)$.

* *Proceedings*, Ser. 2, Vol. 5, p. 275.

Let us consider the special case in which $F(x', x, n)$ is independent of x, say $F(x', x, n) = \phi(x', n)$. We then obtain the following theorem:—

If $f(x')$ have a Lebesgue integral in the interval (a, β); and $\phi(x', n)$ be such that $|\phi(x', n)|$ have a definite upper limit for all values of x' in (a, β), and for all the values of n; and if further $\int_{a_1}^{\beta_1} \phi(x', n)\, dx$ exist and be numerically less than A_n, for all values of a_1, β_1, such that $a \leqslant a_1 < \beta_1 \leqslant \beta$, where A_n is independent of a_1, β_1, and where $\lim_{n=\infty} A_n = 0$, then $\int_{a_1}^{\beta_1} f(x')\, \phi(x', n)\, dx'$ converges to zero, as n is indefinitely increased, uniformly for all values of a_1 and β_1. The number n may be either capable of having all positive values, or may be restricted to have the values in a sequence, for example, in the sequence of positive integers.

A special case of this theorem is that, if $f(x')$ have a Lebesgue integral in the interval $(-\pi, \pi)$, then $\int_a^\beta f(x') \cos nx'\, dx'$, $\int_a^\beta f(x') \sin nx'\, dx'$ converge to zero as n is indefinitely increased, and uniformly for all values of a and β, such that $-\pi \leqslant a < \beta \leqslant \pi$.

The theorems may be generalized so as to apply to the case of a function of any number of variables. As is clear from the theory of Lebesgue integration, the proof of the fundamental theorem is applicable, without any essential modification, to this more general case. It will be sufficient to state the main theorem for the case of a function of three variables, as follows:—

Let $f(x', y', z')$ be a limited, or unlimited, function defined for all points in the space V bounded by a closed surface S, and having a Lebesgue integral through V. Let $F(x', y', z', x, y, z, n)$ be a function defined for all values of (x', y', z') in V, and for all values of (x, y, z) corresponding to the points of a given set G contained in V; and for positive values of n.

Let $F(x', y', z', x, y, z, n)$ satisfy the following conditions:—(1) that $|F(x', y', z', x, y, z, n)|$ does not exceed a definite number $\bar{F}$, for all positions of the points $(x', y', z')(x, y, z)$, such that

$$(x'-x)^2+(y'-y)^2+(z'-z)^2 \geqslant \mu^2,$$

where μ is a fixed positive number, and (x', y', z') belong to V, and (x, y, z) to G; (2) $\int_{(V_1)} F(x', y', z', x, y, z, n)(dx'\, dy'\, dz')$ exists as a Lebesgue integral for every volume V_1 not exterior to V, and bounded by a surface S_1 not exterior to S, and for all values of (x, y, z) corresponding to points

of G such that a sphere with centre (x, y, z) and radius μ has no volume in common with V_1, and that the integral is in absolute value less than A_n, a number independent of V_1 and of x; (3) that $\lim_{n=\infty} A_n = 0$. Then $\int_{(V_1)} f(x', y', z')\, F(x', y', z', x, y, z, n)(dx'\,dy'\,dz')$ converges to zero, as n is indefinitely increased, uniformly for all points (x, y, z) belonging to G and of which the minimum distance from points of V_1 is not less than μ. The convergence is also uniform for all such volumes V_1, under the conditions stated.

In particular, if the integral be taken through the whole volume V with the exception of a sphere of centre (x, y, z) and radius μ (or of the portion of such sphere which is in V), then the convergence is uniform for all points (x, y, z) belonging to G.

It is clear that the statement might be made more general by replacing the volumes V, V_1 by any bounded and measurable sets of points. If H denote the measurable set of points for which the function $f(x', y', z')$ is defined, and in which it has a Lebesgue integral; the set G for which $F(x', y', z', x, y, z, n)$ is defined and satisfies the conditions of the theorem, being contained in H, then the integral of

$$f(x', y', z')\, F(x', y', z', x, y, z, n),$$

taken through a measurable set H_1 contained in H, converges to zero as n is indefinitely increased, subject to the conditions of the theorem, uniformly for all points (x, y, z) belonging to G, and of which the minimum distance from the points of H_1 is $\geqslant \mu$. The convergence is uniform for all such sets H_1. The original statement of the theorems will be, however, sufficient for the purpose of the applications to be made below.

3. The theorems of §§ 1, 2 may be extended to cases in which the given domain of the function is unbounded, provided an additional convergency condition is satisfied. It will be sufficient to give the extension of that case of the theorem in which the set G consists of all the points of the interval (α, β).

Let us assume that $f(x')$ has a Lebesgue integral in every finite interval contained in the unlimited interval $(-\infty, \infty)$. Let it be assumed also that $| F(x', x, n) | < \bar{F}$, for all values of x', x such that $| x'-x | \geqslant \mu$, and for all values of n. Further, let it be assumed that, if K be any arbitrarily chosen positive number, then, if $\beta-\alpha \leqslant K$,

$$\left| \int_\alpha^\beta F(x', x, n)\, dx' \right| < A_n,$$

where A_n depends only on n and K; and for all values of x not interior to the interval $(a-\mu, \beta+\mu)$. Let it also be assumed that for each value of K, $\lim_{n=\infty} A_n = 0$.

Let x be confined to an arbitrarily chosen interval (a_1, β_1). If, then, corresponding to each arbitrarily chosen positive number ϵ, a number $\xi \leqslant a_1-\mu$ can be determined, such that

$$\left| \int_{\xi'}^{\xi} f(x') F(x', x, n)\, dx \right| < \epsilon,$$

for all values of $\xi' < \xi$, and for all values of n; and, if further, a number $\eta \geqslant \beta_1+\mu$ can be so determined that

$$\left| \int_{\eta}^{\eta'} f(x') F(x', x, n)\, dx' \right| < \epsilon,$$

for all values of $\eta' > \eta$, the numbers ξ, η being independent of n and of x, then the integrals

$$\int_{-\infty}^{\xi} f(x') F(x', x, n)\, dx', \quad \int_{\eta}^{\infty} f(x') F(x', x, n)\, dx'$$

exist as
$$\lim_{\xi'=-\infty} \int_{\xi'}^{\xi} f(x') F(x', x, n)\, dx',$$

and
$$\lim_{\eta'=\infty} \int_{\eta}^{\eta'} f(x') F(x', x, n)\, dx',$$

respectively, and neither of them numerically exceeds ϵ. We suppose these conditions to be satisfied for every interval (a_1, β_1) of x.

We have then

$$\int_{-\infty}^{x-\mu} f(x') F(x', x, n)\, dx' = \int_{-\infty}^{\xi} f(x') F(x', x, n)\, dx' + \int_{\xi}^{x-\mu} f(x') F(x', x, n)\, dx'.$$

The variable x being confined to the interval (a_1, β_1), ξ can be so chosen that, for all such values of x, and for all values of n, the first integral on the right-hand side is numerically not greater than ϵ. Moreover, the second integral is, for all sufficiently large values of n, and for all values of x in (a_1, β_1), numerically less than ϵ, in accordance with the theorem of § 1. It has therefore been shewn that

$$\int_{-\infty}^{x-\mu} f(x') F(x', x, n)\, dx'$$

converges uniformly to zero for all values of x in the interval (a_1, β_1). The integral $\int_{x+\mu}^{\infty} f(x') F(x', x, n)\, dx'$ can be similarly shewn to converge

uniformly to zero, for all values of x in (α_1, β_1). In particular, if

$$\int_{-\infty}^{0} |f(x')|\, dx', \quad \int_{0}^{\infty} |f(x')|\, dx'$$

both exist as $\quad \lim_{h=\infty} \int_{-h}^{0} |f(x')|\, dx', \quad \lim_{h=\infty} \int_{0}^{h} |f(x')|\, dx',$

the additional convergency conditions are satisfied. For

$$\left| \int_{\xi'}^{\xi} f(x')\, F(x', x, n)\, dx' \right| \leqslant \bar{F} \int_{\xi'}^{\xi} |f(x')|\, dx',$$

and

$$\left| \int_{\eta}^{\eta'} f(x')\, F(x', x, n)\, dx' \right| \leqslant \bar{F} \int_{\eta}^{\eta'} |f(x')|\, dx';$$

hence ξ, η can be so chosen that for all values of $\xi' < \xi$, and for all values of $\eta' > \eta$, the expressions on the right-hand sides of these inequalities are each $< \epsilon$. The following theorem has thus been established :—

Let $f(x')$ possess a Lebesgue integral in every finite interval, and let $|F(x, x', n)|$ have a finite upper limit for all values of x and x' such that $|x-x'| \geqslant \mu$, and for all the values of n. Further, let it be assumed that the integral of $F(x, x', n)$ in any interval (α, β) whatever, such that $\beta-\alpha \leqslant K$, when K is an arbitrarily chosen positive number, is numerically less than a number A_n dependent only on n and K, for all values of x not interior to the interval $(\alpha-\mu, \beta+\mu)$, and that $\lim_{n=\infty} A_n = 0$, for each value of K. Then, if $\int_{-\infty}^{\infty} |f(x')|\, dx'$ have a definite value as the double limit

$$\lim_{h=\infty,\, k=\infty} \int_{-k}^{h} |f(x')|\, dx'$$

of the Lebesgue integral, the integrals

$$\int_{-\infty}^{x-\mu} f(x')\, F(x', x, n)\, dx', \quad \int_{x+\mu}^{\infty} f(x')\, F(x', x, n)\, dx',$$

both converge to zero as n is indefinitely increased, uniformly for all values of x in any finite interval.

It is clear that a similar theorem may be stated for the case of a function of three variables, or of any number of variables.

It should be observed that for special forms of the function $F(x', x, n)$, the condition that $\int_{-\infty}^{\infty} |f(x')|\, dx'$ exists may be replaced by a less stringent condition depending on the nature of the function $F(x', x, n)$.

THE CONVERGENCE OF A DEFINITE INTEGRAL.

4. Let $F(x', x, n)$ be defined for all values of x' in (α, β), and for all values of x in a set G, which may, in particular, consist of all points in (α, β), or of all points in an interval $(\alpha+\lambda, \beta-\lambda)$, and for positive values of n. It is also assumed that $f(x')$, $F(x', x, n)$ satisfy the conditions of the theorem of § 1, for every sufficiently small positive value of μ.

We propose to consider the limiting value of the integral

$$\int_\alpha^\beta f(x') F(x', x, n)\, dx'$$

as n is indefinitely increased, the integral being assumed to exist for all points x in G. The integral is equivalent to the sum

$$\int_\alpha^{x-\mu} f(x') F(x', x, n)\, dx' + \int_{x+\mu}^\beta f(x') F(x', x, n)\, dx' + \int_{x-\mu}^{x+\mu} f(x') F(x', x, n)\, dx',$$

where the first integral is omitted if $x \leqslant \alpha+\mu$, and the second is omitted if $x+\mu \geqslant \beta$. When $x < \alpha+\mu$, the lower limit in the third integral is replaced by α; and when $x > \beta-\mu$, the upper limit in the third integral is replaced by β. For any fixed value of μ, the first and second integrals converge uniformly to zero for all values of x in G, as n is indefinitely increased.

We therefore consider the integral

$$\int_{x-\mu}^{x+\mu} f(x') F(x', x, n)\, dx',$$

which is equivalent to

$$\int_0^\mu f(x+t) F(x+t, x, n)\, dt + \int_0^\mu f(x-t) F(x-t, x, n)\, dt.$$

Let us assume that, at a particular point x, the two limits $f(x+0)$, $f(x-0)$ have definite finite values; then

$$\int_0^\mu f(x+t) F(x+t, x, n)\, dt$$
$$= f(x+0) \int_0^\mu F(x+t, x, n)\, dt + \int_0^\mu \{f(x+t) - f(x+0)\}\, F(x+t, x, n)\, dt,$$

and $\int_0^\mu f(x-t) F(x-t, x, n)\, dt$

$$= f(x-0) \int_0^\mu F(x-t, x, n)\, dt + \int_0^\mu \{f(x-t) - f(x-0)\}\, F(x-t, x, n)\, dt.$$

If, for a fixed x, belonging to G, the positive number μ can be chosen so small that

$$\int_0^\mu \{f(x-t)-f(x+0)\}\, F(x+t, x, n)\, dt,$$

and

$$\int_0^\mu \{f(x-t)-f(x-0)\}\, F(x-t, x, n)\, dt,$$

are, for all values of n, each numerically less than an arbitrarily chosen positive number ϵ, and if the two limits

$$\lim_{n=\infty} \int_0^\mu F(x+t, x, n)\, dt, \qquad \lim_{n=\infty} \int_0^\mu F(x-t, x, n)\, dt$$

have definite values P, Q independent of μ, we see that

$$\lim_{n=\infty} \int_\alpha^\beta f(x')\, F(x', x, n)\, dx' = Pf(x+0) + Qf(x-0),$$

x being a fixed point in the interior of the interval (α, β).

It has therefore been shewn that *it is sufficient for the convergence of* $\int_\alpha^\beta f(x')\, F(x', x, n)\, dx'$ *for a fixed value of* x, *in the interior of the interval* (α, β), *to the value* $Pf(x+0)+Qf(x-0)$, *that*

$$\lim_{n=\infty} \int_0^\mu F(x+t, x, n)\, dt \quad \textit{and} \quad \lim_{n=\infty} \int_0^\mu F(x-t, x, n)\, dt$$

should have the values P, Q *independent of* μ, *and that*

$$\int_0^\mu \{f(x+t)-f(x+0)\}\, F(x+t, x, n)\, dt,$$

$$\int_0^\mu \{f(x-t)-f(x-0)\}\, F(x-t, x, n)\, dt$$

should both be numerically less than an arbitrarily chosen positive number ϵ, *for a sufficiently small value of* μ, *and for all values of* n. *It is assumed that the conditions of the fundamental convergence theorem are satisfied for a set* G *to which* x *belongs.*

Let us next assume that the function $f(x)$ is such that, for a particular point x in G, a neighbourhood can be found such that the function $f(x')$ is of limited total fluctuation (*à variation bornée*) in that neighbourhood.

We may then replace the function $f(x')$ by $f_1(x')-f_2(x')$, where $f_1(x')$, $f_2(x')$ are monotone in the neighbourhood of the point x.

We have then, by applying the second mean value theorem, for a

sufficiently small value of μ,

$$\int_0^\mu \{f_1(x+t)-f_1(x+0)\}\, F(x+t, x, n)\, dt$$
$$= \{f_1(x+\mu)-f_1(x+0)\} \int_{\mu_1}^\mu F(x+t, x, n)\, dt,$$

where $0 \leqslant \mu_1 < \mu$. A similar equation holds for the function $f_2(x)$.

Let us now assume that the function F is such that $\int_{\mu'}^\mu F(x+t, x, n)\, dt$ is numerically less than a fixed positive number, for all values of μ' such that $0 \leqslant \mu' < \mu$, and for all values of n. The number μ may be so chosen that $f_1(x+\mu)-f_1(x+0)$, $f_2(x+\mu)-f_2(x+0)$ are each less than an arbitrarily chosen positive number. It follows that μ can be so chosen that

$$\int_0^\mu \{f(x+t)-f(x+0)\}\, F(x+t, x, n)\, dt$$

is, for all values of n, less than an arbitrarily chosen positive number. It is clear that μ may be so chosen that, subject to a similar condition, a similar property belongs to

$$\int_0^\mu \{f(x-t)-f(x-0)\}\, F(x-t, x, n)\, dt.$$

It has therefore been shewn that, *for a point x in G, the conditions contained in the last theorem that*

$$\int_0^\mu \{f(x\pm t)-f(x+0)\}\, F(x\pm t, x, n)\, dt,$$

should both be numerically less than an arbitrarily chosen positive number ϵ, for a sufficiently small value of μ, and for all values of n, are satisfied if a neighbourhood of x exists so small that in that neighbourhood $f(x')$ is of limited total fluctuation, provided also μ can be so chosen that the integrals $\int_{\mu_1}^\mu F(x\pm t, x, n)\, dt$ are both numerically less than some fixed positive number for all values of μ_1 such that $0 \leqslant \mu_1 < \mu$, and for all values of n. When the other conditions of the theorem are also satisfied, the integral converges to the value $Pf(x+0)+Qf(x-0)$.

If x coincides with the end-point α of the interval (α, β), that point being assumed to belong to G, $\int_\alpha^\beta f(x')\, F(x', \alpha, n)\, dx'$ converges to the value $f(\alpha+0) \lim\limits_{n=\infty} \int_0^\mu F(\alpha+t, \alpha, n)\, dx'$, provided this expression have a

definite meaning, and provided also that μ can be so chosen that

$$\int_0^{\mu} \{f(a+t)-f(a+0)\} F(x', a, n)\, dx'$$

is numerically less than an arbitrarily chosen positive number for all values of n. This last condition is satisfied, in particular, if a neighbourhood on the right of the point a exists in which $f(x)$ has limited total fluctuation, and if also μ can be so chosen that $\int_{\mu_1}^{\mu} F(a+t, a, n)\, dt$ is numerically less than some fixed positive number, for all values of μ_1 in the interval $(0, \mu)$, and for all values of n. A similar statement may be made for the case $x = \beta$.

5. Having found sufficient conditions for the convergence of the integral $\int_a^{\beta} f(x') F(x', x, n)\, dx'$ at any point x of G, at which $f(x)$ has definite functional limits, we proceed to find sufficient conditions that the convergence of the integral to its limit may be uniform in an interval (a_1, β_1) contained in the interior of (a, β), and in which the function $f(x')$ is continuous. It will be assumed that all points of (a_1, β_1) belong to G.

It is sufficient for such uniform convergence that the two integrals

$$\int_0^{\mu} \{f(x\pm t)-f(x)\}\, F(x\pm t, x, n)\, dt$$

should converge to zero, as μ is indefinitely diminished, uniformly for all values of x in (a_1, β_1), it being assumed also that $\lim_{n=\infty} \int_{-\mu}^{\mu} F(x+t, x, n)\, dt$ exists at each point of (a_1, β_1), independently of the value of μ, and that the convergence to the limit is uniform in (a_1, β_1).

This clearly follows from the discussion in § 4.

If we assume that (a_1, β_1) is contained in the interior of an interval (a_2, β_2) in which $f(x)$ has limited total fluctuation, the function as before being supposed continuous in (a_1, β_1), we see, from the proof of the second theorem in § 4, that *the convergence will be uniform in (a_1, β_1), provided μ can be chosen so small that the integrals $\int_{\mu_1}^{\mu} F(x\pm t, x, n)\, dt$ are both numerically less than some fixed positive number, for all values of μ_1 such that $0 \leqslant \mu_1 < \mu$, and for all values of x in (a_1, β_1), and for all values of n; it being assumed that $\lim_{n=\infty} \int_0^{\mu} F(x\pm t, x, n)\, dt$ exists at each point of (a_1, β_1), and so that the convergence to the limit is uniform in that interval.*

Since (a_1, β_1) is contained in the interior of an interval (a_2, β_2) in which

the total fluctuation of the function $f(x)$ is limited, μ can be so chosen that the interval $(x-\mu, x+\mu)$ is, for each value of x in (α_1, β_1), an interval in which the function has limited total fluctuation.

It has thus been shewn that the uniform convergence of the integral in an interval (α_1, β_1) contained in the interior of (α, β) depends only on the nature of the function $f(x)$ in an interval $(\alpha_1-\epsilon, \beta_1+\epsilon)$ containing (α_1, β_1), where ϵ is arbitrarily small, and not on its nature in the remainder of the interval (α, β); subject, of course, to the condition that the function has a Lebesgue integral in the whole interval (α, β). In particular, the convergence of the integral at a particular point x, depends only on the nature of the function in an arbitrarily small neighbourhood of x. These results are known for the particular case of the convergence of Fourier's series. The result in the case of convergence at a point is due to Riemann.

6. In case the function $F(x', x, n)$ is never negative, the criteria for the convergence of the integral $\int_\alpha^\beta f(x')F(x',x,n)\,dx'$ admit of simplification.

At any point x at which $f(x+0), f(x-0)$ exist and are finite, μ can be chosen so small that $|f(x+t)-f(x+0)|$, $|f(x-t)-f(x-0)|$ are both less than an arbitrarily chosen positive number η, for $0 < t \leqslant \mu$. It follows that, for a properly chosen value of μ,

$$\left| \int_0^\mu \{f(x\pm t)-f(x\pm 0)\}\, F(x\pm t, x, n)\,dt \right| < \eta \int_0^\mu F(x\pm t, x, n)\,dt,$$

and the expression on the right-hand side is arbitrarily small if $\int_0^\mu F(x\pm t, x, n)\,dt$ is less than some fixed finite number for all values of n. We thus obtain the following theorem :—

When $F(x', x, n)$ is never negative, it is sufficient for the convergence of $\int_\alpha^\beta f(x')\,F(x', x, n)\,dx'$, for a fixed value of x in the interior of (α, β), to the value $Pf(x+0)+Qf(x-0)$, that

$$\int_0^\mu F(x+t, x, n)\,dt \quad \textit{and} \quad \int_0^\mu F(x-t, x, n)\,dt$$

should be less than fixed positive numbers for all values of n, and for a sufficiently small value of μ, and that they should have definite limits P, Q independent of μ, when n is indefinitely increased.

If $f(x)$ is continuous in the interval (α_1, β_1), it follows, from the well known property of uniform continuity, that a value of μ can be deter-

mined such that $|f(x\pm t)-f(x\pm 0)| < \eta$, for $0 < t \leqslant \mu$, and for all values of x in (α_1, β_1). If, then,

$$\int_0^\mu F(x+t,\, x,\, n)\, dt \quad \text{and} \quad \int_0^\mu F(x-t,\, x,\, n)\, dt$$

are both less than fixed positive numbers for all values of n, and for all values of x in (α_1, β_1), provided μ be sufficiently small, then the convergence of the integral is uniform in (α_1, β_1). We thus obtain the following theorem :—

When $F(x', x, n)$ is never negative, it is sufficient for the uniform convergence of $\int_\alpha^\beta f(x')\, F(x', x, n)\, dx'$, to $f(x)$ for all values of x in the interval (α_1, β_1) interior to (α, β), where $f(x)$ is continuous in (α_1, β_1), that

$$\int_{-\mu}^{\mu} F(x+t,\, x,\, n)\, dt$$

should be less than a fixed positive number for all values of n, and for all values of x in (α_1, β_1), and for a sufficiently small value of μ; and also that it have a definite limit, independent of μ, for all values of x in (α_1, β_1) when n is indefinitely increased, the convergence to the limit being uniform in (α_1, β_1). It is assumed that G contains (α_1, β_1).

APPLICATIONS OF THE THEORY.

7. As a first application of the preceding theory, let

$$F(x', x, n) = \frac{[1-(x'-x)^2]^n}{\int_0^1 (1-t^2)^n\, dt},$$

where $$0 \leqslant x \leqslant 1, \quad \text{and} \quad 0 \leqslant x' \leqslant 1;$$

n denoting a positive integer. We take G to consist of the interval (0, 1).

If $|x'-x| \geqslant \mu$, we have

$$|F(x', x, n)| \leqslant \frac{(1-\mu^2)^n}{\int_0^1 (1-t^2)^n\, dt} < \frac{\int_0^\mu (1-t^2)^n\, dt}{\mu \int_0^1 (1-t^2)^n\, dt} < \frac{1}{\mu}.$$

Also $$\int_{\alpha_1}^{\beta_1} F(x', x, n)\, dx' = \frac{\int_{\alpha_1}^{\beta_1} [1-(x'-x)^2]^n\, dx'}{\int_0^1 (1-t^2)^n\, dt};$$

and provided x does not lie in the interior of the interval $(\alpha_1-\mu,\ \beta_1+\mu)$, this is less than $\dfrac{(\beta_1-\alpha_1)(1-\mu^2)^n}{\int_0^1 (1-t^2)^n\,dt}$ or than $\dfrac{(1-\mu^2)^n}{\frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n}$, since

$$\int_0^1 (1-t^2)^n\,dt > \int_0^{1/\sqrt{n}} (1-t^2)^n\,dt > \frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n.$$

We may thus take $$A_n = (1-\mu^2)^n\, n^{\frac{1}{2}} \left(1-\frac{1}{n}\right)^{-n},$$

and then $$\lim_{n=\infty} A_n = e \lim_{n=\infty} \frac{n^{\frac{1}{2}}}{(1+\lambda)^n},$$

where $$1+\lambda = \frac{1}{1-\mu^2}.$$

Hence $\lim\limits_{n=\infty} A_n = 0$; and therefore the conditions of the theorem of § 1 are satisfied for each positive value of μ.

Applying the criteria of § 6, we have

$$\int_0^\mu F(x\pm t,\ x,\ n)\,dt = \frac{\int_0^\mu (1-t^2)^n\,dt}{\int_0^1 (1-t^2)^n\,dt} < 1;$$

and the integral may be expressed in the form

$$1-\frac{\int_\mu^1 (1-t^2)^n\,dt}{\int_0^1 (1-t^2)^n\,dt},$$

and it has been shewn above that the limit of this is 1, when n is indefinitely increased, as may be seen by putting $\alpha_1=\mu$, $\beta_1=1$, $x'-x=\mu$. Therefore

$$\lim_{n=\infty} \int_0^\mu F(x\pm t,\ x,\ n)\,dt = 1.$$

We now see that $$\lim_{n=\infty} \frac{\int_0^1 [1-(x'-x)^2]^n f(x')\,dx'}{2\int_0^1 (1-t^2)^n\,dt}$$

converges to the limit $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x in the interior of (0, 1), at which $f(x+0)$, $f(x-0)$ exist, the function $f(x)$ being restricted only by the postulation that it has a Lebesgue integral in the interval

(0, 1). At the points 0, 1 the integral converges to the values $\frac{1}{2}f(1+0)$, $\frac{1}{2}f(1-0)$, provided these limits exist.

Moreover, we see from the second theorem of § 6 that the convergence to the limit $f(x)$ is uniform in any interval (a, b) in the interior of (0, 1), provided $f(x)$ be continuous in (a, b).

This last result has been established by Landau* for the case in which $f(x)$ is continuous in the whole interval (0, 1). He has applied it to prove Weierstrass' fundamental theorem, that if $f(x)$ be continuous in (a, b), a rational integral function $G(x)$ can be determined such that $|f(x)-G(x)|$ is less than a prescribed positive number, for all values of x in the interval (a, b). The proof of this is immediate; for we have only to choose a value of n sufficiently large, to make the rational integral function of x,

$$\frac{\int_0^1 [1-(x'-x)^2]^n f(x')\, dx'}{2\int_0^1 (1-t^2)^n\, dt}$$

differ from $f(x)$ by less than a prescribed positive number, for all points x such that $a \leqslant x \leqslant b$.

This method of proving Weierstrass' theorem may be extended to the case of a function of any number of variables. It will be sufficient to consider the case of three variables.

Let $F(x', y', z', x, y, z, n) = \dfrac{\{1-(x'-x)^2-(y'-y)^2-(z'-z)^2\}^n}{8\pi\int_0^1 (1-t^2)^n\, dt}$;

and let the function $f(x', y', z')$ have a Lebesgue integral in the sphere

$$x'^2+y'^2+z'^2 = 1.$$

As before $\qquad F(x', y', z', x, y, z, n) < \dfrac{1}{8\pi\mu}$,

provided $\qquad (x'-x)^2+(y'-y)^2+(z'-z)^2 \geqslant \mu^2.$

Also $\int F(x', y', z', x, y, z, n)\,(dx'\,dy'\,dz')$ taken through any volume in the given sphere which has no part in common with the sphere of radius μ,

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* See his paper "Ueber die Approximation einer stetigen Funktion durch eine ganze rationale Funktion," *Rendiconti del circ. mat. di Palermo*, Vol. xxv., p. 337.

and centre at (x, y, z), is less than $\dfrac{(1-\mu^2)^n}{6\int_0^1 (1-t^2)^n\,dt}$, or than

$$\frac{\sqrt{n}}{6}(1-\mu^2)^n\left(1-\frac{1}{n}\right)^{-n},$$

which converges to zero as n is indefinitely increased. Therefore the conditions of the theorem in § 2 are satisfied.

To shew that

$$\frac{\int f(x', y', z')\,[1-(x'-x)^2-(y'-y)^2-(z'-z)^2]^n\,(dx'\,dy'\,dz')}{8\pi\int_0^1 (1-t^2)^n\,dt},$$

where the integral in the numerator is taken through the volume

$$x'^2+y'^2+z'^2 = 1,$$

converges to $f(x, y, z)$ uniformly in any volume contained in the interior of the sphere, provided the function is continuous through that volume, we have only to consider the above integral taken through the sphere

$$(x'-x)^2+(y'-y)^2+(z'-z)^2 = \mu^2.$$

If $\quad x' = x+t\sin\theta\cos\phi,\quad y' = y+t\sin\theta\sin\phi,\quad z' = z+t\cos\theta,$

the integral reduces to

$$\frac{\int_0^\mu \phi(x, y, z, t)(1-t^2)^n\,dt}{2\int_0^1 (1-t^2)^n\,dt},$$

where $\phi(x, y, z, t)$ is continuous with respect to (x, y, z) and to t. As before, this integral converges to $\phi(x, y, z, 0)$ or $f(x, y, z)$ uniformly in the given volume through which the function is continuous. Weierstrass' theorem is deduced immediately, as in the case of a function of one variable.

8. Let us consider the limit

$$\lim_{k=0}\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\,e^{-(x'-x)^2/k^2}\,dx',$$

employed by Weierstrass himself, to prove his fundamental theorem.

We assume that $f(x)$ is a function which has a Lebesgue integral in

every finite interval, and is such that $\int_{-\infty}^{\infty} |f(x')|\, dx'$ **exists as the double** limit for $\alpha = -\infty$, $\beta = \infty$, of the Lebesgue integral $\int_{\alpha}^{\beta} |f(x')|\, dx'$.

Writing $k = 1/n$, $F(x', x, n) = \dfrac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}$,

we see that, if $|x'-x| \geqslant \mu$, then

$$F(x', x, n) \leqslant \frac{n}{\sqrt{\pi}} e^{-n^2\mu^2} \leqslant \frac{1}{\mu\pi^{\frac{1}{2}}2^{\frac{1}{2}}} e^{-\frac{1}{2}}.$$

Also
$$\int_{a_1}^{b_1} \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx'$$

in any interval (a_1, b_1) such that x is not interior to the interval $(a_1-\mu,\ b_1+\mu)$, is less than $\dfrac{n}{\sqrt{\pi}}(b_1-a_1)\, e^{-n^2\mu^2}$ or than $\dfrac{n}{\sqrt{\pi}} e^{-n^2\mu^2} K$, where $b_1 - a_1 \leqslant K$; and this converges to zero as n is indefinitely increased. It has thus been shewn that the conditions of the theorem in § 8 are satisfied.

Again,

$$\int_{-\mu}^{\mu} F(x+t, x, n)\, dt = \int_{-\mu}^{\mu} \frac{n}{\sqrt{\pi}} e^{-n^2t^2} dt < \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt < 1;$$

and
$$\lim_{n=\infty} \int_0^{\pm\mu} \frac{n}{\sqrt{\pi}} e^{-n^2t^2} dt = \lim_{n=\infty} \frac{1}{\sqrt{\pi}} \int_0^{\pm n\mu} e^{-t^2} dt = \tfrac{1}{2}.$$

Therefore the conditions of the theorems of § 6 are satisfied.

It has therefore been shewn that, *if $f(x)$ have a Lebesgue integral in every finite interval, and if* $\int_{-\infty}^{\infty} |f(x')|\, dx'$ *exists, then*

$$\frac{1}{k\sqrt{\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-(x'-x)^2/k^2}\, dx'$$

converges, for $k = 0$, to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x at which $f(x+0)$, $f(x-0)$ exist. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous.

It is easy to extend the theorem to the case of the limit

$$\lim_{k=0} \left(\frac{1}{k\sqrt{\pi}}\right)^p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots f(x_1', x_2', \dots, x_p')\, e^{-[(x_1'-x_1)^2+(x_2'-x_2)^2+\dots+(x_p'-x_p)^2]/k^2}\, dx_1'\, dx_2' \dots dx_p';$$

and from this Weierstrass' theorem for continuous functions of p variables can be immediately deduced.

[*Added May 26th*, 1908.]

The condition contained in the statement of the above result, that $\int_{-\infty}^{\infty} |f(x')|\, dx'$ should exist, may be replaced by a much less stringent condition.* Referring to § 8, we see that it is only necessary that when x is confined to an interval (a_1, β_1), ξ and η can be so determined that

$$\int_{\xi'}^{\xi} f(x') \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx' \quad \text{and} \quad \int_{\eta}^{\eta'} f(x') \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx'$$

should both be numerically $< \epsilon$, for all values of $\xi' < \xi$, and $\eta' > \eta$, and for all values of n. We have now

$$\left| \int_{\eta}^{\eta'} f(x') \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx' \right| < \int_{\eta}^{\eta'} |f(x')| \frac{n}{\sqrt{\pi}} e^{-n^2(x'-\beta_1)^2}\, dx'$$

$$< \frac{1}{\sqrt{\pi}} \int_{n\eta}^{n\eta'} \left| f\left(\frac{z'}{n}\right) \right| e^{-(z'-n\beta_1)^2}\, dz'.$$

Now, let it be assumed that, for all values of x greater than some fixed value, the condition $|f(x)| < x^p e^{qx}$ is satisfied, where p and q are fixed positive numbers. Let η be so chosen that $f(z'/n)$ satisfies this condition for $z'/n > \eta$; we have then

$$\left| \int_{\eta}^{\eta'} f(x') \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx' \right| < \frac{1}{\sqrt{\pi}} \int_{n\eta}^{n\eta'} \left(\frac{z'}{n}\right)^p e^{q(z'/n)} e^{-(z'-n\beta_1)^2}\, dz'$$

$$< \frac{1}{\sqrt{\pi}} \int_{n(\eta-\beta_1)}^{n(\eta'-\beta_1)} \left(\beta_1 + \frac{u}{n}\right)^p e^{q\beta_1 + qu/n} e^{-u^2}\, du$$

$$< \frac{1}{\sqrt{\pi}} \int_{\eta-\beta_1}^{\infty} (\beta_1 + u)^p e^{q\beta_1 + qu} e^{-u^2}\, du.$$

Since the integral $\int_{\eta-\beta_1}^{\infty} (\beta_1+u)^p e^{qu-u^2}\, du$ exists, as is well known, η can be so chosen that

$$\frac{1}{\sqrt{\pi}} \int_{\eta-\beta_1}^{\infty} (\beta_1+u)^p e^{q\beta_1+qu-u^2}\, du < \epsilon.$$

Similarly, it can be shewn that if, for sufficiently large negative values of x, the condition $|f(x)| < |x|^p e^{q|x|}$ is satisfied, then

$$\int_{\xi'}^{\xi} f(x') \frac{n}{\sqrt{\pi}} e^{-n^2(x'-x)^2}\, dx'$$

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* That this is the case was suggested to me by Mr. Bromwich.

is numerically $< \epsilon$, for all values of $\xi' < \xi$, and of n, if ξ be properly chosen. We have thus established the following more general theorem :—

If $f(x)$ have a Lebesgue integral in every finite interval, and be such that for $|x| > a$, the condition $|f(x)| < |x|^p e^{q|x|}$ is satisfied, when a, p, q are fixed positive numbers, then $\dfrac{1}{k\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} f(x') e^{-(x'-x)^2/k^2} dx'$ *converges for $k = 0$ to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x at which $f(x+0)$, $f(x-0)$ exist. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous.*

9. Let $s_n(x)$ denote the sum

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \frac{1}{\pi}\sum_{n=1}^{r=n-1}\int_{-\pi}^{\pi} f(x')\cos r(x'-x)\,dx'$$

of the first $2n+1$ terms of Fourier's series. Denoting by $S_n(x)$ the arithmetic mean $(s_1+s_2+\ldots+s_n)/n$, formed in accordance with Césaro's method, it is easily found that

$$S_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x') \left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 dx'.$$

To evaluate $\lim\limits_{n=\infty} S_n(x)$, let

$$F(x', x, n) = \frac{1}{2n\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2.$$

As $|x'-x|$ approaches the extreme value 2π, $F(x', x, n)$ approaches the value $n/2\pi$. Consequently, the conditions of the theorem of § 1 would not be satisfied if G were taken to include the whole interval $(-\pi, \pi)$. The conditions are, however, satisfied if we take the set G of values of x to consist of the points of the interval $(-\pi+\lambda, \pi-\lambda)$, where λ is a fixed positive number as small as we please. In that case $F(x', x, n)$ is less than the greater of the numbers $\dfrac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\mu$, and $\dfrac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\lambda$, which is $\dfrac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\mu$, if we choose μ to be $< \lambda$. The number $\bar{F}$ is then $\dfrac{1}{2\pi}\operatorname{cosec}^2\frac{1}{2}\mu$.

Also

$$\int_{a_1}^{\beta_1} F(x', x, n)\,dx' < \frac{1}{2n\pi}\operatorname{cosec}^2\tfrac{1}{2}\mu\int_{a_1}^{\beta_1}\sin^2\tfrac{1}{2}n(x'-x)\,dx < \frac{1}{n}\operatorname{cosec}^2\tfrac{1}{2}\mu\,;$$

and the limit of this value is zero, when n is indefinitely increased.

Therefore the conditions of the theorem of § 1 are satisfied for every value of μ, such that $0 < \mu \leqslant \lambda$, when G consists of the interval $(-\pi+\lambda, \pi-\lambda)$.

We have also

$$\int_0^\mu F(x \pm t, x, n)\, dt = \frac{1}{2n\pi}\int_0^\mu \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt < \frac{1}{2n\pi}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt < \tfrac{1}{2}.$$

Also $$\lim_{n=\infty}\int_0^\mu F(x \pm t, x, n)\, dt = \lim_{n=\infty}\frac{1}{2n\pi}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt = \tfrac{1}{2};$$

and therefore the conditions of the theorems in § 6 are satisfied.

It has therefore been shewn that, *if $f(x')$ have a Lebesgue integral in the interval $(-\pi, \pi)$, the function $S_n(x)$, formed in accordance with Cesàro's method of arithmetic means, converges to the value*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

at any interior point x at which the functional limits exist. Moreover, the convergence of $S_n(x)$ to the value $f(x)$ is uniform in any interval (a, b) interior to $(-\pi, \pi)$ in which the function is continuous.*

To find $\lim_{n=\infty} S_n(-\pi)$, we have

$$S_n(-\pi) = \frac{1}{2n\pi}\left\{\int_{-\pi}^{-\pi+\mu} + \int_{\pi-\mu}^{\pi} + \int_{-\pi+\mu}^{\pi-\mu}\right\} f(x') \left\{\frac{\sin \frac{1}{2}n(x'+\pi)}{\sin \frac{1}{2}(x'+\pi)}\right\}^2 dx'$$

$$= \frac{1}{2n\pi}\int_0^\mu f(-\pi+t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt + \frac{1}{2n\pi}\int_0^\mu f(\pi-t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt$$

$$+ \frac{1}{2n\pi}\int_{-\pi+\mu}^{\pi-\mu} f(x')\left\{\frac{\sin \frac{1}{2}n(x'+\pi)}{\sin \frac{1}{2}(x'+\pi)}\right\}^2 dt.$$

The limit of the third of these integrals has been shewn to be zero. If

* I take the opportunity of correcting an error which occurs in this connection in my treatise "On the Theory of Functions of a Real Variable, and on Fourier's Series." It is erroneously stated, on p. 712, that the convergence of $S_n(x)$ to the value $\lim_{h=0}\{f(x+h)+f(x-h)\}$ is uniform in any interval in which $f(x)$ is limited, and in which the limit everywhere exists. The source of the error is at the top of p. 711, where the incorrect statement is made, that in any interval (a, b) in which $f(x)$ is limited, and in which $\lim_{h=0}\{f(x+h)+f(x-h)\}$ has everywhere definite values, n may be so chosen that the upper limit of $|F(z)|$ in the interval $\left(0, \frac{\pi}{2n+1}\right)$ is less than ϵ. A sequence of continuous functions $\{S_n(x)\}$ which converges uniformly in any interval must, as is well known, have a continuous limit.

the two limits $f(-\pi+0)$, $f(\pi-0)$ both exist and are finite, we see that

$$\lim S_n(-\pi) = \tfrac{1}{2}\{f(-\pi+0)+f(\pi-0)\}.$$

Clearly $S_n(\pi)$ has the same limit.

10. The preceding theory may also be employed to establish the validity of Fourier's representation of a function in an unlimited interval by means of a single integral, under very general conditions.

Let us consider the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\,\frac{\sin u(x'-x)}{x'-x}\,dx',$$

where u is here written instead of n. It will be assumed that $f(x')$ has a Lebesgue integral in every finite interval. It is unnecessary to assume that $\int_{-\infty}^{\infty} |f(x')|\,dx'$ exists as the double limit of the Lebesgue integral $\int_k^h |f(x')|\,dx'$. It will, in fact, be sufficient to make the more general assumption that $\int_a^{\infty}\left|\frac{f(x')}{x'}\right|dx'$, $\int_{-\infty}^{-a}\left|\frac{f(x')}{x'}\right|dx'$, where a is a positive number, both exist as the limits of Lebesgue integrals. Assuming that x is confined to a finite interval (α_1, β_1), we have

$$\left|\frac{1}{\pi}\int_{\eta}^{\eta'}\frac{f(x')}{x'-x}\sin u(x'-x)\,dx'\right| < \frac{1}{\pi}\int_{\eta}^{\eta'}\left|\frac{f(x')}{x'-x}\right|dx'$$
$$< \frac{1}{\pi}(1+\zeta)\int_{\eta}^{\eta'}\left|\frac{f(x')}{x'}\right|dx',$$

where η is chosen so great that $\frac{x'}{x'-\beta_1} < 1+\zeta$ for $x' \geqslant \eta$, and ζ denotes an arbitrarily chosen positive number.

It now follows that η may be so chosen that, for all values of u, and provided x is confined to the interval (α_1, β_1), $\frac{1}{\pi}\int_{\eta}^{\eta'} f(x')\,\frac{\sin u(x'-x)}{x'-x}\,dx'$ is numerically less than ϵ, for all values of $\eta' > \eta$. It may similarly be shewn that ξ may be so chosen that $\int_{\xi'}^{\xi} f(x')\,\frac{\sin u(x'-x)}{x'-x}\,dx'$ is numerically less than ϵ, for all values of $\xi' < \xi$.

We have now to shew that the conditions of the theorem in § 4 are satisfied. Writing

$$F(x', x, u) = \frac{1}{\pi}\,\frac{\sin u(x'-x)}{x'-x},$$

we have $\qquad | F(x', x, u) | \leqslant \frac{1}{\mu\pi}, \quad \text{for} \quad |x'-x| \geqslant \mu.$

Also we have

$$\int_a^\beta F(x', x, u)\, dx' = \frac{1}{\pi(a-x)} \int_a^\gamma \sin u\,(x'-x)\, dx' + \frac{1}{\pi(\beta-x)} \int_\gamma^\beta \sin u\,(x'-x)\, dx',$$

provided x is exterior to the interval (a, β), where $a \leqslant \gamma \leqslant \beta$. It follows that

$$\left| \int_a^\beta F(x', x, u)\, dx' \right| < \frac{4}{\mu\pi u},$$

if x is not interior to the interval $(a-\mu, \beta+\mu)$; and $4/\mu\pi u$ converges to zero as u is indefinitely increased. It now follows that

$$\frac{1}{\pi} \int_{-\infty}^{x-\mu} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx' \quad \text{and} \quad \frac{1}{\pi} \int_{x+\mu}^{\infty} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'$$

converge to zero as u is indefinitely increased, uniformly for all values of x in any finite interval.

We have now to consider the convergence of

$$\frac{1}{\pi} \int_{x-\mu}^{x+\mu} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'.$$

Let x be confined to an interval (a_1, β_1), and let the criteria provided in §§ 4, 5 be applied. We have then

$$\frac{1}{\pi} \int_0^\mu \frac{\sin ut}{t}\, dt = \frac{1}{\pi} \int_0^{\mu u} \frac{\sin t}{t}\, dt;$$

and this has the limit $\frac{1}{2}$, when u is indefinitely increased.

Also

$$\frac{1}{\pi} \int_{\mu_1}^{\mu} \frac{\sin ut}{t}\, dt = \frac{1}{\pi} \int_{\mu_1 u}^{\mu u} \frac{\sin t}{t}\, dt = \frac{1}{\pi} \int_0^{\mu u} \frac{\sin t}{t}\, dt - \frac{1}{\pi} \int_0^{\mu_1 u} \frac{\sin t}{t}\, dt;$$

and both the last integrals are well known to be numerically less than fixed numbers independent of μ, μ_1 and u.

The following theorem has now been established :—

If $f(x')$ have a Lebesgue integral in every finite interval, and if $\int_a^\infty \left| \frac{f(x')}{x'} \right| dx'$, $\int_{-\infty}^{-a} \left| \frac{f(x')}{x'} \right| dx'$, where a is positive, exist as the limits of Lebesgue integrals $\Big($*this condition being satisfied in particular if $\int_{-}^{\infty} | f(x') |\, dx'$ exists*$\Big)$, *then $\frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'$ converges for*

$u=\infty$, to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point x for which a neighbourhood exists in which $f(x')$ is of limited total fluctuation. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous, and which is in the interior of an interval in which $f(x)$ has limited total fluctuation.

From § 4, we see that a sufficient condition of convergence of the integral at a point x is that

$$\int_0^\mu \frac{f(x+t)-f(x+0)}{t}\sin ut\,dt, \quad \text{and} \quad \int_0^\mu \frac{f(x-t)-f(x-0)}{t}\sin ut\,dt,$$

should converge to zero as u is indefinitely increased. This condition is certainly satisfied if $|f(x+t)-f(x+0)| \leqslant At^{1-a}$, for all sufficiently small values of t, when A, $1-a$ are fixed positive numbers, and if $|f(x-t)-f(x-0)|$ satisfies a similar condition. The conditions are satisfied at a point of continuity of $f(x)$ at which the four derivatives are limited, and generally provided any of the known sufficient criteria for the convergence of Fourier's series at a point are satisfied.

The preceding theory may also be applied to the case of Poisson's integral which occurs in the theory of Fourier's series.

SERIES OF STURM-LIOUVILLE NORMAL FUNCTIONS.

11. The differential equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)\,V=0 \tag{1}$$

occurs in the theory of the conduction of heat in a heterogeneous bar, and in connection with other problems of mathematical physics.

Those solutions of this equation for an interval (a, b) of the variable x which satisfy the boundary conditions

$$\frac{dV}{dx}-hV=0,\ \text{for}\ x=a, \quad \text{and} \quad \frac{dV}{dx}+HV=0,\ \text{for}\ x=b, \tag{2}$$

where h, H are positive constants, were studied by Liouville and by Sturm in a series of memoirs published in the first two volumes of *Liouville's Journal*.

Special cases of the boundary equations are obtained by letting one or both of the constants h, H have the value zero. Other special cases are obtained by supposing h or H, or both of them to be infinite, in which case the corresponding boundary condition is $V=0$.

It is assumed that g, k, l are functions of x which are positive, and do not vanish in the interval (a, b); r is a parameter. It will be further

assumed that g, k have everywhere finite differential coefficients, and that l and $(gk)^{-\frac{1}{2}}$ have limited total fluctuation in (a, b). If we transform the equation (1) by means of the substitutions

$$z = \int_a^x \left(\frac{g}{k}\right)^{\frac{1}{2}} dx, \quad \theta = (gk)^{-\frac{1}{4}}, \quad V = \theta U, \quad r = \rho^2,$$

the differential equation (1) becomes

$$\frac{d^2U}{dz^2} + (\rho^2 - l_1)\, U = 0, \tag{3}$$

where $$l_1 = \frac{1}{\theta (gk)^{\frac{1}{2}}} \left\{ l \left(\frac{k}{g}\right)^{\frac{1}{2}} \theta - \frac{d(gk)^{\frac{1}{2}}}{dz} \frac{d\theta}{dz} - (gk)^{\frac{1}{2}} \frac{d^2\theta}{dz^2} \right\}.$$

The boundary equations (2) become

$$\frac{dU}{dz} - h'U = 0, \text{ for } z = 0, \quad \text{and} \quad \frac{dU}{dz} + H'U = 0, \text{ for } z = \pi, \tag{4}$$

where it is assumed that $$\int_a^b \left(\frac{g}{k}\right)^{\frac{1}{2}} dx = \pi,$$

an equation which is always satisfied if a slight formal change in the variable x be made. The constants h', H' are real, but no longer necessarily positive. We shall suppose that neither h' nor H' is infinite.

Writing the equation (3) in the form

$$\frac{d^2U}{dz^2} + \rho^2 U = l_1 U,$$

we have, as an equation satisfied by a solution of this equation,

$$U = A \cos \rho z + B \sin \rho z + \frac{1}{\left(\frac{d}{dz}\right)^2 + \rho^2}\, l_1 U$$

$$= A \cos \rho z + B \sin \rho z + \frac{1}{2\iota\rho} \left\{ \frac{1}{\frac{d}{dz} - \iota\rho} - \frac{1}{\frac{d}{dz} + \iota\rho} \right\} l_1 U$$

$$= A \cos \rho z + B \sin \rho z + \frac{1}{\rho} \int_0^z l_1' U' \sin \rho (z - \zeta)\, d\zeta,$$

where l_1', U' are what l_1 and U become when ζ is substituted in them for z. If this value of U be substituted in the first of the boundary conditions (4), we find that $B\rho - Ah' = 0$, in order that the condition may be satisfied.

equation for the determination of ρ is of the form

$$\tan \pi\rho = \frac{h'+H'+h''+\dfrac{a}{\rho}}{\rho-\dfrac{a}{\rho}};$$

therefore, for sufficiently large values of ρ,

$$\pi\rho = n\pi + \frac{h'+H'+h''}{\rho} + \frac{a}{\rho^2}, \quad \text{or} \quad \rho = n + \frac{h'+H'+h''}{n\pi} + \frac{a}{n^2}.$$

It then follows that, for all values of n, which represents one of the positive integers,

$$\rho_n = n + \frac{c}{n} + \frac{a}{n^2},$$

where c is the constant $\dfrac{h'+H'+h''}{\pi}$, and a denotes some number of which the absolute value is less than a fixed number independent of n. All the positive roots of the equation for the determination of ρ are given in this form; it is clear that the notation employed enables us to use what is primarily an asymptotic expression, available for large values of n, to represent all the roots.

We shall now employ the expression for ρ_n to express the function U_n, which corresponds to the value ρ_n of ρ, in terms of n and z. We have

$$\cos \rho_n z = \cos nz \left\{1 + \frac{a(z, n)}{n^2}\right\} - \sin nz \left\{\frac{cz}{n} + \frac{a(z, n)}{n^2}\right\},$$

$$\sin \rho_n z = \sin nz \left\{1 + \frac{a(z, n)}{n^2}\right\} + \cos nz \left\{\frac{cz}{n} + \frac{a(z, n)}{n^2}\right\}.$$

Substituting these values in the expression

$$\cos \rho_n z \left\{1 + \frac{a(\rho, z)}{\rho_n^2}\right\} + \sin \rho_n z \left\{\frac{a(z)}{\rho_n} + \frac{a(\rho, z)}{\rho_n^2}\right\},$$

or in the equivalent expression

$$\cos \rho_n z \left\{1 + \frac{a(\rho, z)}{n^2}\right\} + \sin \rho_n z \left\{\frac{a(z)}{n} + \frac{a(\rho, z)}{n^2}\right\},$$

we find that

$$U_n(z) = \cos nz \left\{1 + \frac{a(z, n)}{n^2}\right\} + \sin nz \left\{\frac{a(z)}{n} + \frac{a(z, n)}{n^2}\right\}$$

which is the required expression for $U_n(z)$.

It is easily seen that $\int_0^\pi \{U_n(z)\}^2\,dz$ is of the form $\frac{\pi}{2}+\frac{a(n)}{n^2}$; for

$$\int_0^\pi a(z)\sin 2nz\,dz = \frac{1}{n}\Big[-a(z)\cos 2nz\Big]_0^\pi + \frac{1}{n}\int_0^\pi a'(z)\cos 2nz\,dz,$$

and $a'(z)$ is a limited function, and thus $\int_0^\pi a(z)\sin 2nz\,dz$ is of the form $\frac{a(n)}{n}$.

We take now, as the normal function $\phi_n(z)$,

$$\phi_n(z) = U_n(z)\left[\int_0^\pi \{U_n(z)\}^2\,dz\right]^{-\frac{1}{2}};$$

and thus $\int_0^\pi \{\phi_n(z)\}^2\,dz = 1$, $\int_0^\pi \phi_n(z)\,\phi_{n'}(z)\,dz = 0$, for $n \neq n'$.

It follows, from the above forms for $U_n(z)$, $\phi_n(z)$, that

$$\phi_n(z) = \sqrt{\frac{2}{\pi}}\cos nz\left\{1+\frac{a(z,n)}{n^2}\right\} + \sin nz\left\{\frac{a(z)}{n}+\frac{a(z,n)}{n^2}\right\}.$$

It is necessary for our purposes to find a corresponding expression for $\phi_n'(z)$. We have

$$\begin{aligned}\frac{dU_n}{dz} &= -\rho_n\sin\rho_n z + h'\cos\rho_n z + \int_0^z l_1' U'\cos\rho_n(z-\zeta)\,d\zeta \\ &= -\left[n+\frac{a(z,n)}{n}\right]\left[\sin nz\left\{1+\frac{a(z,n)}{n^2}\right\} + \cos nz\left\{\frac{cz}{n}+\frac{a(z,n)}{n^2}\right\}\right] \\ &\qquad + \left[h'+\frac{a(z,n)}{n}\right]\left[\cos nz\left\{1+\frac{a(z,n)}{n^2}\right\} - \sin nz\left\{\frac{cz}{n}+\frac{a(z,n)}{n^2}\right\}\right] \\ &= -\left\{n+\frac{a(z,n)}{n}\right\}\sin nz + \left\{h'-cz+\frac{a(z,n)}{n}\right\}\cos nz.\end{aligned}$$

On multiplying $\frac{dU_n}{dz}$ by $\left[\int_0^\pi \{U_n(z)\}^2\,dz\right]^{-\frac{1}{2}}$, or $\sqrt{\frac{2}{\pi}}\left\{1+\frac{a(n)}{n^2}\right\}$, we find that

$$\frac{d\phi_n(z)}{dz} = -\sqrt{\frac{2}{\pi}}\left\{n+\frac{a(z,n)}{n}\right\}\sin nz + \sqrt{\frac{2}{\pi}}\left\{h'-cz+\frac{a(z,n)}{n}\right\}\cos nz,$$

the required form for $\phi_n'(z)$.

12. If we write $\lambda_n = \rho_n^2$, the positive numbers $\lambda_1, \lambda_2, \ldots$ are the characteristic values of ρ^2 (*Eigenwerthe*) for the equation (3), with the given boundary conditions, in accordance with the nomenclature of

the theory of integral equations. We may assume that the smallest characteristic number λ_1 is > 0; for if λ_1 were equal to zero, by changing the value of l_1 so that $\rho^2 - l_1$ remained unaltered, we should make λ_1 greater than zero.

Let $\mathfrak{f}_1(z)$, $\mathfrak{f}_2(z)$ be solutions of the equation

$$\frac{d^2u}{dz^2} - l_1 u = 0,$$

which, together with their first two differential coefficients, are continuous in the interval $(0, \pi)$; and such that $\mathfrak{f}_1'(z) - h'\mathfrak{f}_1(z) = 0$, for $z = 0$, and $\mathfrak{f}_2'(z) + H'\mathfrak{f}_2(z) = 0$, for $z = \pi$. The two functions satisfy the relation

$$\mathfrak{f}_1(z)\,\mathfrak{f}_2'(z) - \mathfrak{f}_2(z)\,\mathfrak{f}_1'(z) = -1,$$

if the arbitrary constant factors in $\mathfrak{f}_1(z)$, $\mathfrak{f}_2(z)$ are properly chosen. A function* $K(z, z')$ is defined for the whole interval, by the conditions

$$K(z, z') = \mathfrak{f}_2(z')\,\mathfrak{f}_1(z), \quad \text{for} \quad z \leqslant z',$$

and

$$K(z, z') = \mathfrak{f}_1(z')\,\mathfrak{f}_2(z), \quad \text{for} \quad z \geqslant z'.$$

This function is continuous in the interval $(0, \pi)$ of z, and symmetrical with respect to z and z'; it is the "nucleus" (Kern) for the system of normal functions. Writing

$$\frac{d}{dz} K(z, z') = K'(z, z'),$$

we have $K'(z'-0, z') = \mathfrak{f}_2(z')\,\mathfrak{f}_1'(z')$, and $K'(z'+0, z') = \mathfrak{f}_1(z')\,\mathfrak{f}_2'(z')$;

and therefore $$K'(z'+0, z') - K'(z'-0, z') = -1.$$

The function $K'(z, z')$ is continuous for all values of z not equal to z'. The function $K(z, z')$ clearly satisfies the differential equation

$$\frac{d^2K(z, z')}{dz^2} - l_1 K(z, z') = 0,$$

for every value of z except $z = z'$.

It is known from the theory of integral equations that if the series $\sum_{n=1}^{\infty} \frac{\phi_n(z)\,\phi_n(z')}{\lambda_n}$ is uniformly convergent in the interval $(0, \pi)$, then it represents the function $K(z, z')$.

* See Kneser's "Integralgleichungen und Darstellung willkürlicher Functionen," *Math. Annalen*, Vol. LXIII., p. 483.

13. Let the sum $\phi_1(z)\,\phi_1(z')+\phi_2(z)\,\phi_2(z')+\ldots+\phi_n(z)\,\phi_n(z')$ be denoted by $F(z', z, n)$. In order to apply the theorem of § 1 to the case of this function, we suppose the set G to consist of all the points of the interval $(0, \pi)$ of the variable z.

We have first to verify that $|F(z', z, n)|$ is less than a fixed number, for all values of n, and for all the values of z, z' such that $|z-z'| \geqslant \mu$.

We have, employing the expression for $\phi_n(z)$ found in § 11,

$$F(z', z, n) = \sum_{r=1}^{r=n}\left[\sqrt{\frac{2}{\pi}}\cos rz\left\{1+\frac{\alpha(z, r)}{r^2}\right\}+\sin rz\left\{\frac{\alpha(z)}{r}+\frac{\alpha(z, r)}{r^2}\right\}\right]$$

$$\left[\sqrt{\frac{2}{\pi}}\cos rz'\left\{1+\frac{\alpha(z', r)}{r^2}\right\}+\sin rz'\left\{\frac{\alpha(z')}{r}+\frac{\alpha(z, r)}{r^2}\right\}\right].$$

We have to consider the sums of the various terms in this product. The series

$$\left|\frac{2}{\pi}\sum_{r=1}^{r=n}\cos rz\cos rz'\right| = \frac{1}{\pi}\left|\sum_{r=1}^{r=n}\{\cos r(z-z')+\cos r(z+z')\}\right|$$

$$= \frac{1}{2\pi}\left|\left\{\frac{\sin(2n+1)\dfrac{z-z'}{2}}{\sin\dfrac{z-z'}{2}}+\frac{\sin(2n+1)\dfrac{z+z'}{2}}{\sin\dfrac{z+z'}{2}}-2\right\}\right|$$

$$< \frac{1}{2\pi}(2\operatorname{cosec}\tfrac{1}{2}\mu-2),$$

provided $$|z-z'| \geqslant \mu.$$

The expression

$$\sqrt{\frac{2}{\pi}}\,\alpha(z')\sum_{r=1}^{n}\frac{\cos rz\sin rz'}{r} \quad\text{or}\quad \frac{1}{\sqrt{2\pi}}\,\alpha(z')\sum_{r=1}^{n}\frac{\sin r(z'+z)+\sin r(z'-z)}{r}$$

can be shewn to be numerically less than a fixed number, for all values of z, z' such that $|z-z'| \geqslant \mu$, and for all values of n. For it is known* that the sum $\sum_{1}^{n}\dfrac{\sin rx}{r}$ is given by

$$s_n(x) = s(x)+\int_0^{(n+\frac{1}{2})x}\frac{\sin z}{z}\,dz+\frac{\theta A}{n+\frac{1}{2}},$$

when $s(x)$ is the sum of the convergent series $\sum_{1}^{\infty}\dfrac{\sin rx}{r}$, A is a positive

* See *Theory of Functions of a Real Variable*, p. 649

number independent of x and n, and θ is such that $-1<\theta<1$; provided x is in an interval $(0, b)$, where $b<2\pi$. It thus appears that $|s_n(x)|$ is less than some fixed positive number, provided x is in an interval $(-b, b)$, where $b<2\pi$. The point $x=0$ is a point of non-uniform convergence of the series $\overset{\infty}{\underset{1}{\Sigma}} \frac{\sin rx}{r}$, but the measure of non-uniform convergence is finite, the peaks of the approximation curves representing $s_n(x)$ being all of limited height. It follows that

$$\sum_{r=1}^{r=n} \frac{\sin r(z'+z)}{r}, \quad \sum_{r=1}^{r=n} \frac{\sin r(z'-z)}{r}, \quad \text{for } |z-z'| \geqslant \mu,$$

are both less than fixed numbers independent of z, z', and n. Remembering that $|\alpha(z')|$ is limited, the required result at once follows.

Similarly $\sqrt{\frac{2}{\pi}}\, \alpha(z) \sum_{r=1}^{n} \frac{\cos rz' \sin rz}{r}$ is numerically less than a fixed number independent of a, z and z', for $|z-z'| \geqslant \mu$.

The other terms such as

$$\sum_{r=1}^{r=n} \alpha(z)\,\alpha(z') \frac{\sin rz \sin rz'}{r^2}, \quad \sqrt{\frac{2}{\pi}} \sum_{r=1}^{r=n} \frac{\cos rz \sin rz'\, \alpha(z, r)}{r^2}$$

are absolutely and uniformly convergent, as n is indefinitely increased. It therefore follows that $|F(z', z, n)|$ is less than a fixed number, for all values of n, and for all values of z, z' such that $|z-z'| \geqslant \mu$, and in the interval $(0, \pi)$.

We have next to consider

$$\sum_{r=1}^{n} \int_\alpha^\beta \phi_r(z)\,\phi_r(z')\,dz', \quad \text{or} \quad \int_\alpha^\beta F(z', z, n)\,dz'.$$

This may be written in the form

$$\sum_{r=1}^{n} \frac{\phi_r(z)}{\lambda_r} \int_\alpha^\beta \left\{ l_1' \phi_r(z') - \frac{d^2\phi_r(z')}{dz'^2} \right\} dz',$$

by substituting for $\phi_r(z')$ its value as expressed by the differential equation given in § 12, which it satisfies. This is equivalent to

$$\int_\alpha^\beta l_1' \sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}\,dz' - \sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r'(\beta)}{\lambda_r} + \sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r'(\alpha)}{\lambda_r}.$$

Since
$$\frac{1}{\lambda_r} = \frac{1}{\rho_r^2} = \frac{1}{r^2}\left\{1 + \frac{\alpha(r)}{r^2}\right\},$$

we see that $\sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}$ takes the form

$$\sum_{r=1}^{n} \frac{1}{r^2}\left\{1+\frac{a(r)}{r^2}\right\}\left[\sqrt{\frac{2}{\pi}}\cos rz\left\{1+\frac{a(z,r)}{r^2}\right\}+\sin rz\left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right]$$
$$\left[\sqrt{\frac{2}{\pi}}\cos rz'\left\{1+\frac{a(z',r)}{r^2}\right\}+\sin rz'\left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right].$$

The various terms $\frac{2}{\pi}\sum_{r=1}^{r=n}\frac{\cos rz\cos rz'}{r^2}$, ... converge uniformly, as n is indefinitely increased; therefore the series $\sum_{r=1}^{n}\frac{\phi_r(z)\phi_r(z')}{\lambda_r}$ converges uniformly for all values of z and z' in $(0, \pi)$, and the limit of the sum is consequently $K(z, z')$.

The series $\sum_{r=1}^{r=n}\frac{\phi_r(z)\,\phi_r'(z')}{\lambda_r}$ is, on substituting the forms obtained in § 11, for $\phi_r(z)$, $\phi_r(z')$ and λ_r, of the form

$$\sum_{r=1}^{n} \frac{1}{r^2}\left\{1+\frac{a(r)}{r^2}\right\}\left[\sqrt{\frac{2}{\pi}}\cos rz\left\{1+\frac{a(z,r)}{r^2}\right\}+\sin rz\left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right]$$
$$\left[-\sqrt{\frac{2}{\pi}}\sin rz'\left\{r+\frac{a(z,r)}{r}\right\}+\cos rz\left\{a(z)+\frac{a(z,r)}{r}\right\}\right].$$

The portion $-\left(\frac{2}{\pi}\right)\sum_{r=1}^{n}\frac{\cos rz\sin rz'}{r}$ converges uniformly for all values of z, z', such that $|z-z'| \geqslant \mu$, and the remainder of the series converges uniformly for all values of z, z'. Consequently, the series $\sum_{r=1}^{r=n}\frac{\phi_r(z)\,\phi_r'(\beta)}{\lambda_r}$ and the series $\sum_{r=1}^{r=n}\frac{\phi_r(z)\,\phi_r'(a)}{\lambda_r}$ converge uniformly for all values of z not interior to the interval $(a-\mu, \beta+\mu)$. By a known theorem, the limiting sums of these series are therefore $\frac{dK(z,\beta)}{d\beta}$, $\frac{dK(z,a)}{da}$ respectively.

It has now been shewn that, when x is not interior to the interval $(a-\mu, \beta+\mu)$, the sum

$$\sum_{r=1}^{n}\int_a^\beta \phi_r(z)\,\phi_r(z')\,dz' \quad \text{or} \quad \int_a^\beta F(z', z, n)\,dz'$$

converges uniformly to the value

$$\int_a^\beta l_1' K(z,z')\,dz' - \frac{dK(z,\beta)}{d\beta}+\frac{dK(z,a)}{da},$$

which is equal to $\int_a^\beta \frac{d^2K(z,z')}{dz'^2}\,dz' - \frac{dK(z,\beta)}{d\beta}+\frac{dK(z,a)}{da}$;

and is therefore zero. It thus appears that $\left|\int_a^\beta F(z', z, n)\,dz'\right|$ is less than some fixed number independent of a and β, for all values of z in $(0, \pi)$ which are not interior to the interval $(a-\mu, \beta+\mu)$. For, corresponding to an assigned ϵ, a value n_1 of n can be determined such that

$$\left|\int_a^\beta F(z', z, n)\,dz'\right| < (\beta-a)\,\epsilon+2\epsilon < (\pi+2)\,\epsilon, \quad \text{for} \quad n \geqslant n_1.$$

Therefore the conditions of the theorem in § 1 are satisfied for every value of μ such that $0<\mu<\pi$, the set G consisting of the whole interval $(0, \pi)$.

It follows that, for each value of μ,

$$\int_0^{x-\mu} f(x')\,F(x', x, n)\,dx' \quad \text{and} \quad \int_{x+\mu}^{\pi} f(x')\,F(x', x, n)\,dx'$$

converge to zero, as n is indefinitely increased, uniformly for all values of z in the interval $(0, \pi)$.

14. It will now be shewn that the function

$$F(z', z, n) \equiv \sum_{r=1}^{n} \phi_r(z)\,\phi_r(z')$$

satisfies the conditions for the validity of the theorems in §§ 4, 5.

We have, as in § 13,

$$\int_z^{z+\mu} \sum_1^n \phi_r(z)\,\phi_r(z')\,dz'$$
$$= \int_z^{z+\mu} l_1 \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}\,dz' - \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z+\mu)}{\lambda_r} + \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z)}{\lambda_r}.$$

The series $\sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z)}{\lambda_r}$ consists of parts which converge uniformly for all values of z and z', together with the part $\sum_{r=1}^{n}\left(-\frac{2}{\pi}\right)\frac{\cos rz \sin rz}{r}$, which is equivalent to $-\frac{1}{\pi}\sum_{r=1}^{n}\frac{\sin 2rz}{r}$, and this converges uniformly for all values of z in the interval $(\epsilon, \pi-\epsilon)$ of z, where ϵ is an arbitrarily small positive number. The function to which this sum converges in the interval $(\epsilon, \pi-\epsilon)$ is consequently $\frac{1}{2}\frac{dK(z, z)}{dz}$. Also

$$\lim_{n=\infty} \int_z^{z+\mu} l_1 \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}\,dz' \quad \text{or} \quad \int_z^{z+\mu} \frac{d^2}{dz'^2} K(z, z')\,dz'$$

is equal to $$K'(z, z+\mu) - K'(z, z+0),$$

the convergence to this value being uniform, since $\sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}$ converges uniformly to $K(z, z')$.

We have therefore

$$\lim_{n=\infty} \int_z^{z+\mu} F(z', z, n)\, dz' = \tfrac{1}{2} \frac{dK(z, z)}{dz} - K'(z, z+0),$$

provided z is in the interval $(\epsilon, \pi-\epsilon)$, and the convergence is uniform in this interval. Referring to the notation of § 12, we have

$$\frac{d}{dz} K(z, z) = \mathfrak{k}_1'(z)\,\mathfrak{k}_2(z) + \mathfrak{k}_2'(z)\,\mathfrak{k}_1(z) = K'(z, z+0) + K'(z, z-0);$$

also it has been shewn that

$$K'(z, z+0) - K'(z, z-0) = -1.$$

Therefore $\int_z^{z+\mu} F(z', z, n)\, dz'$ converges to the limit $\frac{1}{2}$, as n is indefinitely increased, uniformly in the interval $(\epsilon, \pi-\epsilon)$.

In a precisely similar manner it may be shewn that $\int_{z-\mu}^{z} F(z', z, n)\, dz'$ converges to $\frac{1}{2}$, uniformly in the interval $(\epsilon, \pi-\epsilon)$.

At the point $z = 0$, we have

$$\lim_{n=\infty} \int_0^{\mu} F(z', 0, n)\, dz'$$
$$= \frac{d}{d\mu} K(0, \mu) - \left[\frac{d}{d\mu} K(0, \mu)\right]_{\mu=0} - \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(\mu)}{\lambda_r} + \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(0)}{\lambda_r}.$$

The series $\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(z')}{\lambda_r}$ converges uniformly in the interval (μ_1, μ_2) of z', where $0 < \mu_1 < \mu < \mu_2$, since the series $\sum_{r=1}^{n} \frac{\sin rz'}{r}$ is uniformly convergent in that interval; therefore the series $\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(\mu)}{\lambda_r}$ converges to the value $\frac{d}{d\mu} K(0, \mu)$. Again,

$$\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(0)}{\lambda_r} = h' \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r(0)}{\lambda_r} = h'K(0, 0).$$

Therefore
$$\begin{aligned}\lim_{n=\infty} \int_0^{\mu} F(z', 0, n)\, dz' &= -\mathfrak{k}_1(0)\,\mathfrak{k}_2'(0) + h'K(0, 0)\\ &= -\mathfrak{k}_1(0)\,\mathfrak{k}_2'(0) + h'\mathfrak{k}_1(0)\,\mathfrak{k}_2(0)\\ &= -\mathfrak{k}_1(0)\,\mathfrak{k}_2'(0) + \mathfrak{k}_1'(0)\,\mathfrak{k}_2(0)\\ &= 1.\end{aligned}$$

It may be shewn, in a similar manner, that

$$\lim_{n=\infty} \int_{\pi-\mu}^{\pi} F(z', \pi, n)\, dz' = 1.$$

We have next to shew that $\int_{\mu_1}^{\mu} F(z \pm t, z, n)\, dt$ are numerically less than a fixed positive number, for all values of μ_1, such that $0 \leqslant \mu_1 < \mu$, and for all values of n; z being any point in the interval $(0, \pi)$.

The value of the integral is

$$\sum_{r=1}^{n} \left[\sqrt{\frac{2}{\pi}} \cos rz + \sin rz \frac{a(z)}{r} + \frac{a(z, r)}{r^2}\right] \left[\sqrt{\frac{2}{\pi}} \frac{\sin rz'}{r} + \frac{a(z, r)}{r^2}\right]_{z'=z+\mu_1}^{z'=z+\mu},$$

for, as before, $\int a(z) \sin rz\, dz$ may be integrated by parts, and the result has the factor $1/r$. Of this, the part

$$\frac{2}{\pi} \Sigma \frac{\cos rz \sin r(z+\mu_1)}{r} \quad \text{or} \quad \frac{1}{\pi} \Sigma \frac{\sin r\mu_1 + \sin (2z+\mu_1)}{r}$$

is numerically less than a fixed positive number, for all values of z and all values of μ_1. The remainder consists of series which are uniformly convergent, and therefore the required result holds.

It has now been verified that the conditions of validity of the theorems of §§ 4, 5 are satisfied.

15. We are now in a position to state the following general theorem, which has been established by the foregoing investigation.

Let $f(z)$ be a function, limited or unlimited, which has a Lebesgue integral in the interval $(0, \pi)$. If the normal functions which satisfy the differential equation

$$\frac{d^2U}{dz^2} + (\rho^2 - l_1)\, U = 0,$$

where l_1 has limited total fluctuation in the interval $(0, \pi)$, and where ρ has values such that the boundary conditions

$$\frac{dU}{dz} - h'U = 0, \textit{ for } z = 0, \quad \textit{and} \quad \frac{dU}{dz} + H'U = 0, \textit{ for } z = \pi,$$

are satisfied, be denoted by $\phi_n(z)$, then the series

$$\sum_{r=1}^{\infty} \phi_r(z) \int_0^{\pi} \phi_r(z') f(z')\, dz'$$

converges to the value $\frac{1}{2}\{f(z+0)+f(z-0)\}$ *at any interior point* z *of the interval* $(0, \pi)$, *at which* $f(z+0), f(z-0)$ *exist and are finite, if a neighbourhood of the point* z *exists in which the function* $f(z)$ *is of limited total fluctuation. In any interval in which* $f(z)$ *is continuous, and which is contained in the interior of an interval in which the function has limited total fluctuation, the convergence of the series to the value* $f(z)$ *is uniform. At the points* $z=0$, $z=\pi$, *the series converges to the values* $f(0+0)$, $f(\pi-0)$, *if the function is of limited total fluctuation in neighbourhoods of these points.*

In order to pass back to the functions which satisfy the equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)\,V=0, \tag{1}$$

with the boundary conditions

$$\frac{dV}{dx}-hV=0, \text{ for } x=a, \qquad \frac{dV}{dx}+HV=0, \text{ for } x=b, \tag{2}$$

we write $$\phi_n(z)=(gk)^{\frac{1}{4}}\,V_n(x);$$

then, since $$dz=\left(\frac{g}{k}\right)^{\frac{1}{2}}dx,$$

we have $$\int_a^b gV_n(x)\,V_{n'}(x)\,dx=0, \text{ for } n\neq n',$$

and $$\int_a^b g\,\{V_n(x)\}^2\,dx=1.$$

Writing $\chi(x)$ for $f(z)$, the series becomes

$$\sum_{n=1}^{\infty}(gk)^{\frac{1}{4}}\,V_n(x)\int_a^b g'\,V_n(x')\,\frac{\chi(x')}{(g'k')^{\frac{1}{4}}}\,dx'.$$

If we now write $F(x)$ for $\chi(x)(gk)^{-\frac{1}{4}}$, and remember the assumption that g and k are such that $(gk)^{-\frac{1}{4}}$ has limited total fluctuation, and is continuous in (a, b), we obtain the following theorem :—

Let $F(x)$ *be a limited or unlimited function which has a Lebesgue integral in* (a, b). *Let* $V_n(x)$ *be the function which satisfies the equation* (1), *and is such that*

$$\int_a^b g\,\{V_n(x)\}^2\,dx=1,$$

and corresponds to the value r_n of r, found so that the boundary conditions (2) *are satisfied. Then, it being assumed that $(gk)^{-\frac{1}{4}}$ has limited total fluctuation in (a, b), the series*

$$\Sigma V_n(x) \int_a^b g' V_n(x') F(x') dx'$$

converges to the value $\frac{1}{2}\{F(x+0)+F(x-0)\}$ at any interior point of (a, b) at which the functional limits have definite finite values, and which is such that the function has limited total fluctuation in some neighbourhood of the point. In any interval in which $F(x)$ is continuous, and which is contained in the interior of another interval in which it has limited total fluctuation, the convergence of the series to the value $F(x)$ is uniform. The series converges to the values $F(a+0)$, $F(b-0)$ at the points $x = a$, $x = b$, if there exist neighbourhoods of these points in which the function has limited total fluctuation.

We have not considered the cases in which h or H is infinite, or in which both are infinite. The investigation in that case is of a precisely similar character, the details being slightly different on account of the somewhat different form of the functions $\phi_n(z)$.

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16. In the preceding investigation, the differential equation (1) has no singular points in the interval (a, b). As an example of a case in which there are singular points at the ends of the interval, the case of the series

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} f(x') P_n(x') dx'$$

will be here considered. The normal functions $\sqrt{\frac{2n+1}{2}} P_n(x)$ satisfy Legendre's equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1) P_n(x) = 0.$$

Let
$$F(x', x, n) = \sum_{r=0}^{r=n} \frac{2r+1}{2} P_r(x) P_r(x');$$

then, by a well known formula of summation, we have the expression

$$F(x', x, n) = \frac{n+1}{2} \frac{P_{n+1}(x) P_n(x') - P_n(x) P_{n+1}(x')}{x-x'}.$$

We shall now verify that the conditions of the theorem of § 1 are satisfied for the interval $(-1+\epsilon, 1-\epsilon)$ of x, where ϵ is an arbitrarily chosen positive number. We cannot, in this case, apply the theorem to the interval $(-1, +1)$.

It is known that in the interval $(-1+\epsilon, 1-\epsilon)$, the value of $P_n(x)$ is given by

$$\left(\frac{2}{n\pi \sin\theta}\right)^{\frac{1}{2}} \left[\cos\left\{(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} + \frac{a(n,\theta)}{n}\right],$$

for every value of $n\ (>0)$; where $x = \cos\theta$, and $a(n,\theta)$ represents a function which is in absolute value less than some fixed number, for all values of $n\ (>0)$, and for all values of x in the interval $(-1+\epsilon, 1-\epsilon)$.

This value of $P_n(x)$ is clearly of the form $\dfrac{a(n,x)}{n^{\frac{1}{2}}}$.

If $|x-x'| \geqslant \mu$, we have

$$|F(x', x, n)| < \frac{n+1}{2\mu}\,\frac{1}{\sqrt{n\,(n+1)}}\,|a(n, x, x')| < \frac{1}{\mu}\,|a(n, x, x')|,$$

provided x, x' are in the interval $(-1+\epsilon, 1-\epsilon)$. Hence, in this interval, $|F(x', x, n)|$ is less than a fixed number, for all values of n, x', and x.

Again,

$$\int_{\alpha_1}^{\beta_1} F(x', x, n)\,dx' = \frac{n+1}{2}\left\{P_{n+1}(x)\int_{\alpha_1}^{\beta_1}\frac{P_n(x')}{x-x'}\,dx' - P_n(x)\int_{\alpha_1}^{\beta_1}\frac{P_{n+1}(x')}{x-x'}\,dx'\right\}$$

$$= \frac{n+1}{2}\left[P_{n+1}(x)\left\{\frac{1}{x-\alpha_1}\int_{\alpha_1}^{k_1}P_n(x')\,dx' + \frac{1}{x-\beta_1}\int_{k_1}^{\beta_1}P_n(x')\,dx'\right\}\right.$$

$$\left. - P_n(x)\left\{\frac{1}{x-\alpha_1}\int_{\alpha_1}^{k_2}P_{n+1}(x')\,dx' + \frac{1}{x-\beta_1}\int_{k_2}^{\beta_1}P_{n+1}(x')\,dx'\right\}\right],$$

where k_1 and k_2 are numbers such that $\alpha_1 \leqslant k_1 \leqslant \beta_1$, $\alpha_1 \leqslant k_2 \leqslant \beta_2$; and x is not interior to the interval $(\alpha_1-\mu, \beta_1+\mu)$.

If we employ the known formula

$$(2n+1)\,P_n(x') = \frac{dP_{n+1}(x')}{dx'} - \frac{dP_{n-1}(x')}{dx'},$$

we then find that

$$\int_{\alpha_1}^{\beta_1} F(x',\, x,\, n)\, dx'$$

$$= \frac{n+1}{2}\left[\frac{P_{n+1}(x)}{2n+1}\left\{\frac{1}{x-\alpha_1}[P_{n+1}(k_1)-P_{n+1}(\alpha_1)-P_{n-1}(k_1)+P_{n-1}(\alpha_1)]\right.\right.$$

$$\left.+\frac{1}{x-\beta_1}[P_{n+1}(\beta_1)-P_{n+1}(k_1)-P_{n-1}(\beta_1)+P_{n-1}(k_1)]\right\}$$

$$-\frac{P_n(x)}{2n+3}\left\{\frac{1}{x-\alpha_1}[P_{n+2}(k_2)-P_{n+2}(\alpha_1)-P_n(k_2)+P_n(\alpha_1)]\right.$$

$$\left.\left.+\frac{1}{x-\beta_1}[P_{n+2}(\beta_1)-P_{n+2}(k_2)-P_n(\beta_1)+P_n(k_2)]\right\}\right].$$

Hence we have, by using the form $\frac{a(n,\, x)}{n^{\frac{1}{2}}}$ for $P_n(x)$,

$$\left|\int_{\alpha_1}^{\beta_1} F(x',\, x,\, n)\, dx'\right| < \frac{1}{\mu n}\,|\,a(n)\,|,$$

where $(\alpha_1,\, \beta_1)$ is any interval in the interval $(-1+\epsilon,\, 1-\epsilon)$, and is not interior to the interval $(\alpha_1-\mu,\, \beta_1+\mu)$. Thus $\left|\int_{\alpha_1}^{\beta_1} F(x',\, x,\, n)\,dx'\right|$ is less than a fixed number independent of α_1, β_1; and this number converges to zero as n is indefinitely increased. It has therefore been shewn that the conditions of validity of the general convergence theorem of § 1 are satisfied for every value of $\mu > 0$.

17. The limit of $\int_{-1}^{-1+\epsilon} f(x')\, F(x',\, x,\, n)\, dx'$ will now be investigated.

It will be assumed that x is such that $x+1-\epsilon \geqslant \mu$. We have then

$$\int_{-1}^{-1+\epsilon} f(x')\, F(x',x,n)\, dx'$$

$$= \frac{n+1}{2}\,\frac{1}{x+1}\int_{-1}^{-1+\epsilon_1} f(x')\,[P_{n+1}(x)\, P_n(x')-P_n(x)\, P_{n+1}(x')]\, dx'$$

$$+\frac{n+1}{2}\,\frac{1}{x+1-\epsilon_1}\int_{-1+\epsilon_1}^{-1+\epsilon} f(x')\,[P_{n+1}(x)\, P_n(x')-P_n(x)\, P_{n+1}(x')]\,dx',$$

where ϵ_1 is a number such that $0 \leqslant \epsilon_1 \leqslant \epsilon$.

Let us now assume that $f(x')$ is monotone and limited in the interval

$(-1, -1+\epsilon)$; we have then

$$\int_{-1}^{-1+\epsilon_1} f(x')\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$
$$=f(-1+0)\int_{-1}^{-1+\epsilon_2}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$
$$+f(-1+\epsilon_1-0)\int_{-1+\epsilon_2}^{-1+\epsilon_1}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx',$$

where ϵ_2 is such that $0 \leqslant \epsilon_2 \leqslant \epsilon_1$. Now

$$(n+1)\int_{-1}^{-1+\epsilon_2} P_{n+1}(x)\,P_n(x')\,dx$$
$$=\frac{n+1}{2n+1}P_{n+1}(x)\left[P_{n+1}(-1+\epsilon_2)-P_{n-1}(-1+\epsilon_2)\right],$$

and the expression on the right-hand side is numerically less than $2\,|\,P_{n+1}(x)\,|$, which converges to zero as n is indefinitely increased, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. It may similarly be shewn that

$$(n+1)\int_{-1}^{-1+\epsilon_2} P_n(x)\,P_{n+1}(x')\,dx'$$

has the same property. A precisely similar proof establishes also that

$$(n+1)\int_{-1+\epsilon_2}^{-1+\epsilon_1}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

has the same property; therefore

$$\frac{n+1}{x+1}\int_{-1}^{-1+\epsilon_1} f(x')\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

converges to zero, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$.

Similarly also, it may be shewn that the other part of the expression for $\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$ converges uniformly to zero. Since a function with limited total fluctuation is the difference of two monotone functions, it can now be seen that, if $f(x')$ is of limited total fluctuation in the interval $(-1, -1+\epsilon)$, then $\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$ converges to zero, as n is indefinitely increased, uniformly for all values of x in the interval

$(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. Also, if $f(x')$ is of limited total fluctuation in the interval $(1-\epsilon,\ 1)$, a precisely similar proof establishes that

$$\int_{1-\epsilon}^{1} f(x')\, F(x',\ x,\ n)\, dx'$$

converges uniformly to zero, for all values of x in the same interval as before.

18. To prove that the conditions of the theorems in §§ 4, 5 are satisfied, we have

$$\int_x^1 \sum_{r=0}^{n} \frac{2r+1}{2} P_r(x)\, P_r(x')\, dx' = \tfrac{1}{2} \sum_{r=1}^{n} P_r(x)\, [P_{r-1}(x) - P_{r+1}(x)] + \tfrac{1}{2}(1-x)$$

$$= \tfrac{1}{2}[1 - P_n(x)\, P_{n+1}(x)].$$

Therefore

$$\int_x^{x+\mu} F(x',\ x,\ n)\, dx'$$

$$= \tfrac{1}{2}[1 - P_n(x)\, P_{n+1}(x)] - \int_{x+\mu}^{1-\epsilon} F(x',\ x,\ n)\, dx' - \int_{1-\epsilon}^{1} F(x',\ x,\ n)\, dx'.$$

It has been shewn in § 17 that $\int_{1-\epsilon}^{1} F(x',\ x,\ n)\, dx'$ converges uniformly to zero, as n is indefinitely increased, for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. It has been shewn in § 16, that $\int_{x+\mu}^{1-\epsilon} F(x',\ x,\ n)\, dx'$ converges uniformly to zero, for all values of x in the interval $(-1+\epsilon,\ 1-\epsilon)$, the conditions of the fundamental convergence theorem being satisfied. Also $P_n(x)\, P_{n+1}(x)$ converges uniformly to zero, for all values of x in the interval $(-1+\epsilon,\ 1-\epsilon)$. It therefore follows that $\int_x^{x+\mu} F(x',\ x,\ n)\, dx'$ converges to the value $\tfrac{1}{2}$, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$ of x. Similarly it can be shewn that $\int_{x-\mu}^{x} F(x',\ x,\ n)\, dx'$ converges to the value $\tfrac{1}{2}$, uniformly for all values of x in the same interval. We have next to shew that

$$\left| \int_{x+\mu_1}^{x+\mu} F(x',\ x,\ n)\, dx' \right|$$

is less than some fixed finite number for all values of μ_1 such that

$0 \leqslant \mu_1 \leqslant \mu$, for all values of n, and for all values of x in the interval $(-1+\epsilon,\ 1-\epsilon)$, the number μ being taken to be $< \epsilon$.

The integral

$$\int_{x+\mu_1}^{x+\mu} F(x', x, n)\, dx' \quad \text{or} \quad \sum_{r=0}^{n} \frac{2r+1}{2} P_r(x) \int_{x+\mu_1}^{x+\mu} P_r(x')\, dx'$$

is equivalent to

$$\tfrac{1}{2} \sum_{r=1}^{n} P_r(x)[P_{r+1}(x+\mu) - P_{r-1}(x+u) - P_{r+1}(x+\mu_1) + P_{r-1}(x+\mu_1)] + \tfrac{1}{2}(\mu - \mu_1).$$

Writing $x = \cos\theta$, $x+\mu = \cos\theta'$, $x+\mu_1 = \cos\theta''$, and substituting for $P_r(x)$ the value

$$\left(\frac{2}{r\pi \sin\theta}\right)^{\frac{1}{2}} \left[\cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r^2}\right],$$

with the corresponding values of $P_{r+1}(\cos\theta')$, $P_{r-1}(\cos\theta')$, $P_{r+1}(\cos\theta'')$ and $P_{r-1}(\cos\theta'')$, we obtain an aggregate of terms of which the first is

$$\Sigma \tfrac{1}{2} \left(\frac{2}{r\pi \sin\theta}\right)^{\frac{1}{2}} \left(\frac{2}{(r+1)\pi \sin\theta'}\right)^{\frac{1}{2}} \left[\cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r}\right]$$
$$\times \left[\cos\left\{(r+\tfrac{3}{2})\theta' - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r+1}\right].$$

This consists partly of series with $\dfrac{1}{r^{\frac{1}{2}}(r+1)^{\frac{3}{2}}}$ or $\dfrac{1}{r^{\frac{3}{2}}(r+1)^{\frac{1}{2}}}$ as factors of the general term, and which converge uniformly, and partly of the series

$$\Sigma \left(\frac{1}{r(r+1)\pi^2 \sin\theta \sin\theta'}\right)^{\frac{1}{2}} \cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} \cos\left\{(r+\tfrac{3}{2})\theta' - \frac{\pi}{4}\right\},$$

and this is expressible as the sum of four series which, apart from factors independent of r which are less than fixed numbers, are of the forms

$$\sum_1^n \frac{\cos r(\theta+\theta')}{\sqrt{r(r+1)}},\quad \sum_1^n \frac{\sin r(\theta+\theta')}{\sqrt{r(r+1)}},\quad \sum_1^n \frac{\cos r(\theta-\theta')}{\sqrt{r(r+1)}},\quad \sum_1^n \frac{\sin r(\theta-\theta')}{\sqrt{r(r+1)}}.$$

It is known* that these series all converge uniformly, for all values of θ and θ' such that $\theta+\theta'$ and $|\,\theta-\theta'\,|$ are in an interval interior to the interval $(0, 2\pi)$, and this condition is satisfied if x is in the interval $(-1+\epsilon,\ 1-\epsilon)$, and if $\mu < \epsilon$.

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* See *Theory of Functions of a Real Variable*, p. 729.

The part of the expression which depends on μ_1 is of the form

$$\tfrac{1}{2}\Sigma P_r(x)\left[\sqrt{\frac{2}{(r-1)\pi\sin\theta''}}\left\{\cos\left((r-\tfrac{1}{2})\theta''-\frac{\pi}{4}\right)+\frac{a(r,\theta'')}{r}\right\}\right.$$
$$\left.-\sqrt{\frac{2}{(r+1)\pi\sin\theta''}}\left\{\cos(r+\tfrac{3}{2})\theta''-\frac{\pi}{4}+\frac{a(r,\theta'')}{r}\right\}\right]$$

or
$$\Sigma P_r(x)\sqrt{\frac{2}{r\pi\sin\theta''}}\left[\sin\theta''\sin\left\{(r+\tfrac{1}{2})\theta''-\frac{\pi}{4}\right\}+\frac{a(r,\theta'')}{r}\right].$$

When the value of $P_r(x)$ is substituted, we obtain an aggregate of terms, of which the only one which requires special examination is

$$\frac{2}{\pi}\frac{1}{\sqrt{\sin\theta\sin\theta''}}\sin\theta''\sum_1^n\frac{\cos\left\{(r+\frac{1}{2})\theta-\frac{\pi}{4}\right\}\sin\left\{(r+\frac{1}{2})\theta''-\frac{\pi}{4}\right\}}{r}$$

or
$$\frac{1}{\pi}\sqrt{\frac{\sin\theta''}{\sin\theta}}\sum_1^n\frac{\sin(r+\frac{1}{2})(\theta''-\theta)-\cos(r+\frac{1}{2})(\theta''+\theta)}{r}.$$

Now $\sum_1^n\frac{\sin(r+\frac{1}{2})(\theta''-\theta)}{r}$, although it does not converge uniformly in the neighbourhood of $\theta''-\theta=0$, can easily be shewn to have a value which is numerically less than a fixed number, for all values of n, θ, θ''. The series $\sum_1^n\frac{1}{r}\cos(r+\frac{1}{2})(\theta''+\theta)$ converges uniformly in an interval of $\theta''+\theta$, which is interior to the interval $(0, 2\pi)$, and is therefore numerically less than a fixed number.

It has now been shewn that $\int_{x+\mu_1}^{x+\mu}F(x', x, n)\,dx'$ is numerically less than a fixed number independent of n, x, μ, and μ_1, provided $0\leqslant\mu_1\leqslant\mu<\epsilon$, if x is in the interior of the interval $(-1+\epsilon, 1-\epsilon)$.

That $\int_{x-\mu}^{x-\mu_1}F(x', x, n)\,dx'$ has the same property, can be proved in the same manner. It has now been shewn that the theorems of §§ 4, 5 are applicable to an interval enclosed in the interior of the interval $(-1, 1)$.

19. The investigations in §§ 16–18, are sufficient to establish the following theorem:—

Let $f(x)$ be a function which, whether limited or unlimited, has a Lebesgue integral in the interval $(-1, 1)$, and is such that in sufficiently

small neighbourhoods of the points -1, 1, *the function is of limited total fluctuation* (*à variation bornée*).

The series
$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} f(x') P_n(x')\, dx'$$
converges at any point x *interior to the interval* $(-1, 1)$ *to the value* $\frac{1}{2}\{f(x+0)+f(x-0)\}$, *if a neighbourhood of the point* x *exists in which the function is of limited total fluctuation.*

In any interval in which $f(x)$ *is continuous, and which is contained in the interior of another interval in which the function has limited total fluctuation, the convergence of the series to the value* $f(x)$ *is uniform.*

The condition that the function should be of limited total fluctuation in neighbourhoods of the points -1, 1, although sufficient, is not necessary. I propose, in a later communication, to replace this condition by a much less stringent one.

THE INFLUENCE OF VISCOSITY ON THE OSCILLATIONS OF SUPERPOSED FLUIDS

By W. J. HARRISON.

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IN this paper two problems of hydrodynamics are attacked with a view to discovering the influence of viscosity, the solutions being known for the case of non-viscous fluids.

The first is the case of two fluids of infinite depth, where it is found to a first approximation that the modulus of decay is of the order $1/\sqrt{\nu}$.

The second is the case of a fluid of finite depth superposed on a fluid of infinite depth. There are two modes; in the first the modulus of decay is of the order $1/\nu$, and in the second it is of the order $1/\sqrt{\nu}$.

In the problems dealt with in this paper the fluids are at rest, except for the wave-motion. In a subsequent paper I shall publish some results dealing with wave-motion at the surface of a stream of viscous fluid.

Waves at the Interface between Two Viscous Fluids of Infinite Depth.

1. Take the origin in the undisturbed surface of separation, and the axis of y vertically upwards.

For the lower fluid we have

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y},$$

$$v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x},$$

where $$\phi = Ae^{ky}e^{\iota kx+at},$$

$$\psi = He^{\lambda y}e^{\iota kx+at},$$

and $$\lambda^2 = k^2 + \frac{a}{\nu}.*$$

* See Lamb's *Hydrodynamics*, p. 564.

For the upper fluid
$$u' = \frac{\partial\phi'}{\partial x} + \frac{\partial\psi'}{\partial y},$$
$$v' = \frac{\partial\phi'}{\partial y} - \frac{\partial\psi'}{\partial x},$$
where
$$\phi' = A'e^{-ky}e^{ikx+at},$$
$$\psi' = H'e^{-\lambda' y}e^{ikx+at},$$
and
$$\lambda'^2 = k^2 + \frac{a}{\nu'}.$$

The kinematical conditions at the interface are
$$u = u', \qquad v = v';$$
these give
$$ikA + \lambda H = \quad ikA' - \lambda' H',$$
$$kA - ikH = -kA' - ikH';$$
since
$$u = ikAe^{ky} + \lambda He^{\lambda y},$$
$$v = kAe^{ky} - ikHe^{\lambda y},$$
$$u' = \quad ikA'e^{-ky} - \lambda' H'e^{-\lambda' y},$$
$$v' = -kA'e^{-ky} - ikH'e^{-\lambda' y}.$$

We have tacitly dropped the factor e^{ikx+at}.

The dynamical conditions are the continuity of p_{xy}, p_{yy} across the interface.

Now
$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - g\eta,$$
where η is the elevation of the interface; and therefore
$$\frac{\partial\eta}{\partial t} = v_{(y=0)};$$
therefore
$$\eta = \frac{1}{a}(kA - ikH) = \frac{1}{a}(-kA' - ikH').$$

Hence
$$\left(\frac{p_{yy}}{\rho}\right)_{y=0} = -\frac{p}{\rho}+2\nu\frac{\partial v}{\partial y}$$
$$= aA+\frac{g}{a}(+kA-ikH)+2\nu\,(k^2A-\iota k\lambda H),$$
$$\left(\frac{p'_{yy}}{\rho'}\right)_{y=0} = aA'+\frac{g}{a}(-kA'-\iota kH')+2\nu'\,(k^2A'+\iota k\lambda' H'),$$
$$\left(\frac{p_{xy}}{\rho\nu}\right)_{y=0} = \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{y=0}$$
$$= 2\iota k^2A+(\lambda^2+k^2)\,H,$$
$$\left(\frac{p'_{xy}}{\rho'\nu'}\right)_{y=0} = -2\iota k^2A'+(\lambda'^2+k^2)\,H'.$$

Hence we obtain the remaining conditions
$$\rho\left\{aA+\frac{g}{a}(kA-\iota kH)+2\nu(k^2A-\iota k\lambda H)\right\}$$
$$= \rho'\left\{aA'+\frac{g}{a}(-kA'-\iota kH')+2\nu'\,(k^2A'+\iota k\lambda' H')\right\},$$
and
$$\rho\nu\,\{2\iota k^2A+(\lambda^2+k^2)\,H\} = \rho'\nu'\,\{-2\iota k^2A'+(\lambda'^2+k^2)\,H'\}.$$

After eliminating A, A', H, H' from these four equations, we obtain the period equation
$$4k^3(\nu\rho-\nu'\rho')(k-\lambda)(k-\lambda')+4k^2a(\rho\nu-\rho'\nu')\,[\rho\,(k-\lambda')-\rho'\,(k-\lambda)]$$
$$+\rho^2(a^2+gk)(k-\lambda')+\rho'^2(a^2-gk)(k-\lambda)$$
$$-\rho\rho'[2ak^2+a^2(\lambda+\lambda')+gk\,(\lambda-\lambda')] = 0.$$

In this equation a is still implicitly contained in λ and λ'; after rationalisation the equation is found to be of the tenth degree in a.

If we take ν and ν' to be small, and include only the most important terms, we obtain
$$(\rho\lambda'+\rho'\lambda)\,[(\rho+\rho')\,a^2+gk\,(\rho-\rho')] = 0,$$
or
$$a^2 = -\frac{gk(\rho-\rho')}{\rho+\rho'}.$$

This is the known result for non-viscous fluids, as we should have expected.*

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* See § 2.

To proceed to a higher approximation we write

$$\alpha = \alpha_0 + \beta,$$

where
$$\alpha_0 = \pm i \sqrt{\frac{gk(\rho-\rho')}{\rho+\rho'}},$$

in the equation

$$4k^3(\nu\rho-\nu'\rho')\lambda\lambda + \rho^2(\alpha^2+gk)(k-\lambda')$$
$$+\rho'^2(\alpha^2-gk)(k-\lambda)-\rho\rho'[2\alpha k^2+\alpha^2(\lambda+\lambda')+gk(\lambda-\lambda')] = 0.$$

We find
$$\beta = -\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{2k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')\{\rho\sqrt{\nu}+\rho'\sqrt{\nu'}\}} \frac{1\pm i}{\sqrt{2}}.$$

We see that both the change in the velocity due to viscosity and the reciprocal of the modulus of decay depend on terms of the order $\sqrt{\nu}$ to a first approximation. When there is only one fluid the modulus of decay of the amplitude is of the order $1/\nu$, and the change in the velocity is of the order ν^2. The difference is due to this fact. When there is wave motion at the interface between two non-viscous fluids, the tangential velocities at the interface are different; in the viscous motion they must be the same. Hence, for dynamical reasons, we should expect a change of the nature obtained above. We have the result that, in general, wave motion at the interface between two fluids dies away much more rapidly than in the case of a single fluid. The difference is especially marked for great wave-lengths. Nevertheless the change in the velocity is small compared with the velocity.

To proceed to a still higher approximation, we write

$$\alpha = \alpha_0 + \beta + \gamma$$

in terms chosen suitably from the period equation.

We easily find
$$\gamma = -\frac{2k^2(\nu^2\rho^3+\nu'^2\rho'^3)}{(\rho+\rho')\{\rho\sqrt{\nu}+\rho'\sqrt{\nu'}\}^2}.$$

Hence to this order we have, as our final value for α,

$$\alpha = \pm i\left[\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{2}} - \left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{\sqrt{2}k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\right]$$
$$-\left[\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{\sqrt{2}k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})} + 2k^2\frac{\nu^2\rho^3+\nu'^2\rho'^3}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})^2}\right].$$

When $\rho'=0$, $\nu'=0$, we have

$$\alpha = \pm i\sqrt{gk} - 2\nu k^2,$$

which is known to be the result for a single fluid of infinite depth.

2. We have said above that the result, that to a first approximation the period is the same as that for the motion of non-viscous fluids of the same type, is according to expectation. This was reasoned not from the nature of the present problem, but from the known results of similar problems. An objection might be raised to this conclusion on the ground that the boundary conditions are totally different from those employed in the non-viscous motion, and are apparently contradictory if $\nu = \nu' = 0$. The answer to the latter objection is that, even if H and H' become zero, λ and λ' become infinite and the equations indeterminate. A similar objection to the first would apply to the work on page 571 of Lamb's *Hydrodynamics*, where he discusses the effect of oil on water waves, and also to the work of Basset in treating the case of a fluid of finite depth. In both these cases the boundary conditions are different, and yet to a first approximation the period is the same as for non-viscous motion of the same type. We are not questioning the physical truth of the assumption that there is no slipping at the interface, but the correctness of the result on this assumption. The final court of appeal is the analysis itself. But a physical answer may perhaps be given along the following lines. In the non-viscous motion there is a vortex sheet at the interface of strength $-2kc\beta \cos kx$ (Lamb's *Hydrodynamics*, p. 354), where β is the amplitude of the surface waves. This vortex sheet does not exist in the viscous motion. Now this difference between the two motions may be made as small as we please by sufficiently diminishing β, without at the same time affecting the average tangential velocity. Hence, unless the period of the motion of the viscous fluids is to depend on the amplitude, even when squares of the amplitude are neglected, it must be the same as the period for the non-viscous fluids, when the viscosity is neglected, dynamically but not kinematically. A more rigid formulation could be given, but in a general way probably this will suffice.

It is interesting to notice that in the work mentioned above, on the effect of oil on water waves, the modulus of decay depends on $1/\sqrt{\nu}$ as in the present case.

3. We can include the effect of capillarity at the interface by writing $g(\rho-\rho')+k^2T$ instead of $g(\rho-\rho')$ in our results. When the wave-length is small capillarity has a very great effect in causing the decay of the motion. However, when the wave-length is small our approximations are not sufficiently good, as they would, if continued, be in the form of a series of ascending powers of k. When the wave-length is small the effect of capillarity in causing decay of the motion would be more evident from the

term arising from the next approximation to that which we have already written down.

In the table given below the results are shown for the case of air over water. The c.g.s. system of units is used, and the following data: $\rho = 1$, $\rho' = \cdot 00129$; $\nu = \cdot 0109$, $\nu' = \cdot 139$, for water and air at 17° C. respectively; $T = 74$.

Wave-length.	1 cm.	10	100	1000
v_0	12·48	39·46	124·79	394·62
v_c	24·90	40·05	124·81	394·62
v	24·89	40·04	124·81	394·62
τ_0	1·162″	1′ 56·2″	3 hrs. 12′ 39·4″	321 hrs. 5′ 40″
τ	1·125″	1′ 34·1″	1 hr. 21′ 40·6″	36 hrs. 50′ 36″
τ_c	1·106″	1′ 34·0″	1 hr. 21′ 40·3″	36 hrs. 50′ 34″

v_0 is the wave-velocity in centimetres per second without viscosity and capillarity, v_c the velocity with capillarity only, v the velocity with both.

τ_0 is the modulus of decay of the water alone; τ, of water and air without capillarity; τ_c, of water and air with capillarity.

In the above table we notice the great influence of the air in damping waves of great wave-length.

Fluid of Finite Depth Superposed on a Fluid of Infinite Depth.

4. We suppose the upper fluid to have a free upper surface.

For the lower fluid we assume

$$\phi = Ae^{ky}e^{\iota kx+at},$$

$$\psi = iHe^{\lambda y}e^{\iota kx+at},$$

where

$$\lambda^2 = k^2 + \frac{a}{\nu}.$$

For the upper fluid we assume

$$\phi' = (B \cosh ky + C \sinh ky)\, e^{\iota kx+at},$$

$$\psi' = i\,(K \cosh \lambda' y + L \sinh \lambda' y)\, e^{\iota kx+at},$$

where

$$\lambda'^2 = k^2 + \frac{a}{\nu'}.$$

From these we derive the velocities by the formulæ

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y},$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}.$$

The kinematical conditions are the continuity of the component velocities across the interface. The dynamical conditions are the continuity of pressures and tractions across the interface and the free surface. Writing these down, and eliminating the constants, we obtain the period equation

$$\begin{vmatrix} P, & p, & -(\lambda'^2+k^2)\rho'\nu', & -\frac{gk}{a}\rho, & 0, & -1 \\ Q, & q, & 0, & -2\nu'\rho'k\lambda', & -\lambda', & 0 \\ R, & r, & 0, & -\rho'a-2\nu'k^2\rho', & -k, & 0 \\ S, & s, & -2k^2\rho'\nu', & -\frac{gk}{a}\rho', & 0, & -1 \\ 0, & 0, & 2k^2\rho\nu, & \rho\left(a+2\nu k^2+\frac{gk}{a}\right), & k, & 1 \\ 0, & 0, & (\lambda^2+k^2)\rho\nu, & \frac{gk}{a}\rho+2\nu\rho k\lambda, & \lambda, & 1 \end{vmatrix} = 0,$$

where

$$P = (\lambda'^2+k^2)\cosh\lambda' h,$$

$$Q = (\lambda'^2+k^2)\sinh\lambda' h,$$

$$R = 2k^2 \sinh kh,$$

$$S = 2k^2 \cosh kh,$$

$$p = \frac{kg}{a}\cosh\lambda' h + 2\nu' k\lambda' \sinh\lambda' h,$$

$$q = \frac{kg}{a}\sinh\lambda' h + 2\nu' k\lambda' \cosh\lambda' h,$$

$$r = a\cosh kh + \frac{gk}{a}\sinh kh + 2\nu' k^2 \cosh kh,$$

$$s = a\sinh kh + \frac{gk}{a}\cosh kh + 2\nu' k^2 \sinh kh,$$

and h is the depth of the upper fluid.

When the viscosity is small the terms of greatest importance are those of the type $\lambda'^3 \cosh \lambda' h$. Including only terms of this order, in which we put $\tanh \lambda' h = 1$, we have the period equation

$$a^4 [\rho \cosh kh + \rho' \sinh kh] + a^2 gk\rho [\cosh kh + \sinh kh] \\ + g^2 k^2 (\rho - \rho') \sinh kh = 0,$$

as in the absence of viscosity.

Thus there are two modes, for one

$$a^2 + gk = 0,$$

and for the other

$$a^2 (\rho \cosh kh + \rho' \sinh kh) + gk(\rho - \rho') \sinh kh = 0.$$

In the first mode the tangential velocity is continuous across the interface. Hence in this mode we should expect that the first approximation to the dissipation terms in a would be of the order ν, and that the change in the velocity of propagation would be of the order ν^2; in the second mode we should expect both of these approximations to be of the order $\sqrt{\nu}$. This will be seen to be the case.

The terms of next importance are those of the type $\lambda'^2 \cosh \lambda' h$. If we put $a^2 + gk = 0$ in these, we find them vanish identically. Hence to obtain the next approximation to the first mode we have to take the terms of order $\lambda' \cosh \lambda' h$. We obtain

$$a = \pm i \sqrt{gk} - 2k^2 \frac{\rho\nu \cosh kh + 2(\rho'\nu' - \rho\nu) \sinh kh}{\rho \cosh kh + (2\rho' - \rho) \sinh kh}.$$

We notice that when $\rho = \rho'$, $\nu = \nu'$, then

$$a = \pm i \sqrt{gk} - 2k^2 \nu,$$

as is known.

Proceeding to the next approximation in the case of the second mode, we obtain

$$a = a_0 + \frac{k\sqrt{\nu\nu'}(\rho - \rho')}{(\rho\sqrt{\nu} + \rho'\sqrt{\nu'})\, a_0^{\frac{1}{2}}}$$

$$\times \frac{\{a^4(\rho - \rho') \sinh kh + gka^2\rho(\sinh kh + \cosh kh) + g^2k^2(\rho \cosh kh + \rho' \sinh kh)\}}{\{4a_0^3(\rho \cosh kh + \rho' \sinh kh) + 2gk\rho a_0 (\cosh kh + \sinh kh)\}},$$

where $$a_0^2 = -\frac{gk(\rho - \rho') \sinh kh}{\rho \cosh kh + \rho' \sinh kh}.$$

Hence
$$a = \pm i\sqrt{\frac{gk(\rho-\rho')\sinh kh}{\rho\cosh kh+\rho'\sinh kh}}$$
$$\mp i\frac{k\sqrt{\nu\nu'}\{gk(\rho-\rho')\}^{\frac{1}{4}}}{\sqrt{2}\,(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\left[\frac{\rho\cosh kh+\rho'\sinh kh}{\sinh kh}\right]^{\frac{3}{4}}\times P$$
$$-\frac{k\sqrt{\nu\nu'}\{gk(\rho-\rho')\}^{\frac{1}{4}}}{\sqrt{2}\,(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\left[\frac{\rho\cosh kh+\rho'\sinh kh}{\sinh kh}\right]^{\frac{3}{4}}\times P,$$
where
$$P = \frac{[\rho^3(\cosh kh-\sinh kh)+4\rho\rho'\sinh kh\,\{\rho+\rho'(\sinh^2 kh+\cosh^2 kh)\}]}{[-4(\rho-\rho')\sinh kh+2\rho(\cosh kh+\sinh kh)]\,(\rho\cosh kh+\rho'\sinh kh)^2}.$$

When h is infinite, we obtain our former results.

5. In the first mode when kh is small, *i.e.*, when the wave-length is large compared with the depth of the upper fluid, we have
$$a = \pm i\sqrt{gk}-2k^2\nu.$$
The modulus of decay is thus the same as that of the lower fluid alone.

When kh is large, $\quad a = \pm i\sqrt{gk}-2k^2\nu'.$

The modulus of decay is thus the same as that of the upper fluid alone.

In the second mode, when kh is small, the modulus of decay depends on $\sinh^{\frac{3}{4}} kh/k^{\frac{1}{4}}$, or on $h^{\frac{3}{4}}/k^{\frac{1}{2}}$; and therefore it increases much less rapidly with the wave-length than in general.

When kh is large we obtain our former results for two fluids of infinite depth.

6. In the second mode the upper surface will in general be disturbed less than the common interface, and the waves set up by any disturbance will be of this type to a predominating extent, particularly if the difference between ρ and ρ' be small. Such waves as these are those referred to by Ekman in the *Scientific Results of the Norwegian North Polar Expedition.* He remarks that a ship moving in the Norwegian Fiords experiences great resistance owing to considerable waves being set up at the common interface of the layer of fresh water and the sea-water. Such waves would be very quickly damped, and would therefore drain a great amount of energy from the ship.

We append some numerical results illustrating the case of fresh water

over sea-water of infinite depth. We take $\rho'/\rho = \frac{35}{36}$, and $\nu = \nu'$. The fact that we take $\nu = \nu'$ makes no important difference.

Wave-length.	$\frac{1}{10}$	1	10	100	1000 cms.
τ	·013″	1·27″	2′ 7″	3 hrs. 31′ 2″	351 hrs. 43′ 36″
τ_∞	·083″	1·5″	26″	7′ 46″	2 hrs. 18′ 15″
τ_1	·083″	1·4″	16″	1′ 18″	6′ 28″
τ_{10}	·083″	1·5″	26″	4′ 48″	23′ 15″
τ_{100}	·083″	1·5″	26″	7′ 45″	1 hr. 25′ 12″

τ is the modulus of decay for the first mode,

τ_∞ that for the second mode when kh is large,

τ_1 ,, ,, ,, $h = 1$ cm.,

τ_{10} ,, ,, ,, $h = 10$ cms.,

τ_{100} ,, ,, ,, $h = 100$ cms.

The very rapid decay when kh is small is very striking, even for the large wave-lengths.

ON THE ORDERING OF THE TERMS OF POLARS AND TRANSVECTANTS OF BINARY FORMS

By L. ISSERLIS.

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Abstract.

1. The reduction of transvectants depends on the possibility of inserting between any two terms of a transvectant a series of others such that any two consecutive terms possess the property which is described technically as "adjacence." It is asserted without proof that this is possible by Gordan (*Vorlesungen über Invariantentheorie*, Zweiter Band, § 42, S. 44), by Clebsch (*Binäre Formen*, § 53, S. 185), and by Grace and Young (*Algebra of Invariants*, Art. 50, p. 51). In endeavouring to prove this possibility I have succeeded in giving a method of arranging the terms of any polar or transvectant in a single series, such that any two consecutive terms are in the technical sense "adjacent." The results, so far as I can express them in the space at my disposal, are as follows.

In the following, numerical coefficients are immaterial and are omitted.

2. *To Order the Terms of the Polar* $P = \left(y\,\dfrac{\partial}{\partial x}\right)^r (a_{1_x}^{n_1}\, a_{2_x}^{n_2} \ldots a_{p_x}^{n_p})$.

Let D_s denote an operator which polarizes powers of a_{s_x} only, with regard to y, so that

$$\left(y\,\frac{\partial}{\partial x}\right)^r = (D_1+D_2+\ldots+D_p)^r.$$

Then the ordered development of $(D_1+D_2)^r$ is

$$D_1^r,\; D_1^{r-1}D_2,\; D_1^{r-2}D_2^2,\; \ldots,\; D_2^r,$$

and that of $(D_2+D_1)^r$ the same written in reverse order. $(D_1+D_2+D_3)^r$ is ordered as follows: in the ordered development of $(D_1+D_2)^r$, in the first term in which D_2 occurs replace it by (D_2+D_3); in the second term in

which it occurs, by (D_3+D_2); in the third by D_2+D_3, and so on. Similarly $(D_1+D_2+\ldots+D_p)^r$ is obtained from $(D_1+D_2+\ldots+D_{p-1})^r$, by replacing D_{p-1} in the first term in which it occurs by $D_{p-1}+D_p$, and in the second term in which it occurs by D_p+D_{p-1}.

3. *On the Ordering of a Mixed Polar.*

Let the mixed polar be

$$\left(y\frac{\partial}{\partial x}\right)^{r_\kappa}_{y=y_\kappa}\left(y\frac{\partial}{\partial x}\right)^{r_{\kappa-1}}_{y=y_{\kappa-1}}\cdots\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1} b_{1_x}^{m_1} b_{2_x}^{m_2}\ldots b_{q_x}^{m_q},$$

and let $A_1+A_2+A_3+\ldots$ be the result of ordering

$$\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1} b_{1_x}^{m_1}\ldots b_{q_x}^{m_q} \text{ (by § 2)};$$

then, putting

$$\left(y\frac{\partial}{\partial x}\right)^{r_2} = (D_1+D_2+\ldots+D_q)^{r_2} = D_1^{r_2}+\ldots+D_t^{r_2} \text{ say,}$$

the mixed polar $$\left(y\frac{\partial}{\partial x}\right)^{r_2}\left(y\frac{\partial}{\partial x}\right)^{r_1} b_{1_x}^{m_1}\ldots b_{q_x}^{m_q}$$

is ordered when expanded in the form

$$(D_1^{r_2}+\ldots+D_t^{r_2})A_1+(D_t^{r_2}+\ldots+D_1^{r_2})A_2+\ldots = B_1+B_2+B_3+\ldots \text{ say,}$$

and the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_3}_{y=y_3}\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1} b_{1_x}^{m_1}\ldots b_{q_x}^{m_q}$$

is obtained from $B_1+B_2+B_3+\ldots$ in a manner similar to that in which $B_1+B_2+\ldots$ is obtained from $A_1+A_2+\ldots$. A similar method applies to the general case.

4. *The Ordering of the Terms of a Transvectant.*

Symbolically $$T = (a_{1_x}^{n_1} a_{2_x}^{n_2}\ldots a_{p_x}^{n_p},\ b_{1_x}^{m_1} b_{2_x}^{m_2}\ldots b_{q_x}^{m_q})^r$$

may be written

$$T = \Sigma a_{1_x}^{n_1-\lambda_1}\ldots a_{p}^{n_p-\lambda_p}\left(y_1\frac{\partial}{\partial x}\right)^{\lambda_1}\cdots\left(y_p\frac{\partial}{\partial x}\right)^{\lambda_p} b_{1_x}^{m_1}\ldots b_{q_x}^{m_q},$$

$$\left.\begin{aligned} y_{s_1} &= -a_{s_2} \\ y_{s_2} &= a_{s_1}\end{aligned}\right\},\quad \Sigma\lambda = r,$$

where the factors preceding the polarizing operators are the factors independent of y of a term of the polar $(a_{1_x}^{n_1} \dots a_{p_x}^{n_p})_{y^r}$. Let $D_{\kappa s}$ be an operator which polarizes b_{s_x} only, with regard to y_κ; then we may write

$$T = \Sigma a_{1_x}^{n_1-\lambda_1} \dots a_{p_x}^{n_p-\lambda_p} (D_{11}+D_{12}+\dots+D_{1q})^{\lambda_1} \text{ operating on}$$
$$(D_{21}+D_{22}+\dots+D_{2q})^{\lambda_2} \text{ operating on}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$(D_{p1}+D_{p2}+\dots+D_{pq})^{\lambda_p} b_{1_x}^{m_1} \dots b_{q_x}^{m_q}.$$

The terms of the transvectant will be ordered if we expand the product of operators in the above so that (i.) it starts with $D_{11}^{\lambda_1} D_{21}^{\lambda_2} \dots D_{p1}^{\lambda_p}$, (ii.) it ends with $D_{1q}^{\lambda_1} D_{2q}^{\lambda_2} \dots D_{pq}^{\lambda_q}$, and (iii.) any two consecutive terms are either of the form ΔD_{uv}, ΔD_{uw}, or of the form $\Delta D_{uv} D_{tw}$, $\Delta D_{uw} D_{tv}$, where Δ is the same product of D's in both. I first show how to satisfy these conditions when developing

$$A = (D_{11}+D_{12})^{\lambda_1} (D_{21}+D_{22})^{\lambda_2} \dots (D_{p1}+D_{p2})^{\lambda_p},$$

and then $B = (D_{11}+D_{12}+D_{13})^{\lambda_1} (D_{21}+D_{22}+D_{23})^{\lambda_2} \dots (D_{p1}+D_{p2}+D_{p3})^{\lambda_p}$

is deduced by replacing in the last term of A each term of the form $D_{\kappa 2}$ by $D_{\kappa 2}+D_{\kappa 3}$, but in the term before by $D_{\kappa 3}+D_{\kappa 2}$, and in the term before that by $D_{\kappa 2}+D_{\kappa 3}$, and so on. Similarly

$$(D_{11}+\dots+D_{14})^{\lambda_1} \dots (D_{p1}+\dots+D_{p4})^{\lambda_p}$$

is deduced from B, and so on.

The development of A to satisfy the conditions (i.), (ii.), (iii.) is simply effected when all the λ's or all the λ's but one are even, but is a little complicated when more than one of the λ's are even. A method is given in the paper by which the development is effected in all cases. It depends ultimately on the proper arrangements of products of the form

$$(a_1+a_2)(b_1+b_2),\ (a_1+a_2)(b_1+b_2)(c_1+c_2),\ \dots .$$

The first of these is arranged by means of the scheme

$$a_1 \begin{cases} b_1 \\ b_2 \end{cases}$$
$$a_2 \begin{cases} b_1 \\ b_2 \end{cases}$$

which means $a_1 b_1$, $a_1 b_2$, $a_2 b_1$, $a_2 b_2$.

The second is arranged by means of the scheme

$$a_1 \begin{cases} b_1 \begin{cases} c_1 \\ c_2 \end{cases} \\ b_2 \begin{cases} c_2 \\ c_1 \end{cases} \end{cases}$$

$$a_2 \begin{cases} b_1 \begin{cases} c_1 \\ c_2 \end{cases} \\ b_2 \begin{cases} c_1 \\ c_2 \end{cases} \end{cases}$$

which means $a_1 b_1 c_1$, $a_1 b_1 c_2$, $a_1 b_2 c_2$, $a_1 b_2 c_1$, $a_2 b_1 c_1$, $a_2 b_1 c_2$, $a_2 b_2 c_1$, $a_2 b_2 c_2$.

The rule for constructing the schemes is that in any vertical column, the pair of letters which occur have their order reversed after each entry, except that the last two entries are alike.

THE MULTIPLICATION OF CONDITIONALLY CONVERGENT SERIES

By G. H. HARDY.

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1. Although much has been written concerning the multiplication of series according to Cauchy's rule, the last word has not yet been said upon the subject, and a number of interesting questions connected with it remain unanswered. In this paper I prove a few simple theorems which I believe to be new. In § 4 I prove that a sufficient condition for the multiplication of two convergent series Σa_n, Σb_n is that na_n and nb_n should each tend to zero as n tends to infinity. In § 8 I generalise this result by showing (by the aid of slightly more elaborate analysis) that it is sufficient that the absolute values of na_n and nb_n should have an upper limit. In § 7 I establish a generalisation of a somewhat different kind, showing that the conditions

$$n\phi(n)\, a_n \to 0, \qquad \frac{nb_n}{\phi(n)} \to 0,$$

where $\phi(n)$ is one of a general class of functions of which $\log n$ is typical, are sufficient.

I have also (§ 13) stated and indicated the proofs of some corresponding theorems for integrals, and I have added (§§ 12, 10) a generalisation of Mertens' theorem and new proofs of some results of Pringsheim's concerning series of a special form. I have thought it worth while to add this last section, although it contains no new results, because the class of series to which it refers is the most natural and important of all, and because, so far as I know, the results have never yet been proved with anything like the simplicity which is desirable and attainable.

I wish to state explicitly that I have not proved, either positively or negatively, but particularly negatively, as much as I think ought to be capable of proof. In § 11 I indicate some questions which seem to me of considerable interest, but which I am at present unable to answer.

2. I shall adopt the notation of Mr. Bromwich's *Infinite Series* (pp. 82 *et seq.*); *i.e.*, I shall denote by A, B, C the series

$$a_1+a_2+a_3+\ldots,\quad b_1+b_2+b_3+\ldots,\quad c_1+c_2+c_3+\ldots,$$

where
$$c_n = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1.$$

I shall also use the letters A, B, C in equations or inequalities to denote the *sums* of the series, when they are convergent; and I shall denote the sums of the first n terms of the series by A_n, B_n, C_n, so that, *e.g.*,

$$A_n = a_1+a_2+\ldots+a_n.$$

3. The classical results in connection with the multiplication of series are the following:—

(1) **Abel's Theorem.**—*If all three series are convergent, then* $C = AB$.

(2) **Cauchy's Theorem.**—*If* A *and* B *are absolutely convergent, then* C *is absolutely convergent.*

(3) **Mertens' Theorem.**—*If* A *is absolutely and* B *conditionally convergent, then* C *is convergent.*

In addition to these results, a number of theorems have been proved by Pringsheim, Voss, and Cajori.* These relate to the case in which A and B are conditionally convergent, but one at least becomes absolutely convergent when its terms are associated in certain groups, the number in each group being less than some fixed number. I shall return to some of the simplest and most important of these theorems later on.

4. THEOREM A.—*If* A *and* B *are convergent, and*

$$na_n \to 0, \qquad nb_n \to 0$$

as $n \to \infty$, *then* C *is convergent.*

The proof is very simple. For

$$C_n = a_1B_n+a_2B_{n-1}+\ldots+a_nB_1 = a_1B_n+a_2B_{n-1}+\ldots+a_NB_{n+1-N}$$
$$+a_{N+1}B_{n-N}+a_{N+2}B_{n-N-1}+\ldots+a_nB_1.$$

Applying Abel's partial summation lemma to the first line, we obtain

$$C_n - A_N B_{n+1-N} = A_1b_n+A_2b_{n-1}+\ldots+A_{N-1}b_{n+2-N}$$
$$+B_1a_n+B_2a_{n-1}+\ldots+B_{n-N}a_{N+1}.$$

* For references, see Bromwich, *Infinite Series*, p. 87.

If N is such a function of n that N and $n-N$ tend to infinity with n, then

$$A_N B_{n+1-N} \to AB. \tag{1}$$

This is certainly the case if $Gn < N < Hn$, where G and H are constants, and $0 < G < H < 1$. But then

$$| A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < K(N-1)\beta,$$
$$| B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} | < K(n-N)\alpha,$$

where K is a constant, and α and β are the greatest of the moduli of

$$a_{N+1},\ a_{N+2},\ \ldots,\ a_n \qquad b_{n+2-N},\ b_{n+3-N},\ \ldots,\ b_n$$

respectively. In virtue of the restriction imposed upon N, we have

$$N-1 < \lambda n, \qquad n-N < \lambda n,$$

where λ is a constant. And we can choose n_0 so that

$$| n\alpha | < \epsilon/\lambda K, \qquad | n\beta | < \epsilon/\lambda K,$$

for $n \geqslant n_0$. It follows that for $n \geqslant n_0$, we have

$$\begin{cases} | A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < \epsilon, \\ | B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} \ | < \epsilon, \end{cases} \tag{2}$$

and from (1) and (2) the conclusion follows.

5. This theorem is not of very wide application, the range of series which are only conditionally convergent, and yet satisfy the condition $na_n \to 0$, being of course comparatively narrow. The simplest of such series are those of the type

$$\frac{1}{\phi(1)} - \frac{1}{2\phi(2)} + \frac{1}{3\phi(3)} - \ldots,$$

where $\phi(n)$ is any function which tends steadily to infinity with n, but (like $\log n$ or $\log n \log\log n$) so slowly that the series is not absolutely convergent. Or, again, the series

$$\Sigma n^{-1-ai} \quad (a \gtrless 0)$$

is known* to oscillate finitely, so that, if $\phi(n)$ is any function which tends steadily to infinity with n, the series

$$\Sigma \frac{1}{n^{1+ai}\,\phi(n)}, \quad \Sigma \frac{\cos(a \log n)}{n\phi(n)}, \quad \Sigma \frac{\sin(a \log n)}{n\phi(n)}$$

are convergent. This result may be extended (as in Mr. Bromwich's paper printed earlier in this volume) to such series as

$$\Sigma \frac{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(a \log_{k+1} n)}{n \log n \log_2 n \ldots \log_k n\, \phi(n)},$$

where $\log_2 n = \log\log n, \quad \log_3 n = \log\log_2 n, \quad \ldots,$

and generally to series of the type

$$\Sigma \frac{f'(n)}{\phi(n)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} \{f(n)\},$$

* See Landau, *Crelle*, Bd. cxxv., pp. 105–7, for references in connection with this series.

where $f(n)$ is a function of n such that $f(n)$, $f'(n)$ are monotonic, $f(n) \to \infty$, $f'(n) \to 0$, and

$$\Sigma \{f'(n)\}^2$$

is convergent. Another interesting type is

$$\Sigma \frac{\Gamma(i+n)}{\Gamma(1+n)} \frac{1}{\phi(n)}.$$

The theorem, however, seems to me of some interest in spite of its comparatively narrow range of applicability, on account of the simplicity of the conditions and the fact that no use whatever is made of the notion of absolute convergence. All of Pringsheim's theorems depend on the possibility of securing absolute convergence in one at least of the series A, B by the insertion of brackets in some prescribed manner.

6. Series for which $na_n \to 0$ have another interesting property first discovered by Tauber.* The converse of Abel's theorem on the continuity of power series holds for them—that is to say, the convergence of Σa_n may be *deduced* from the equations

$$\lim na_n = 0, \qquad \lim_{x \to 1} \Sigma a_n x^n = A.$$

The fact that the simplest proof of Abel's theorem on the multiplication of series is derived from his theorem on the continuity of power series suggests that Theorem A might be deduced from Tauber's theorem. But this proves not to be the case, for the equations

$$\lim na_n = 0, \qquad \lim nb_n = 0,$$

do *not* involve

$$\lim nc_n = 0.$$

Suppose, *e.g.*, that

$$a_n = b_n = \frac{(-1)^n}{(n+1)\sqrt{\{\log(n+1)\}}},$$

so that

$$c_n = (-1)^n \sum_1^n \frac{1}{(r+1)(n+2-r)\sqrt{\{\log(r+1)\log(n+2-r)\}}}.$$

It is easy to see that, if n is odd, the value of r which makes $\log(r+1)\log(n+2-r)$ greatest is $r = \frac{1}{2}(n+1)$, so that

$$c_n > \frac{1}{\log\{\frac{1}{2}(n+3)\}} \sum_1^n \frac{1}{(r+1)(n+2-r)} = \frac{2}{(n+3)\log\{\frac{1}{2}(n+3)\}} \sum_2^{n+1} \frac{1}{r} > \frac{K}{n},$$

and nc_n certainly does not tend to zero. In fact this line of argument suffices to prove that C is convergent only when the more stringent conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

are satisfied.

7. Theorem A may be generalised as follows. It is easy to verify that if $\psi(n)$ is any function of the form

(1) $$(\log n)^\alpha (\log\log n)^\beta (\log\log\log n)^\gamma \ldots$$

which tends to infinity with n, then

$$\lim_{n\to\infty} \frac{\psi\left\{\dfrac{n}{\psi(n)}\right\}}{\psi(n)} = 1;$$

we may indeed replace the $\psi(n)$ which occurs inside the curly bracket by any other function of n of the same type as $\psi(n)$.

* For references see Bromwich, *Infinite Series*, p. 251.

THEOREM B.—*If A and B are convergent, and*

$$n\psi(n)\,a_n \to 0, \qquad \frac{nb_n}{\psi(n)} \to 0,$$

where $\psi(n)$ *is any function of n of the form* (1), *then C is convergent.*

We have, as in § 1 above,

$$C_n - A_N B_{n+1-N} = A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N}$$
$$+ B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1},$$

and
$$|C_n - A_N B_{n+1-N}| < K\{(n-N)\alpha + (N-1)\beta\},$$

where α and β are the greatest of the moduli of $a_{N+1}, a_{N+2}, \ldots, a_n$ and $b_{n+2-N}, b_{n+3-N}, \ldots, b_n$ respectively.

We choose N to be of the same order of greatness as $n/\psi(n)$. Then, given ϵ, we can choose n so that

$$\alpha < \frac{\epsilon}{(N+1)\,\psi(N+1)}, \qquad \beta < \frac{\epsilon\psi(n+2-N)}{n+2-N},$$

and so
$$|C_n - A_N B_{n+1-N}| < K\epsilon\left\{\frac{n}{N\psi(N)} + \frac{N\psi(n)}{n}\right\}$$

$$< K\epsilon\left[1 + \frac{\psi(n)}{\psi\left\{\frac{n}{\psi(n)}\right\}}\right] < K\epsilon.*$$

From this the theorem follows. The simplest and most interesting case is that in which

$$n(\log n)^a\,a_n \to 0, \qquad \frac{nb_n}{(\log n)^a} \to 0,$$

where $0 \leqslant a \leqslant 1$ (if $a > 1$ the first series is absolutely convergent and the result is a mere corollary from Mertens' theorem).

8. Another generalisation of Theorem A, in a somewhat different direction, is the following:—

THEOREM C.—*If A and B are convergent, and*

$$|na_n| < K, \qquad |nb_n| < K,$$

for all values of n, then C is convergent.

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* Of course K is not the same constant in all these inequalities.

It is known* that

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n} = AB.$$

It is also known† that, if a series Σc_n is such that $(C_1+C_2+\ldots+C_n)/n$ has a limit as $n \to \infty$, then the necessary and sufficient condition for the convergence of the series is

$$\lim \frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = 0:$$

this indeed follows at once from the identity

$$\frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = \frac{n+1}{n} C_n - \frac{C_1+C_2+\ldots+C_n}{n}.$$

Let us denote the sums

$$a_1+2a_2+\ldots+na_n, \quad b_1+2b_2+\ldots+nb_n, \quad c_1+2c_2+\ldots+nc_n$$

by $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_n$ respectively. It is easy to verify the identity

$$\begin{aligned}\bar{C}_n+C_n = \; & a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1 \\ & +b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1.\end{aligned}$$

Also $$C_1+C_2+\ldots+C_n = n(AB+\gamma_n),$$

where $\gamma_n \to 0$ as $n \to \infty$, and so

$$\frac{C_n}{n} = \frac{AB}{n} + \gamma_n - \frac{n-1}{n}\gamma_{n-1} \to 0.$$

It follows that the necessary and sufficient condition for the convergence of C is that

(1) $$(X+Y)/n \to 0,$$

where

(2) $$\begin{cases} X = a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1, \\ Y = b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1. \end{cases}$$

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* Bromwich, *Infinite Series*, p. 83.

† This result is due to Tauber and Pringsheim. See Bromwich, *Infinite Series*, p. 251, for references.

This condition can be written in a variety of different forms. Thus, applying Abel's lemma to X and Y, we obtain

$$(3) \qquad \begin{cases} X = b_1 A_n + 2b_2 A_{n-1} + \ldots + nb_n A_1, \\ Y = a_1 B_n + 2a_2 B_{n-1} + \ldots + na_n B_1. \end{cases}$$

Further, if we put $\quad A_n = A + \epsilon_n, \quad B_n = B + \eta_n,$

so that $\epsilon_n \to 0$, $\eta_n \to 0$, we see that

$$X = A\bar{B}_n + b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \ldots + nb_n \epsilon_1,$$

$$Y = B\bar{A}_n + a_1 \eta_n + 2a_2 \eta_{n-1} + \ldots + na_n \eta_1.$$

Since $\bar{A}_n/n$, $\bar{B}_n/n$ each tend to zero, we see that the necessary and sufficient condition for the convergence of C is that

$$(4) \qquad (X' + Y')/n \to 0,$$

where

$$(5) \quad X' = b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \ldots + nb_n \epsilon_1, \quad Y' = a_1 \eta_n + 2a_2 \eta_{n-1} + \ldots + na_n \eta_1.$$

But, if $|na_n| < K$ and $|nb_n| < K$, it is clear that

$$\left| \frac{X'}{n} \right| < K \frac{|\epsilon_1| + |\epsilon_2| + \ldots + |\epsilon_n|}{n} \to 0,$$

and similarly $\qquad |Y'/n| \to 0.$

Hence the theorem is established.

9. The simplest example of the use of this theorem is obtained by applying it to the series

$$\pm \frac{1}{a} \pm \frac{1}{a+b} \pm \frac{1}{a+2b} \pm \ldots \pm \frac{1}{a+nb} \pm \ldots.$$

We see that *any* two series of this type, whatever be the law of arrangements of the signs, may be multiplied together, provided only they are convergent. A simple example is obtained by squaring the series

$$(1) \qquad 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} - \ldots,$$

in which the number of terms in each group of signs increases by one at each step. That the series is convergent is easily proved by observing that if we subtract from it the series

$$(1) \qquad 1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{7} - \ldots$$

we obtain an absolutely convergent series, and that the series (2) is convergent and equal to

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(\nu+1)}{\frac{1}{2}\nu(\nu+1)+1}.$$

The corresponding series in which the numbers of terms in the groups are 1^k, 2^k, 3^k, ..., where k is any positive integer, is also convergent. On the other hand, if the numbers are k, k^2, k^3, ..., the series oscillates, behaving very much like the oscillatory series

$$\sum \frac{\cos(a \log n)}{n}, \quad \sum \frac{\sin(a \log n)}{n}.$$

10. It will be convenient to give at this stage the simple proof of some of Pringsheim's results to which I alluded in § 1. The most important case, and the only one which I shall consider here, is that in which

$$a_n = (-1)^{n-1} a_n, \quad b_n = (-1)^{n-1} \beta_n,$$

where a_n and β_n are positive and decreasing. The generalisations of Cajori are rather artificial, and it seems to me worth while to establish the really important results in as simple a way as possible; and Pringsheim's own proofs are far from being the simplest possible.*

Pringsheim's results may be stated thus: *if a_n, β_n tend steadily to zero, we have the following alternative sets of conditions for the multiplication of*

$$\Sigma(-1)^{n-1} a_n, \quad \Sigma(-1)^{n-1} \beta_n,$$

by Cauchy's rule :—

(1) *it is* **necessary and sufficient** *that*

$$\gamma_n = |c_n| = a_1\beta_n + a_2\beta_{n-1} + \ldots + a_n\beta_1 \to 0;$$

(2) *it is* **necessary and sufficient** *that*

$$(a_1+a_2+\ldots+a_n)\beta_n \to 0, \quad (\beta_1+\beta_2+\ldots+\beta_n)a_n \to 0;$$

(3) *it is* **sufficient but not necessary** *that*

$$\Sigma a_n \beta_n$$

should be convergent;

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* A simpler proof of one of them is given by Mr. Bromwich, *Infinite Series*, pp. 86, 87. Even this proof does not seem to me as simple as it may be made.

(4) *it is* **necessary but not sufficient** *that*

$$\Sigma(a_n\beta_n)^{1+s}$$

should be convergent for any positive value of s.

These results may be proved as follows. We observe first that, if

$$A_n = A+(-1)^n\rho_n, \qquad B_n = B+(-1)^n\sigma_n,$$

we have $$0<\rho_n<a_{n+1}, \qquad 0<\sigma_n<\beta_{n+1}.$$

Also $$C_n = a_1B_n+a_2B_{n-1}+\ldots+a_nB_1,$$

$$(-1)^n(C_n-A_nB) = a_1\sigma_n-a_2\sigma_{n-1}+\ldots+(-1)^{n-1}a_n\sigma_1,$$

and so $|C_n-A_nB|<a_1\beta_{n+1}+a_2\beta_n+\ldots+a_n\beta_2=\gamma_{n+1}-a_{n+1}\beta_1$.

From this it follows that the condition $\gamma_n\to 0$ is *sufficient* to ensure $C_n\to AB$, and that the condition is *necessary* is obvious. This establishes Pringsheim's theorem (1).

Again $\gamma_n = a_1\beta_n+a_2\beta_{n-1}+\ldots+a_n\beta_1>(a_1+\ldots+a_n)\beta_n$,

and similarly $$\gamma_n>(\beta_1+\ldots+\beta_n)a_n.$$

Hence the conditions (2) are *necessary.*

Also, if $\nu=\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, according as n is odd or even, we have

$$\gamma_n = a_1\beta_n+a_2\beta_{n-1}+\ldots+a_n\beta_1<(a_1+a_2+\ldots+a_\nu)\beta_{n+1-\nu}$$
$$+(\beta_1+\beta_2+\ldots+\beta_{n-\nu})a_{\nu+1},$$

and from this it follows that the conditions (2) are *sufficient.*

Finally, if $\Sigma a_n\beta_n$ is convergent, we can choose μ_0 so that

$$a_\mu\beta_\mu+a_{\mu+1}\beta_{\mu+1}+\ldots+a_\nu\beta_\nu<\epsilon \quad (\mu_0<\mu<\nu),$$

and, *a fortiori*, $$(a_\mu+a_{\mu+1}+\ldots+a_\nu)\beta_\nu<\epsilon \quad (\mu_0<\mu<\nu).$$

But when μ is fixed we can choose ν_0, so that

$$(a_1+a_2+\ldots+a_{\mu-1})\beta_\nu<\epsilon \quad (\nu_0<\nu),$$

and so $$(a_1+a_2+\ldots+a_\nu)\beta_\nu<2\epsilon \quad (\nu_0<\nu).$$

Similarly, we can prove that the second of the conditions (2) is satisfied. Hence condition (3) is *sufficient;* that it is not necessary has been shown by Pringsheim by an example.*

* The example is given by $a_n=\beta_n=\{(n+1)\log(n+1)\}^{-1}$.

Finally, as regards (4), I have nothing to add to Pringsheim's own proof. Since

$$(a_1+a_2+\ldots+a_n)\,\beta_n > na_n\beta_n,$$

the condition

$$na_n\beta_n \to 0$$

is *necessary*. Thus

$$n^{1+s}(a_n\beta_n)^{1+s} \to 0,$$

and so $\Sigma(a_n\beta_n)^{1+s}$ is convergent; *i.e.*, (4) is a *necessary* condition.

11. Theorems A, B, and C, taken in connection with Pringsheim's theorems, suggest questions of some interest to which I am unable at present to give a definite answer.

Let us, for simplicity, consider the special problem of the multiplication of the two series

$$\pm 1^{-s} \pm 2^{-s} \pm 3^{-s} \pm \ldots, \qquad \pm 1^{-t} \pm 2^{-t} \pm 3^{-t} \pm \ldots,$$

where all that is known about the signs of the terms is that they are such as to ensure the convergence of each series.

If $0 < s \leqslant \frac{1}{2}$, $0 < t \leqslant \frac{1}{2}$, or more generally, if s, t and $s+t$ are all positive and not greater than unity, we can certainly choose the signs so that A and B are convergent and C oscillatory. It is enough to take the alternating series $1^{-s}-2^{-s}+\ldots$, $1^{-t}-2^{-t}+\ldots$. The modulus γ_n of the n-th term of the product series is

$$\sum_{r=1}^{n} r^{-s}(n+1-r)^{-t},$$

which tends to infinity with n, if $s+t<1$, and to the finite limit*

$$\int_0^1 \frac{dx}{x^s(1-x)^{1-s}} = \frac{\pi}{\sin s\pi},$$

if $s+t=1$.

On the other hand, if $s=1$, $t=1$, Theorem C shows that the product series is convergent for *all* arrangements of the signs. But the argument by which it was proved does not appear to be capable of extension.

Now let us consider such a case as that in which $s=t=\frac{3}{4}$, or $s=\frac{1}{2}$, $t=1$. Then either (a) the product series is always convergent, or (b) it is possible to choose the signs so that the product series is oscillatory. My own opinion is that (b) is true; *i.e.*, that when $s+t>1$, but at least one

* In connection with the representation of infinite integrals as the limits of finite sums, see a paper by Mr. Bromwich and myself, *Quarterly Journal*, Vol. XXXIX., p. 222.

of s and t is less than 1, we can make A and B convergent and C oscillatory by a proper choice of signs. But I am unable to support this conclusion by an actual example. I wish merely to point out the considerable margin of uncertainty that still remains. In all such cases as these, of course, Pringsheim's results show that the product of the *alternating* series is convergent.

It is easy to see that examples of the kind desired are not likely to be very readily found. For the conditions

$$\sqrt{n}\, a_n \to 0, \qquad \sqrt{n}\, b_n \to 0$$

are sufficient to ensure $c_n \to 0,$*

since $|c''|$ can never be greater than in the alternating case. Moreover, the series Σc_n is *in any case* summable by Cesàro's mean value, *i.e.*,

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n}$$

exists. Now series whose n-th term tends to zero, and which are summable, but not convergent, certainly exist—examples are given by the series

$$\Sigma \frac{\sin\sqrt{n}}{\sqrt{n}}, \quad \Sigma \frac{\cos\sqrt{n}}{\sqrt{n}}, \quad \Sigma \frac{(-1)^{[\sqrt{n}]}}{\sqrt{n}}.$$

But such examples are not particularly obvious, much less is it obvious how to construct examples in which the general term is of the form of the general term of the product of two convergent series.

12. I take this opportunity of also stating the following generalisation of Mertens' theorem, which I have not seen before, although it is not strictly relevant to the main purpose of the paper.

If A is absolutely convergent, and B is a finitely oscillating series whose n-th term tends to zero, then C is a finitely oscillating series; and if the limits of oscillation of B are β_1 and β_2, those of C are $A\beta_1$ and $A\beta_2$.

To prove this, we go back to the equation

$$\begin{aligned} C_n - A_N B_{n+1-N} = {} & A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} \\ & + B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1}. \end{aligned}$$

Let us suppose that A is absolutely convergent, and that $|B_\nu| < K$ for all values of ν.

* It will be remembered (§ 6) that the conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

ensure $nc_n \to 0$.

First choose N_0 so that

$$(1) \qquad |a_{N+1}|+|a_{N+2}|+\ldots<\epsilon/K,$$

for $N \geqslant N_0$. *A fortiori*, we have also

$$(2) \qquad |A-A_N|<\epsilon/K.$$

When any value of N greater than N_0 has been determined, we can choose n_0 so that

$$(3) \qquad |A_1 b_n+A_2 b_{n-1}+\ldots+A_{N-1} b_{n+2-N}|<\epsilon,$$

for $n \geqslant n_0$. From (1), (2), and (3) it follows that

$$|C_n-A_N B_{n+1-N}|<2\epsilon,$$
$$|C_n-AB_{n+1-N}|<3\epsilon,$$

for $n \geqslant n_0$, which establishes the result. In the particular case in which $\beta_1=\beta_2$, we obtain Mertens' theorem. It should be observed that the theorem is *not* true if the condition $b_n \rightarrow 0$ is removed. Suppose, for example, that $a_n>0$, and form the product of

$$a_1+a_2+a_3+\ldots, \qquad 1-1+1-\ldots.$$

We easily see that
$$C_{2n}=a_2+a_4+\ldots+a_{2n},$$
$$C_{2n+1}=a_1+a_3+\ldots+a_{2n+1},$$

so that C oscillates, but not between the limits prescribed by the theorem. In particular the product of

$$\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots, \qquad 1-1+1-\ldots,$$

converges to the sum 1.

13. I shall conclude by stating the theorems for integrals which are analogous to some of those for series discussed in the preceding pages. But, as these theorems are of much less importance, I shall only outline the proofs.

Suppose that $a(x)$ and $b(x)$ are continuous functions, such that

$$\int_0^\infty a(x)\,dx, \qquad \int_0^\infty b(x)\,dx$$

are convergent and have the values A, B. And let

$$c(x)=\int_0^x a(t)\,b(x-t)\,dt=\int_0^x a(x-t)\,b(t)\,dt.$$

$$A(x)=\int_0^x a(t)\,dt, \quad B(x)=\int_0^x b(t)\,dt, \quad C(x)=\int_0^x c(t)\,dt.$$

Then it is easy to prove the formulæ

$$C(x) = \int_0^x A(t)\, b(x-t)\, dt = \int_0^x A(x-t)\, b(t)\, dt$$
$$= \int_0^x a(t)\, B(x-t)\, dt = \int_0^x a(x-t) B(t)\, dt,$$
$$\int_0^x C(t)\, dt = \int_0^x A(t) B(x-t)\, dt = \int_0^x A(x-t) B(t)\, dt.$$

It is moreover easy to prove that, if $A(x)$ and $B(x)$ tend, as $x \to \infty$, to limits A and B, then

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x A(t)\, B(x-t)\, dt = AB.$$

It follows that :—

(1) *If* $\int_0^\infty a(x)\, dx = A, \qquad \int_0^\infty b(x)\, dx = B,$

then $\lim_{x\to\infty} \frac{1}{x} \int_0^x dt \int_0^t du \int_0^u a(w)\, b(u-w)\, dw = AB.$

This is the analogue of Cesàro's theorem that

$$(C_1 + C_2 + \ldots + C_n)/n \to AB,$$

whenever A and B are convergent.

From this the analogue of Abel's theorem follows at once; viz.,

(2) *If* $\int_0^\infty dx \int_0^x a(t)\, b(x-t)\, dt$

is convergent, its value is AB.

There is no difficulty whatever in establishing the analogues of Cauchy's and Mertens' theorems, viz., that

(3) *If A and B are absolutely convergent, so is C;*

(4) *If A is absolutely and B conditionally convergent, C is (absolutely or conditionally) convergent.*

Corresponding to Theorem A we have

(5) *If A and B are convergent, and $xa(x) \to 0$, $xb(x) \to 0$, as $x \to \infty$, then C is convergent.*

Corresponding to Theorem B, we have

(6) *If* $\phi(x) = (\log x)^{\alpha} (\log_2 x)^{\beta} \ldots (\log_k x)^{\kappa} \to \infty$ *with* x, *and*

$$x\phi(x)\, a(x) \to 0, \qquad \frac{xb(x)}{\phi(x)} \to 0,$$

then C *is convergent.*

Finally, we can show that the necessary and sufficient conditions for the convergence of

$$\int_0^{\infty} f(x)\, dx,$$

are

$$\text{(i.)} \quad \frac{1}{x}\int_0^x dt \int_0^t f(u)\, du \to 0,$$

$$\text{(ii.)} \quad \frac{1}{x}\int_0^x tf(t)\, dt \to 0;$$

and from this we can deduce the analogue of Theorem C, viz.

(7) *If* $|\, xa(x)\,| < K$, $|\, xb(x)\,| < K$, *then* C *is convergent.*

RELATIONS BETWEEN THE DIVISORS OF THE FIRST n NATURAL NUMBERS

(SECOND PAPER.)

By J. W. L. GLAISHER.

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Introduction. § 1.

1. This paper is a continuation of one having the same title which was read before the Society on May 14th, 1891, and was published in the *Proceedings*, Vol. XXII., pp. 359–410. It was written in the summer of 1891, but was laid aside while the second part (relating to the divisors δ') was still incomplete. During the present year I have revised the manuscript of the first part, which relates to the divisors $(-1)^{d-1}d$, and have put into shape the second part. I have left the first part nearly as it stood, the principal change being that I have reduced the number of examples.* In the second part I have followed the sequence of the formulæ as originally worked out and have added no new results.

The theorems that relate to actual divisors seem to me, like those in the previous paper, to be very curious, as they connect divisors of numbers which differ by fixed intervals.

At the end of the paper a list of the paragraph-headings is given, forming a table of contents of the paper.

PART I.—THEOREMS RELATING TO DIVISORS, UNEVEN DIVISORS HAVING THE POSITIVE SIGN AND EVEN DIVISORS THE NEGATIVE SIGN. §§ 2–45.

Formulæ derived from the Function zc. §§ 2, 3.

2. Proceeding as in § 25 of the previous paper, we have

$$-\rho\,\mathrm{zc}\,\rho x = \frac{2q^{\frac{1}{4}}\sin x + 6q^{\frac{9}{4}}\sin 3x + 10q^{\frac{25}{4}}\sin 5x + \dots}{2q^{\frac{1}{4}}\cos x + 2q^{\frac{9}{4}}\cos 3x + 2q^{\frac{25}{4}}\cos 5x + \dots},$$

* In the paper as originally written the sections were numbered in continuation of those of the previous paper; but, appearing after so long an interval, I have thought it better that the sections of this paper should have a separate numbering.

and also

$$-\rho\,\mathrm{zc}\,\rho x = \frac{\sin x}{\cos x} + \frac{4q^2}{1-q^2}\sin 2x - \frac{4q^4}{1-q^4}\sin 4x + \frac{4q^6}{1-q^6}\sin 6x - \dots$$

$$= \frac{\sin x}{\cos x} + 4\Sigma_1^\infty \zeta(\sin 2nx)\, q^{2n},$$

where $\zeta\phi(n)$ denotes the sum

$$(-1)^{d_1-1}\phi(d_1) + (-1)^{d_2-1}\phi(d_2) + \dots + (-1)^{d_f-1}\phi(d_f),$$

$d_1, d_2, \dots, d_f$ being all the divisors of n. In this expression any term $\phi(d)$ has the positive or negative sign according as d is uneven or even, so that $\zeta\phi(n)$ is equal to the sum of the ϕ's of the uneven divisors of n diminished by the sum of the ϕ's of the even divisors of n.

We thus obtain the identical relation

$$\frac{\sin x}{\cos x} + 4\Sigma_1^\infty \zeta(\sin 2nx)\, q^n = \frac{\sin x + 3q\sin 3x + 5q^3 \sin 5x + \dots}{\cos x + q\cos 3x + q^3\cos 5x + \dots},$$

whence, by following exactly the process of § 26 of the previous paper, we find, corresponding to the first formula of § 27,

$$\{1-(1-2\cos x)\,q + (1-2\cos x + 2\cos 2x)\,q^3 - \dots\}\ \Sigma_1^\infty \zeta(\sin nx)\, q^n$$

$$= \sin n \,.\, q - (\sin x - 2\sin 2x)\,q^3 + (\sin x - 2\sin 2x + 3\sin 3x)\,q^6 - \dots.$$

3. Equating the coefficients of q^n, we see that the expression

$$\Sigma_n(-1)^{d-1}\sin dx - \Sigma_{n-1}(-1)^{d-1}\{\sin dx - \sin(d-1)x - \sin(d+1)x\}$$

$$+\Sigma_{n-3}(-1)^{d-1}\{\sin dx - \sin(d-1)x - \sin(d+1)x + \sin(d-2)x + \sin(d+2)x\}$$

$$+\dots$$

is equal to zero, or to

$$(-1)^{g-1}\{\sin x - 2\sin 2x + 3\sin 3x - \dots + (-1)^{g-1} g\sin gx\},$$

according as n is not a triangular number, or is equal to the triangular number $\tfrac{1}{2}g(g+1)$.

Definition of the Group Symbol G'. § 4.

4. It is convenient to denote by $G'_r\{\phi(d), \psi(d), \dots\}$ the group of numbers

$$\begin{matrix} (-1)^{d_1-1}\phi(d_1), & (-1)^{d_2-1}\phi(d_2), & \dots, & (-1)^{d_f-1}\phi(d_f), \\ (-1)^{d_1-1}\psi(d_1), & (-1)^{d_2-1}\psi(d_2), & \dots, & (-1)^{d_f-1}\psi(d_f), \\ \dots & \dots & \dots & \dots \end{matrix}$$

$d_1, d_2, \dots, d_f$ being all the divisors of r.

Comparing G' with G, defined in § 2 of the previous paper, we see that $G'_r\{\phi(d), \psi(d), \ldots\}$ is the same as $G\{(-1)^{d-1}\phi(d), (-1)^{d-1}\psi(d), \ldots\}$.

It is supposed that $-G'_r\{\phi(d), \psi(d), \ldots\}$ has the same meaning as $G'_r\{-\phi(d), -\psi(d), \ldots\}$.

Theorem relating to the Actual Divisors of n, $n-1$, $n-3$, ...
§ 5.

5. The formula in § 3, regarded as a theorem relating to the actual divisors, shows that the numbers

$$G'_n(d)-G'_{n-1}\{d, -(d\pm 1)\}+G'_{n-3}\{d, -(d\pm 1), d\pm 2\}-\ldots,$$

all cancel each other unless n is a triangular number $\frac{1}{2}g(g+1)$, in which case there remain uncancelled

one 1, two (-2)'s, three 3's, four (-4)'s, ..., $g(\pm g)$'s,

if g is uneven, and these numbers with the sign reversed if g is even.

This, however, is not a new result, for it differs from the theorem in § 3 of the previous paper only by the reversal of the signs of the even numbers. It is evident that in that theorem, which relates to the mutual cancellation of numbers, we are at liberty to change the sign of all the even numbers, or indeed of all the numbers of any given form.

General Theorems connecting ζ_m, ζ_{m-2}, ... § 6.

6. By taking $G'_r(d)$ to be $\Sigma_r(-1)^{d-1}d^m$, m being uneven, or by equating the coefficients of x^m in the formula of § 3, we find that

$$\Sigma_n(-1)^{d-1}d^m-\Sigma_{n-1}(-1)^{d-1}\{d^m-(d-1)^m-(d+1)^m\}$$
$$+\Sigma_{n-3}(-1)^{d-1}\{d^m-\ldots+(d+2)^m\}-\ldots$$

is equal to zero, if n is not a triangular number, and is equal to

$$(-1)^{g-1}\{1^{m+1}-2^{m+1}+3^{m+1}-\ldots+(-1)^{g-1}g^{m+1}\},$$

if $$n=\tfrac{1}{2}g(g+1).$$

Denoting by $\zeta_m(n)$ the excess of the sum of the m-th powers of the uneven divisors of n over the sum of the m-th powers of the even divisors

of n, we thus obtain the general theorem

$$\begin{aligned}
&\zeta_m(n)+\zeta_m(n-1)+\zeta_m(n-3)+\zeta_m(n-6)+\dots \\
&+2\,(m)_2\,\{\zeta_{m-2}(n-1)+(2^2-1^2)\zeta_{m-2}(n-3)+(3^2-2^2+1^2)\,\zeta_{m-2}(n-6)+\dots\} \\
&+2\,(m)_4\,\{\zeta_{m-4}(n-1)+(2^4-1^4)\zeta_{m-4}(n-3)+(3^4-2^4+1^4)\zeta_{m-4}(n-6)+\dots\} \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&+2m\,\{\zeta(n-1)+(2^{m-1}-1^{m-1})\zeta(n-3)+(3^{m-1}-2^{m-1}+1^{m-1})\,\zeta(n-6)+\dots\} \\
&\qquad\qquad = [g^{m+1}-(g-1)^{m+1}+(g+2)^{m+1}-\dots+(-1)^{g-1}\,1^{m+1}],
\end{aligned}$$

where $(m)_r$ denotes the number of combinations of m things r together, and the square brackets indicate that the quantity included by them only occurs when $n=\frac{1}{2}g\,(g+1)$; otherwise the term in [ ] is to be replaced by zero.

This ζ-theorem corresponds to the σ-theorem of § 11 of the previous paper, viz.,

$$\begin{aligned}
&\sigma_m(n)-3\sigma_m(n-1)+5\sigma_m(n-3)-7\sigma_m(n-6)+\dots \\
&-2\,(m)_2\,\{\sigma_{m-2}(n-1)+(2^2+1^2)\,\sigma_{m-2}(n-3)+(3^2+2^2+1^2)\,\sigma_{m-2}(n-6)+\dots\} \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&-2m\,\{\sigma(n-1)+(2^{m-1}+1^{m-1})\,\sigma(n-3)+(3^{m-1}+2^{m-1}+1^{m-1})\,\sigma(n-6)+\dots\} \\
&\qquad\qquad = [(-1)^{g-1}\,\{g^{m+1}+(g-1)^{m+1}+(g-2)^{m+1}+\dots+1^{m+1}\}].
\end{aligned}$$

The particular Cases $m=1,\ 3,\ 5.$ §§ 7, 8.

7. By putting $m=1$ in the general ζ-theorem, we have

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\dots=[g^2-(g-1)^2+\dots+(-1)^{g-1}\,1^2],$$

which is the same as the theorem quoted in § 68 of the previous paper, for the right-hand member is equal to $[\frac{1}{2}g\,(g+1)]$, so that we may write the theorem

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\dots=0,$$

if we assign to $\zeta(0)$ the value $-n$.

Putting $m=3$, we find in the same manner

$$\begin{aligned}
&\zeta_3(n)+\zeta_3(n-1)+\zeta_3(n-3)+\zeta_3(n-6)+\dots \\
&+6\,\{\zeta(n-1)+3\zeta(n-3)+6\zeta(n-6)+10\zeta(n-10)+\dots\} \\
&\qquad\qquad = [\tfrac{1}{2}(g^2+g)(g^2+g-1)].
\end{aligned}$$

We may dispense with the additional term (the right-hand side of the

equation being then always zero), if we assign to $\zeta_3(0)$ and $\zeta(0)$ the values n and $-\frac{1}{3}n$ respectively.

Similarly, putting $m = 5$, we find

$$\begin{aligned}
&\zeta_5(n)+\zeta_5(n-1)+\zeta_5(n-3)+\zeta_5(n-6)+\dots \\
&+20\,\{\zeta_3(n-1)+3\zeta_3(n-3)+6\zeta_3(n-6)+\dots\} \\
&+10\,\{\zeta(n-1)+15\zeta(n-3)+66\zeta(n-6)+\dots\} \\
&\qquad\qquad = [\tfrac{1}{2}(g^2+g)\,\{(g^2+g)^2-3(g^2+g)+3\}].
\end{aligned}$$

The general term in the third series is of the form $t(2t-1)\,\zeta(n-t)$, where t is a triangular number.

We may dispense with the additional term by putting

$$\zeta_5(0) = -3n, \quad \zeta_3(0) = \tfrac{1}{5}n, \quad \zeta(0) = -\tfrac{1}{5}n.$$

8. Using t to denote any triangular number, we may write the three formulæ :—

$$\Sigma\zeta(n-t) = [n],$$

$$\Sigma\zeta_3(n-t)+6\Sigma t\zeta(n-t) = [2n^2-n],$$

$$\Sigma\zeta_5(n-t)+20\Sigma t\zeta_3(n-t)+10\Sigma(2t^2-t)\,\zeta(n-t) = [4n^3-6n^2+3n],$$

the summations extending from $t = 0$ to $t =$ the triangular number next inferior to n.

As already mentioned, we may dispense with the additional term (the right-hand side of the equation being then zero), if we extend the summation so as to include n, when n is a triangular number, and put $\zeta(0) = -n$ in the first formula, $\zeta_3(0) = n$ and $\zeta(0) = -\frac{1}{3}n$ in the second, and $\zeta_5(0) = -3n$, $\zeta_3(0) = \frac{1}{5}n$, $\zeta(0) = -\frac{1}{5}n$ in the third.

Second Theorem relating to the Actual Divisors of $n, n-1, n-3, \dots$.
§§ 9–12.

9. It is easily seen that the series $\Sigma_1^\infty \sigma(\sin nx)q^n$ is converted into $\Sigma_1^\infty \zeta(\sin nx)q^n$ by replacing x by $\pi-x$. It is unnecessary, therefore, to work out independently the ζ-formulæ which give rise to the divisor theorems, as we may deduce them from the corresponding σ-formulæ obtained in the previous paper. Thus, for example, the formula at the end of § 2 might have been deduced from the first formula in § 27 of the previous paper by substituting $\pi-x$ for x.

If the original formula involves $\sigma(\sin 2nx)$ instead of $\sigma(\sin nx)$, it is evident that we are to replace x by $\frac{1}{2}\pi - x$.

10. Putting $\frac{1}{2}\pi - x$ for x in the first formula of § 28 of the previous paper, we obtain the equation

$$\begin{aligned}2(\cos x + q\cos 3x + q^3\cos 5x + \ldots)\, \Sigma_1^\infty \zeta(\sin 2nx)\, q^n \\ = (\sin 3x + \sin x)\, q + (2\sin 5x + \sin 3x - \sin x)\, q^3 \\ + (3\sin 7x + \sin 5x - \sin 3x + \sin x)\, q^6 + \ldots,\end{aligned}$$

the general term on the right-hand side being

$$g\sin(2g+1)x + \sin(2g-1)x - \sin(2g-3)x + \ldots + (-1)^{g-1}\sin x.$$

By equating the coefficients of q^n, we thus obtain the formula

$$\begin{aligned}&\Sigma_n(-1)^{d-1}\{\sin(2d-1)x + \sin(2d+1)x\} \\ &+\Sigma_{n-1}(-1)^{d-1}\{\sin(2d-3)x + \sin(2d+3)x\} \\ &+\Sigma_{n-3}(-1)^{d-1}\{\sin(2d-5)x + \sin(2d+5)x\} + \ldots\end{aligned}$$

$= 0$ if n is not a triangular number, and

$= g\sin(2g+1)x + \sin(2g-1)x - \sin(2g-3)x + \ldots + (-1)^{g-1}\sin x$

if n is the triangular number $\frac{1}{2}g(g+1)$.

11. This equation shows that the numbers

$$G'_n(2d \pm 1) + G'_{n-1}(2d \pm 3) + G'_{n-3}(2d \pm 5) + \ldots,$$

all cancel each other unless n is a triangular number $\frac{1}{2}g(g+1)$, in which case

$g(2g+1)$'s, one $(2g-1)$, one $-(2g-3)$, one $(2g-5)$, ..., one (± 1) remain uncancelled.

In forming the scheme of numbers given by this formula, we take the divisors d of n and form the numbers $2d \pm 1$, we take the divisors d of $n-1$ and form the numbers $2d \pm 3$, and so on; we then change the sign of all the numbers derived from the even divisors.

Thus, taking $n = 9$, and writing the divisors themselves in the middle line, enclosed in brackets, we form the scheme

$$\begin{array}{llll} 3,\ 7,\ 19, & 5,\ 7,\ 11,\ 19, & 7,\ \ 9,\ 11,\ 17, & 9,\ 13, \\ (1,\ 3,\ \ 9), & (1,\ 2,\ \ 4,\ \ 8), & (1,\ \ 2,\ \ 3,\ \ 6), & (1,\ \ 3), \\ 1,\ 5,\ 17, & -1,\ 1,\ \ 5,\ 13, & -3,\ -1,\ \ 1,\ \ 7, & -5,\ -1; \end{array}$$

we then change the signs of all the numbers which stand above or below the even divisors, thus obtaining the numbers

$$\begin{array}{l} 3,\,7,\,19,\quad 5,\,-7,\,-11,\,-19,\quad 7,\,-9,\,11,\,-17,\quad 9,\;13, \\ 1,\,5,\,17,\,-1,\,-1,\quad -5,\,-13,\,-3,\quad 1,\;1,\quad -7,\,-5,\,-1, \end{array}$$

which cancel one another completely, as they should do, since 9 is not a triangular number.

As a second example, taking $n = 10$, so that $g = 4$, we first form the scheme

$$\begin{array}{cccccccccccc} 3, & 5, & 11, & 21, & 5, & 9, & 21, & 7, & 19, & 9, & 11, & 15, \\ (1, & 2, & 5, & 10), & (1, & 3, & 9), & (1, & 7), & (1, & 2, & 4), \\ 1, & 3, & 9, & 19, & -1, & 3, & 15, & -3, & 9, & -5, & -3, & 1, \end{array}$$

giving, after changing the signs of the numbers above and below the even divisors,

$$\begin{array}{cccccccccccc} 3, & -5, & 11, & -21, & 5, & 9, & 21, & 7, & 19, & 9, & -11, & -15, \\ 1, & -3, & 9, & -19, & -1, & 3, & 15, & -3, & 9, & -5, & 3, & -1, \end{array}$$

in which the numbers that remain after the cancellation are

$$9,\ 9,\ 9,\ 9,\ 7,\ -5,\ 3,\ -1.$$

12. It will be observed that we may derive the final system of numbers directly from the divisors if, in writing down the latter, we attach the negative sign to the even divisors. Thus, in the case of $n = 10$, we may form at once the scheme

$$\begin{array}{cccccccccccc} 3, & -3, & 11, & -19, & 5, & 9, & 21, & 7, & 19, & 9, & 3, & -1, \\ (1, & -2, & 5, & -10), & (1, & 3, & 9), & (1, & 7), & (1, & -2, & -4), \\ 1, & -5, & 9, & -21, & -1, & 3, & 15, & -3, & 9, & -5, & -11, & -15, \end{array}$$

in which the number above any number e in the middle or divisor line, whether e be positive or negative, is $2e+1$, and the number below it is $2e-1$.

Comparison between the corresponding σ and ζ Divisor Theorems. §§ 13–15.

13. It is interesting to compare the theorem in § 11 with the corresponding result in § 29 of the previous paper. The latter theorem

which was obtained from a σ-formula involving cosines relates to the absolute magnitude (irrespective of sign) of the numbers derived from the divisors, but the actual numbers which occur are the same.

Take, for example, $n = 7$, which is not a triangular number. In the case of the σ-theorem (§ 29 of the previous paper),* we first form the scheme

$$\begin{array}{cccc} 3,\ 15, & 5,\ 7,\ 9,\ 15, & 7,\ 9,\ 13, & 9, \\ (1,\ 7), & (1,\ 2,\ 3,\ 6), & (1,\ 2,\ 4), & (1), \\ 1,\ 13, & 1,\ 1,\ 3,\ 9, & 3,\ 1,\ 3, & 5, \end{array}$$

in which the upper line consists of the values of $2d+1, \ldots$, and the lower line of the values of $2d-1, \ldots$, all taken with the positive sign, *i.e.*, when the number is negative the sign is to be changed and so made positive. We then change the signs of the numbers in the upper line of the first group, in the lower line of the second group, in the upper line of the third group, and so on, thus obtaining the numbers

$$\begin{array}{cccccccccc} -3, & -15, & 5, & 7, & 9, & 15, & -7, & -9, & -13, & 9, \\ 1, & 13, & -1, & -1, & -3, & -9, & 3, & 1, & 3, & -5, \end{array}$$

all of which cancel one another.

In the case of the ζ-theorem, following the procedure of § 11, we first form the scheme

$$\begin{array}{cccc} 3,\ 15, & 5,\ 7,\ 9,\ 15, & 7,\ \ 9,\ 13, & 9, \\ (1,\ 7), & (1,\ 2,\ 3,\ 6), & (1,\ \ 2,\ 4), & (1), \\ 1,\ 13, & -1,\ 1,\ 3,\ 9, & -3,\ -1,\ 3, & -5, \end{array}$$

in which the upper line consists of the values of $2d+1, \ldots$, and the lower line of the values of $2d-1, \ldots$, the latter having their proper signs. We then change the signs of the numbers in the columns in which the even divisors occur, thus obtaining the numbers

$$\begin{array}{cccccccccc} 3, & 15, & 5, & -7, & 9, & -15, & 7, & -9, & -13, & 9, \\ 1, & 13, & -1, & -1, & 3, & -9, & -3, & 1, & -3, & -5, \end{array}$$

all of which cancel one another.

14. If we start in both cases with the scheme containing the $2d+1, \ldots$ and $2d-1, \ldots$ numbers (the latter with their proper signs), we see that

* This mode of statement of the σ-theorem is slightly changed from § 29 of the previous paper for the sake of the comparison.

in the one process we first change the signs of the negative numbers (so as to make all the signs positive), and then change the signs of the numbers on the upper and lower lines of each group alternately, while in the other process we merely change the signs in the columns depending upon the even divisors.

Taking as another example $n = 6$, which is the triangular number corresponding to $g = 3$, we form the scheme

$$\begin{array}{cccc} 3,\ 5,\ 7,\ 13, & 5,\ 13, & 7,\ 11, \\ (1,\ 2,\ 3,\ \ 6), & (1,\ \ 5), & (1,\ \ 3), \\ 1,\ 3,\ 5,\ 11, & -1,\ \ 7, & -3,\ \ 1, \end{array}$$

in which the upper line contains the numbers $2d+1$, $2d+3$, ..., and the lower line the numbers $2d-1$, $2d-3$, ... (with their proper signs).

The σ process gives

$$\begin{array}{cccccccc} -3, & -5, & -7, & -13, & 5, & 13, & -7, & -11, \\ 1, & 3, & 5, & 11, & -1, & -7, & 3, & 1, \end{array}$$

leaving uncancelled -7, -7, -7, 5, 3, 1; and the ζ process gives

$$\begin{array}{cccc} 3,\ -5,\ 7,\ -13, & 5,\ 13, & 7,\ 11, \\ 1,\ -3,\ 5,\ -11, & -1,\ \ 7, & -3,\ \ 1, \end{array}$$

leaving uncancelled 7, 7, 7, 5, -3, 1.

15. The numbers yielded by the two processes differ only by the reversal of sign of those which are of the form $4k+3$. The reason for this may be seen as follows.

Considering first the upper line of the scheme, which contains the values of $2d+1$, $2d+3$, ..., we see that, in the case of n, even values of d produce numbers of the form $4k+1$, and uneven values of d produce numbers of the form $4k+3$; in the case of $n-1$, even d's produces numbers of the form $4k+3$, and uneven d's produce numbers of the form $4k+1$, and so on.* Thus, writing the forms of the numbers derived from even d's first, and the forms of those derived from uneven d's second, we

* Of course when n, or $n-1$, ... is uneven there are no even divisors, but this does not affect the argument,

see that the forms of the numbers in the upper line are

$$\begin{array}{c|c|c|l} 4k+1,\ 4k+3 & 4k+3,\ 4k+1 & 4k+1,\ 4k+3 & \dots \\ (n) & (n-1) & (n-3) & \end{array}$$

By the first process we obtain numbers of the forms

$$-(4k+1),\ -(4k+3)\ |\ 4k+3,\ 4k+1\ |\ -(4k+1),\ -(4k+3)\ |\ \dots,$$

and by the second process numbers of the forms

$$-(4k+1),\ (4k+3)\ |\ -(4k+3),\ 4k+1\ |\ -(4k+1),\ 4k+3\ |\ \dots,$$

so that the results differ only by a change of sign in the numbers of the form $4k+3$. Passing now to the lower line, which contains the values of $2d-1$, $2d-3$, ... (with their proper signs), an exactly similar argument applies to the positive numbers; but in the case of the negative numbers, the same reasoning shows that the numbers given by the two processes have the same sign when of the form $4k+3$, and opposite signs when of the form $4k+1$. If, therefore, we change the signs of all the negative numbers in the lower line (as is done in the first process), we always obtain by the two processes numbers which have the same sign when of the form $4k+1$, and have opposite signs when of the form $4k+3$.

Second General Theorem connecting ζ_m, ζ_{m-2}, ... §§ 16, 17.

16. By replacing $G_r'(d)$ by $\Sigma_r(-1)^{d-1}d^m$, m being uneven, in the formula of § 11, or by equating the coefficients of x^m in the formula of § 10, we obtain the general numerical theorem

$$\zeta_m(n)+\zeta_m(n-1)+\zeta_m(n-3)+\zeta_m(n-6)+\dots$$

$$+\frac{(m)_2}{2^2}\{\zeta_{m-2}(n)+3^2\zeta_{m-2}(n-1)+5^2\zeta_{m-2}(n-3)+7^2\zeta_{m-2}(n-6)+\dots\}$$

$$+\frac{(m)_4}{2^4}\{\zeta_{m-4}(n)+3^4\zeta_{m-4}(n-1)+5^4\zeta_{m-4}(n-3)+7^4\zeta_{m-2}(n-6)+\dots\}$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

$$+\frac{m}{2^{m-1}}\{\zeta(n)+3^{m-1}\zeta(n-1)+5^{m-1}\zeta(n-3)+7^{m-1}\zeta(n-6)+\dots\}$$

$$=\left[\frac{g(2g+1)^m+(2g-1)^m-(2g-3)^m+\dots+(-1)^{g-1}1^m}{2^{m+1}}\right],$$

the square brackets indicating, as before, that the quantity included only occurs when $n=\frac{1}{2}g(g+1)$.

17. The corresponding σ-theorem, which was given in § 32 of the previous paper, is

$$\sigma_m(n)-3\sigma_m(n-1)+5\sigma_m(n-3)-7\sigma_m(n-6)+\dots$$
$$+\frac{(m)_2}{2^2.3}\{\sigma_{m-2}(n)-3^3\sigma_{m-2}(n-1)+5^3\sigma_{m-4}(n-3)-7^3\sigma_{m-2}(n-6)+\dots\}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+\frac{1}{2^{m-1}}\{\sigma(n)-3^m\sigma(n-1)+5^m\sigma(n-3)-7^m\sigma(n-6)+\dots\}$$
$$=\left[(-1)^{g-1}\frac{g(2g+1)^{m+1}-(2g-1)^{m+1}-(2g-3)^{m+1}-\dots-1^{m+1}}{2^{m+1}(m+1)}\right].$$

The particular Cases $m=1, 3, 5$. §§ 18, 19.

18. Putting $m=1$ in the general formula of § 16, we find

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\dots=[\tfrac{1}{2}g(g+1)],$$

which is the formula quoted in § 68 of the previous paper, and already obtained in § 7 of the present paper. We may dispense with the additional term by putting $\zeta(0)=-n$.

Putting $m=3$, we find

$$\zeta_3(n)+\zeta_3(n-1)+\zeta_3(n-3)+\zeta_3(n-6)+\dots$$
$$+\tfrac{3}{4}\{\zeta(n)+3^2\zeta(n-1)+5^2\zeta(n-3)+7^2\zeta_3(n-6)+\dots\}$$
$$=[\tfrac{1}{8}(g^2+g)(4g^2+4g-1)].$$

We may dispense with the additional term by putting

$$\zeta_3(0)=\tfrac{1}{2}n, \qquad \zeta(0)=-\tfrac{1}{3}n.$$

Putting $m=5$, we find

$$\zeta_5(n)+\zeta_5(n-1)+\zeta_5(n-3)+\zeta_5(n-6)+\dots$$
$$+\tfrac{5}{2}\{\zeta_3(n)+3^2\zeta_3(n-1)+5^2\zeta_3(n-3)+7^2\zeta_3(n-6)+\dots\}$$
$$+\tfrac{5}{16}\{\zeta(n)+3^4\zeta(n-1)+5^4\zeta(n-3)+7^4\zeta(n-6)+\dots\}$$
$$=[\tfrac{1}{32}(g^2+g)\{16(g^2+g)^2-8(g^2+g)+13\}].$$

We may dispense with the additional term by putting

$$\zeta_5(0)=-n, \qquad \zeta_3(0)=\tfrac{1}{10}n, \qquad \zeta(0)=-\tfrac{1}{5}n.$$

19. Using the notation employed in § 18 of the previous paper—viz., putting $t=\frac{1}{2}i(i+1)$, so that t is the i-th triangular number, and putting

$c = 2i+1$, so that t and c are connected by the relation $t = \frac{1}{8}(c^2-1)$, we may write the three preceding formulæ in the form

$$\Sigma\zeta(n-t) = [n],$$

$$\Sigma\zeta_3(n-t)+\tfrac{3}{4}\Sigma c^2\zeta(n-t) = [2n^2-\tfrac{1}{4}n],$$

$$\Sigma\zeta_5(n-t)+\tfrac{5}{2}\Sigma c^2\zeta_3(n-t)+\tfrac{5}{16}\Sigma c^4\zeta(n-t) = [4n^3-n^2+\tfrac{13}{16}n],$$

where, as in § 8, the summations extend from $t = 0$ to $t =$ the triangular number next inferior to n. We may, of course, dispense with the additional term by extending the summation so as to include n, and assigning to $\zeta(0)$, to $\zeta(0)$ and $\zeta_3(0)$, ... the same values as in the preceding section.

The Notations H, H', H'', ..., H_2, H_4, ... for the Series. §§ 20, 21.

20. It is convenient to have a compendious notation for representing the classes of series which occur in §§ 8 and 19. Let, therefore,

$$H\phi(n) = \phi(n)+\phi(n-1)+\phi(n-3)+\phi(n-6)+\ldots,$$

$$H'\phi(n) = \phi(n-1)+3\phi(n-3)+6\phi(n-6)+\ldots,$$

$$H''\phi(n) = \phi(n-1)+3^2\phi(n-3)+6^2\phi(n-6)+\ldots,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

so that, in general, $\qquad H^{(r)}\phi(n) = \Sigma_{t=0}^{t=w} t^r\phi(n-t),$

w being the triangular number next inferior to n.

Also, let

$$H_2\phi(n) = \phi(n)+3^2\phi(n-1)+5^2\phi(n-3)+7^2\phi(n-6)+\ldots,$$

$$H_4\phi(n) = \phi(n)+3^4\phi(n-1)+5^4\phi(n-3)+7^4\phi(n-6)+\ldots,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

so that, in general, $\qquad H_{2r}\phi(n) = \Sigma_{t=0}^{t=w} c^{2r}\phi(n-t).$

It is supposed that $H_0\phi(n)$ and $H^{(0)}\phi(n)$ are the same as $H\phi(n)$.

21. Since $t = \frac{1}{8}(c^2-1)$, we may easily express H', H'', ... in terms of H_2, H_4, ...; for

$$H'\phi(n) = \Sigma t\phi(n-t) = \tfrac{1}{8}\Sigma(c^2-1)\,\phi(n-t) = \tfrac{1}{8}\{H_2\phi(n)-H\phi(n)\}.$$

Similarly $\qquad H''\phi(n) = \frac{1}{64}(H_4-2H_2+H)\,\phi(n),$

and, in general,

$$H^{(r)}\phi(n) = \frac{1}{8^r}\left\{H_{2r} - rH_{2r-2} + \frac{r(r-1)}{2!}H_{2r-4} - \ldots + (-1)^r H\right\}\phi(n).$$

Conversely, $H_2\phi(n) = \Sigma(1+8t)\,\phi(n-t) = H\phi(n) + 8H'\phi(n)$,

$$H_4\phi(n) = (H + 16H' + 64H'')\,\phi(n),$$

and, in general,

$$H_{2r}\phi(n) = \left\{H + r\,.\,8H' + \frac{r(r-1)}{2!}8^2H'' + \ldots + 8^rH^{(r)}\right\}\phi(n).^*$$

Relations between the ζ-formulæ of §§ 8 *and* 18. §§ 22, 23.

22. Expressed in the H notation, the formulæ of § 8 are

$$H\zeta(n) = [n],$$

$$H\zeta_3(n) + 6H'\zeta(n) = [2n^2 - n],$$

$$H\zeta_5(n) + 20H'\zeta_3(n) + 20H''\zeta(n) - 10H'\zeta(n) = [4n^3 - 6n^2 + 3n],$$

and those of § 19 are

$$H\zeta(n) = [n],$$

$$H\zeta_3(n) + \tfrac{3}{4}H_2\zeta(n) = [2n^2 - \tfrac{1}{4}n],$$

$$H\zeta_5(n) + \tfrac{5}{2}H_2\zeta_3(n) + \tfrac{5}{16}H_4\zeta(n) = [4n^3 - n^2 + \tfrac{13}{16}n].$$

23. The two sets of formulæ are deducible the one from the other, for taking the second formula of the first set and substituting $\frac{1}{8}(H_2 - H_1)$ for H', we obtain

$$H\zeta_3(n) + \tfrac{3}{4}H_2\zeta(n) = \tfrac{3}{4}H\zeta(n) + [2n^2 - n] = [\tfrac{3}{4}n] + [2n^2 - n] = [2n^2 - \tfrac{1}{4}n],$$

which is the second formula of the second set.

---

* I found a similar notation useful in the treatment of the series which occur in connection with the σ-formulæ—viz., putting

$$J\phi(n) = \Sigma_{t=0}^{t=\infty}(-1)^{\frac{1}{2}(c-1)}c\phi(n-t),$$

$$J^{(r)}\phi(n) = (-1)^r\Sigma_{t=0}^{t=\infty}(-1)^{\frac{1}{2}(c-1)}ct^r\phi(n-t),$$

and

$$J_{2r+1}\phi(n) = \Sigma_{t=0}^{t=\infty}(-1)^{\frac{1}{2}(c-1)}c^{2r+1}\phi(n-t)$$

(so that $J^{(0)}$ and J_0 are the same as J), we obtain relations between J', J'', ..., and J_1, J_3, ..., which are exactly analogous to the above relations between H'. H'', ... and H, H_2, ... (*Messenger of Mathematics*, Vol. XXI., 1891, pp. 54–64).

if p be uneven, where $\frac{1}{2}p(p+1)$ is the triangular number next superior to n.

The square brackets in the formula indicate that the absolute value of the quantity included is to be taken.*

25. This result is not a new one, as it differs from the corresponding theorem in § 46 of the previous paper only by a reversal of the signs of the numbers derived from the even divisors. It is, however, interesting to examine the new form of the theorem.

Taking, as an example, $n = 6$, so that $p = 4$, we form the scheme

$$\begin{array}{cccc|ccccc|ccccc} 2, & 3, & 4, & 7 & 3, & 7, & 3, & 4, & 6 & 4, & 6, & 4, & 5, & 4 \\ (1, & 2, & 3, & 6) & (1, & 5), & (1, & 2, & 4) & (1, & 3), & (1, & 2), & (1) \\ -0, & -1, & -2, & -5 & -1, & -3, & -1, & -0, & -2 & -2, & -0, & -2, & -1, & -2 \end{array}$$

in which the central line of the first group contains the divisors of 6, the central line of the second group the divisors of 5, 4, and the central line of the third group the divisors of 3, 2, 1. We add 1 to the divisors in the first group, 2 to those in the second, and 3 to those in the third, writing the numbers above the divisors; and we subtract 1 from the first group, 2 from the second, and 3 from the third, writing the numbers below the divisors and prefixing the negative sign to all which have not that sign already. We then change the signs in the columns derived from the even divisors, thus obtaining the numbers

$$\begin{array}{cccccccccccccc} 2, & -3, & 4, & -7, & 3, & 7, & 3, & -4, & -6, & 4, & 6, & 4, & -5, & 4 \\ -0, & 1, & -2, & 5, & -1, & -3, & -1, & 0, & 2, & -2, & -0, & -2, & 1, & -2 \end{array}$$

all of which cancel one another with the exception of

$$-0,\ -2,\ -2,\ 4,\ 4,\ 4.$$

26. We may exhibit the above process in a slightly more compendious form as follows.

Starting, as before, with the divisors separated into groups of one, two, three, ..., we place the numbers $d+1$, $d+2$, ... above or below according as d is uneven or even; and we place the numbers $d-1$, $d-2$, ... (attending only to their absolute magnitude) below or above according as d is uneven or even. The theorem then asserts that, after cancelling the numbers common to the upper and lower lines, there will remain in the

* But, of course, $[\phi(d)]$, when occurring in a G'-group, is affected like all other quantities in the group (§ 4) by the sign $(-1)^{d-1}$.

upper line $(p-1)\,p$'s, and in the lower line one 0, two 2's, two 4's, ..., two $(p-2)$'s or two 1's, two 3's, ..., two $(p-2)$'s, according as p is even or uneven.

As an example, take $n = 8$, so that $p = 4$ as before. The scheme is

2, 1, 3, 7	3, 9,	3, 0, 5, 4	4, 8,	4, 1, 1,	4, 6	5, 2,	5
(1, 2, 4, 8)	(1, 7),	(1, 2, 3, 6)	(1, 5),	(1, 2, 4),	(1, 3)	(1, 2),	(1)
0, 3, 5, 9	1, 5,	1, 4, 1, 8	2, 2,	2, 5, 7,	2, 0	3, 6,	3

in which, after the cancelling, three 4's remain in the upper line and 0, 2, 2 in the lower.

27. Comparing this form of the theorem with that given in §§ 48 and 49 of the previous paper, it will be seen that the only difference in the formation of the two schemes is that, taking, for example, the numbers $d+1$, $d+2$, ..., these are placed in the σ-scheme, above and below in the alternate sets, and that in the ζ-scheme they are placed above and below according as the divisor is uneven or even. The final result (after the cancelling) is slightly more regular in the new form, as the $(p-1)\,p$'s are always left in the upper line.

Theorem connecting ζ_m, ζ_{m-2}, ..., the arguments being the first n Natural Numbers. §§ 28–30.

28. The ζ-theorem deducible from § 24 is

$$
\begin{aligned}
&\zeta_m(n)+2\,\{\zeta_m(n-1)+\zeta_m(n-2)\}+3\,\{\zeta_m(n-3)+\zeta_m(n-4)+\zeta_m(n-5)\}+\dots \\
&+\tfrac{1}{3}(m)_2\,[\zeta_{m-2}(n)+2^3\,\{\zeta_{m-2}(n-1)+\zeta_{m-2}(n-2)\}+3^3\,\{\dots\}+\dots] \\
&+\tfrac{1}{5}(m)_4\,[\zeta_{m-4}(n)+2^5\,\{\zeta_{m-4}(n-1)+\zeta_{m-4}(n-2)\}+3^5\,\{\dots\}+\dots] \\
&\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\
&+\tfrac{1}{3}(m)_2[\zeta_3(n)+2^{m-2}\,\{\zeta_3(n-1)+\zeta_3(n-2)\}+3^{m-2}\,\{\dots\}+\dots] \\
&+\zeta(n)+2^m\,\{\zeta(n-1)+\zeta(n-2)\}+3^m\,\{\dots\}+\dots \\
&\qquad=\frac{1}{2(m+1)}\left\{p^{m+2}+p^{m+1}-2\begin{pmatrix}1^{m+1}+3^{m+1}+\dots+p^{m+1}\ (p\text{ uneven}) \\ 2^{m+1}+4^{m+1}+\dots+p^{m+1}\ (p\text{ even})\end{pmatrix}\right\},
\end{aligned}
$$

where, as before, $\tfrac{1}{2}p\,(p+1)$ is the triangular number next superior to n.

It follows, from § 53 of the previous paper, that the right-hand member

of this equation, for all values of p, is equal to

$$\frac{p^{m+2}}{2(m+2)} - B_1 p^m + 2^2 \frac{(m)_2}{3} \frac{B_2}{2} p^{m-2} - 2^4 \frac{(m)_4}{5} \frac{B_3}{3} p^{m-4} + \dots \pm 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)},$$

where B_r denotes the r-th Bernoullian number.

29. The corresponding σ-theorem (§ 54 of the previous paper) is

$$\sigma_m(n) - 2\{\sigma_m(n-1)+\sigma_m(n-2)\} + 3\{\sigma_m(n-3)+\sigma_m(n-4)+\sigma_m(n-5)\} - \dots$$

$$+\tfrac{1}{3}(m)_2 \left[\sigma_{m-2}(n) - 2^3\{\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)\} + 3^3\{\dots\} - \dots\right]$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$+\sigma(n) - 2^m\{\sigma(n-1)+\sigma(n-2)\} + 3^m\{\dots\} - \dots$$

$$= (-1)^p \left\{ \frac{p^{m+2}}{2(m+2)} - B_1 p^m + 2^2 \frac{(m)_2}{3} \frac{B_2}{2} p^{m-2} - \dots \pm 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)} p \right\}.$$

Thus the two formulæ only differ in the signs of the alternate sets of terms, and, when p is uneven, in the sign of the right-hand member.

30. As in the case of the σ-theorem, we may represent all the terms of the right-hand member, excepting only the first $\frac{p^{m+2}}{2(m+2)}$, by including the terms $\zeta(0)$, $\zeta_3(0)$, ..., $\zeta_m(0)$, and assigning to them the values

$$\zeta(0) = B_1, \quad \zeta_3(0) = -2^2 \frac{B_2}{2}, \quad \zeta_5(0) = 2^4 \frac{B_3}{3},$$

$$\dots, \quad \zeta_m(0) = (-1)^{\frac{1}{2}(m-1)} 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)},$$

which are independent of both m and n.

These values are equal and opposite in sign to those assigned to $\sigma(0)$, ..., $\sigma_m(0)$ in § 55 of the previous paper.

The Particular Cases $m = 1, 3, 5$. § 31.

31. Putting $m = 1$, we find

$$\zeta(n) + 2\{\zeta(n-1)+\zeta(n-2)\} + 3\{\zeta(n-3)+\zeta(n-4)+\zeta(n-5)\} + \dots$$

$$= \tfrac{1}{6}p(p^2-1),$$

which was given in Vol. xv. (1884), p. 118, of the *Proceedings* of this Society, and also in *Proc. Camb. Phil. Soc.*, Vol. v. (1884), p. 117.

Similarly, putting $m = 3$,

$$\zeta_3(n)+2\,\{\zeta_3(n-1)+\zeta_3(n-2)\}+3\,\{\dots\}+\dots$$
$$+\zeta(n)+2^3\{\zeta(n-1)+\zeta(n-2)\}+3^3\{\dots\}+\dots = \tfrac{1}{30}p(p^2-1)(3p^2-2),$$

and, putting $m = 5$,

$$\zeta_5(n)+2\,\{\zeta_5(n-1)+\zeta_5(n-2)\}+3\,\{\dots\}+\dots$$
$$+\tfrac{10}{3}[\zeta_3(n)+2^3\{\zeta_3(n-1)+\zeta_3(n-2)\}+3^3\{\dots\}+\dots$$
$$+\quad \zeta(n)+2^5\{\zeta(n-1)+\zeta(n-2)\}+3^5\{\dots\}+\dots$$
$$= \tfrac{1}{126}p(p^2-1)(9p^4-12p^2+16).$$

Second Theorem connecting the Actual Divisors of all Numbers from unity to n. §§ 32, 33.

32. From § 57 of the previous paper by substituting $\pi-x$ for x, we find

$$\{1+2\cos x(q+q^2)+(1+2\cos 2x)(q^3+q^4+q^5)$$
$$+(2\cos x+2\cos 3x)(q^6+\dots+q^9)+\dots\}\,\Sigma_1^\infty\,\zeta(\sin nx)\,q^n$$
$$= \sin x(q+q^2)+2\sin 2x(q^3+q^4+q^5)+(\sin x+3\sin 3x)(q^6+\dots+q^9)+\dots .$$

This equation shows that the numbers given by the formula

$$G'_n(d)+(G_{n-1}+G'_{n-2})(d\pm 1)+(G'_{n-3}+G'_{n-4}+G'_{n-5})(d,\ d\pm 2)$$
$$+(G'_{n-6}+\dots+G'_{n-9})(d\pm 1,\ d\pm 3)+\dots,$$

all cancel one another with the exception of

one 1, three 3's, five 5's, ..., $(p-1)(p-1)$'s, if p be even,

and two 2's, four 4's, six 6's, ..., $(p-1)(p-1)$'s, if p be even,

where, as before, $\frac{1}{2}p(p+1)$ is the triangular number next superior to n.

33. This result differs from the theorem of § 58 of the previous paper only by the reversal of the signs of the numbers derived from the even divisors, and the new form of the theorem stands to the old in the same relation as the theorem of § 24 stands to that of § 46 of the previous paper.

Second Theorem connecting ζ_m, ζ_{m-2}, ... of all Numbers from unity to n.
§ 34.

34. The ζ-theorem deducible from § 32 is

$$\zeta_m(n)+2\{\zeta_m(n-1)+\zeta_m(n-2)\}+3\{\zeta_m(n-3)+\zeta_m(n-4)+\zeta_m(n-5)\}+\dots$$
$$+2(m)_2\{\zeta_{m-2}(n-1)+\zeta_{m-2}(n-2)+2^2(\text{next three})+(1^2+3^2)(\text{next four})+\dots\}$$
$$+2(m)_4\{\zeta_{m-4}(n-1)+\zeta_{m-4}(n-2)+2^4(\quad\text{,,}\quad)+(1^4+3^4)(\quad\text{,,}\quad)+\dots\}$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$+2m\{\zeta(n-1)+\zeta(n-2)+2^{m-1}(\text{next three})+(1^{m-1}+3^{m-1})(\text{next four})+\dots\}$$
$$=\tfrac{1}{2}\frac{l^{m+2}}{m+2}+\tfrac{1}{2}l^{m+1}+(m+1)_1B_1l^m-2^2(m+1)_3\frac{B_2}{2}l^{m-2}+\dots,$$

where $\frac{1}{2}l(l+1)$ is the triangular number next inferior to n, or equal to n according as n is not or is a triangular number.

This formula differs from the corresponding σ-formula (§ 61 of the previous paper) only in the signs of the alternate sets of terms, and, when l is uneven, in the sign of the term on the right-hand side.

The Particular Cases $m = 1$ and 3. § 35.

35. Putting $m = 1$, we find

$$\zeta(n)+2\{\zeta(n-1)+\zeta(n-2)\}+3\{\zeta(n-3)+\zeta(n-4)+\zeta(n-5)\}+\dots$$
$$=\tfrac{1}{6}l^3+\tfrac{1}{2}l^2+\tfrac{1}{3}l,$$

and substituting $p-1$ for l, the right-hand member becomes $\frac{1}{6}(p^3-p)$ agreeing with § 31.

Putting $m = 3$,

$$\zeta_3(n)+2\{\zeta_3(n-1)+\zeta_3(n-2)\}+3\{\dots\}+\dots$$
$$+6\left\{\zeta(n-1)+\zeta(n-2)+\frac{3^3-3}{6}(\text{next three})+\frac{4^3-4}{6}(\text{next four})+\dots\right\}$$
$$=\tfrac{1}{10}l^5+\tfrac{1}{2}l^4+\tfrac{2}{3}l^3-\tfrac{4}{15}l,$$

that is,
$$\zeta_3(n)+2\{\zeta_3(n-1)+\zeta_3(n-2)\}+3\{\dots\}+\dots$$
$$+(2^3-2)\{\zeta(n-1)+\zeta(n-2)\}+(3^3-3)\{\dots\}+(4^3-4)\{\dots\}+\dots$$
$$=\tfrac{1}{30}(3p^5-10p^3+7p),$$

which is easily verified by means of the formulæ in § 31.

of all the even numbers), though in its new form it is different in appearance and simpler in statement.

The theorem of § 65 of the previous paper is that the numbers given by the formula

$$G_n\{d\pm 0,\ -(d\pm 1)\}-G_{n-1}\{d\pm 1,\ -(d\pm 2)\}+G_{n-3}\{d\pm 2,\ -(d\pm 3)\}-\dots$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g(g+1)$, in which case $(g+1)(-g)$'s and $g(g+1)$'s, or $(g+1)g$'s and $g\{-(g+1)\}$'s remain uncancelled, according as g is even or uneven. Zeros are not to be taken account of.

Writing the formula in the form

$$G_n\{d\pm 0,\ -(d\pm 1)\}+G_{n-1}\{-(d\pm 1),\ d\pm 2\}+G_{n-3}\{d\pm 2,\ -(d\pm 3)\}+\dots,$$

and applying it to the case $n=10$, we form the same scheme as before and change the signs of the numbers in the two outside lines in the first group, in the two inside lines in the second group, in the two outside lines in the third group, and so on. We thus obtain the numbers—

-2,	-3,	-6,	-11,	3,	5,	11,	-4,	-10,	5,	6,	8
1,	2,	5,	10,	-2,	-4,	-10,	3,	9,	-4,	-5,	-7
1,	2,	5,	10,	0,	-2,	-8,	-1,	5,	2,	1,	-1
0,	-1,	-4,	-9,	-1,	1,	7,	2,	-4,	-3,	-2,	0

These numbers differ from those obtained in § 37 only by a change of sign of the even numbers, and by comparing the formulæ

$$G'_n(d\pm 0,\ d\pm 1)+G'_{n-1}(d\pm 1,\ d\pm 2)+\dots,$$

and $$G_n\{d\pm 0,\ -(d\pm 1)\}+G_{n-1}\{-(d\pm 1),\ d\pm 2\}+\dots,\text{*}$$

it is evident that this is true generally.

39. It may be noted that the theorem of § 65 of the previous paper combined with that in § 35 of the present paper, shows that, if n is not a triangular number, the numbers given (after the cancelling) by each of the four following formulæ (in which d denotes any divisor, δ any uneven divisor, and D any even divisor) are all the same:—

(i.) $G_n(d\pm 0)+G_{n-1}(d\pm 2)+G_{n-3}(d\pm 2)+G_{n-6}(d\pm 4)+\dots,$

(ii.) $G_n(d\pm 1)+G_{n-1}(d\pm 1)+G_{n-3}(d\pm 3)+G_{n-6}(d\pm 3)+\dots,$

(iii.) $G_n(\delta\pm 0,\ \delta\pm 1)+G_{n-1}(\delta\pm 1,\ \delta\pm 2)+G_{n-3}(\delta\pm 2,\ \delta\pm 3)+\dots,$

(iv.) $G_n(D\pm 0,\ D\pm 1)+G_{n-1}(D\pm 1,\ D\pm 2)+G_{n-3}(D\pm 2,\ D\pm 3)+\dots.$

* $G_n\{\phi(d),\ \psi(d),\ \dots\}$ denotes the group of numbers $\phi(d_1),\ \phi(d_2),\ \dots,\ \psi(d_1),\ \psi(d_2),\ \dots,$ where $d_1,\ d_2,\ \dots$ are all the divisors of n (see § 2 of the previous paper).

In (i.) and (ii.) the arguments, except in the first term of (i.), are the same in pairs.

In the formula (iv.) half, or about half, of the terms give no numbers, for of the numbers n, $n-1$, $n-3$, ..., half, or about half, must be uneven, and so can have no even divisors.

Taking, as an example, $n = 7$, the four formulæ give the schemes

$$\begin{array}{lcccc}
\text{(i.)} & 1,\ 7, & 3,\ 4,\ 5,\ 8, & 3,\ 4,\ 6, & 5 \\
 & (1,\ 7), & (1,\ 2,\ 3,\ 6), & (1,\ 2,\ 4), & (1) \\
 & 1,\ 7, & -1,\ 0,\ 1,\ 4, & -1,\ 0,\ 2, & -3\ ;
\end{array}$$

$$\begin{array}{lcccc}
\text{(ii.)} & 2,\ 8, & 2,\ 3,\ 4,\ 7, & 4,\quad 5,\ 7, & 4 \\
 & (1,\ 7), & (1,\ 2,\ 3,\ 6), & (1,\quad 2,\ 4), & (1) \\
 & 0,\ 6, & 0,\ 1,\ 2,\ 5, & -2,\ -1,\ 1, & -2\ ;
\end{array}$$

$$\begin{array}{lcccc}
\text{(iii.)} & 2,\ 8, & 3,\ 5, & 4, & 5 \\
 & 1,\ 7, & 2,\ 4, & 3, & 4 \\
 & (1,\ 7), & (1,\ 3), & (1), & (1) \\
 & 1,\ 7, & 0,\ 2, & -1, & -2 \\
 & 0,\ 6, & -1,\ 1, & -2, & -3\ ;
\end{array}$$

$$\begin{array}{lcc}
\text{(iv.)} & 4,\ 8, & 5,\ 7 \\
 & 3,\ 7, & 4,\ 6 \\
 & (2,\ 6), & (2,\ 4) \\
 & 1,\ 5, & 0,\ 2 \\
 & 0,\ 4, & -1,\ 1.
\end{array}$$

In all four cases the numbers 8, 7, 7, 6, 5, 5, 4, 4, 4, 3, 2, 1 remain uncancelled.

40. By putting $\pi+x$ for x in the trigonometrical equation at the beginning of § 66 of the previous paper, we find that the numbers given by the formula

$$G'_n \left\{ \begin{array}{c} d+1 \\ -[d-1] \end{array} \right\} + G'_{n-1} \left\{ \begin{array}{cc} -(d+1), & d+2 \\ {[d-1]}, & -[d-2] \end{array} \right\}$$

$$+ G'_{n-3} \left\{ \begin{array}{cc} -(d+2), & d+3 \\ {[d-2]}, & -[d-3] \end{array} \right\} + \ldots,$$

where, in the first group, we add and subtract 2; in the second, we add and subtract 1 and 3; in the third, we add and subtract 2 and 4; in the fourth, 3 and 5; and so on; the groups containing the divisors of one, two, three, ... numbers.

We then change the signs of the even numbers in the first group, of the uneven numbers in the second group, of the even numbers in the third group, and so on. We thus obtain the numbers

$$\begin{array}{r}
4, -5, \quad 6, -9, \quad 4, \quad 8, \quad 5, -6, -8, \quad 5, \quad 7, \quad 5, -6, \quad 6 \\
3, \quad 9, \quad 2, -3, \quad 4, -7, \quad 2, \quad 6, \quad 3, -4, -6, \quad 3, \quad 5, \quad 3, -4, \quad 4 \\
-1, -5, -0, \quad 1, -2, \quad 5, -0, -4, -1, \quad 0, \quad 2, -1, -1, -1, \quad 0, -2 \\
-2, \quad 1, -0, \quad 3, -2, -2, -3, \quad 2, \quad 0, -3, -1, -3, \quad 2, -4
\end{array}$$

which cancel one another with the exception of

$$-1, \ -1, \ -1, \ -1, \ 3, \ 5, \ 5, \ 5,$$

which are left.

For all values of n either the even or the uneven numbers cancel one another completely.

General Formula connecting ζ_m, ζ_{m-2}. §§ 42, 43.

42. The recurring formula derived from § 35 seems deserving of notice on account of the simple form of the additional term.

Putting

$$G'_n(d, d, d\pm 1) = \Sigma_n(-1)^{d-1}\{d^m+d^m+(d-1)^m+(d+1)^m\},$$

where m is uneven, the theorem of § 35 shows that

$$\begin{aligned}
&\Sigma_n(-1)^{d-1}\{d^m+d^m+(d-1)^m+(d+1)^m\} \\
&+\Sigma_{n-1}(-1)^{d-1}\{(d-1)^m+(d+1)^m+(d-2)^m+(d+2)^m\} \\
&+\Sigma_{n-3}(-1)^{d-1}\{(d-2)^m+(d+2)^m+(d-3)^m+(d+3)^m\}+\dots \\
&\qquad\qquad = [g(g+1)^m+(g+1)g^m],
\end{aligned}$$

giving the ζ-formula

$$\begin{aligned}
&\zeta_m(n)+\zeta_m(n-1)+\zeta_m(n-3)+\zeta_m(n-6)+\dots \\
&+\tfrac{1}{2}(m)_2\{\zeta_{m-2}(n)+(1^2+2^2)\zeta_{m-2}(n-1)+(2^2+3^2)\zeta_{m-2}(n-3)+\dots\} \\
&+\tfrac{1}{2}(m)_4\{\zeta_{m-4}(n)+(1^4+2^4)\zeta_{m-4}(n-1)+(2^4+3^4)\zeta_{m-4}(n-3)+\dots\} \\
&\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&+\tfrac{1}{2}m\{\zeta(n)+(1^{m-1}+2^{m-1})\zeta(n-1)+(2^{m-1}+3^{m-1})\zeta(n-3)+\dots \\
&\qquad\qquad = [\tfrac{1}{2}\{g(g+1)^m+(g+1)g^m\}],
\end{aligned}$$

in which the coefficients in the series involving ζ_{m-2r} are

$$1^{2r}+2^{2r},\ 2^{2r}+3^{2r},\ 3^{2r}+4^{2r},\ 4^{2r}+5^{2r},\ \dots .$$

The right-hand member of this equation $= [\frac{1}{2}n\,\{g^{m-1}+(g+1)^{m-1}\}]$, and since, when $n=\frac{1}{2}g(g+1)$, the coefficient of $\zeta(0)$ in the last series of the formula is $g^{m-1}+(g+1)^{m-1}$, it is evident that we may dispense with the additional term if we assign to $\zeta(0)$ the value $-n/m$, the other quantities $\zeta_3(0)$, $\zeta_5(0)$, ..., $\zeta_m(0)$ all retaining their zero values.

43. As particular cases, putting $m=1$, we find

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\dots=0,$$

where $\zeta(0)=-n$.

Putting $m=3$, we find

$$\zeta_3(n)+\zeta_3(n-1)+\zeta_3(n-3)+\zeta_3(n-6)+\dots$$
$$+\tfrac{3}{2}\,\{\zeta(n)+(1^2+2^2)\,\zeta(n-1)+(2^2+3^2)\,\zeta(n-3)+\dots\}=0,$$

where $\zeta(0)=-\frac{1}{3}n$; and, putting $m=5$,

$$\zeta_5(n)+\zeta_5(n-1)+\zeta_5(n-3)+\zeta_5(n-6)+\dots$$
$$+5\,\{\zeta_3(n)+(1^2+2^2)\,\zeta_3(n-1)+(2^2+3^2)\,\zeta_3(n-3)+\dots\}$$
$$+\tfrac{5}{2}\,\{\zeta(n)+(1^4+2^4)\,\zeta(n-1)+(2^4+3^4)\,\zeta(n-3)+\dots\},$$

where $\zeta(0)=-\frac{1}{5}n$.

These formulæ may be easily deduced from those in § 22.

The corresponding σ-formula. §§ 44, 45.

44. The corresponding σ-formula may be deduced from § 64 of the previous paper, by putting

$$G_n\{d,\,d,\,-(d\pm1)\}=\Sigma_n\{d^{m+2}+d^{m+2}-(d-1)^{m+2}-(d+1)^{m+2}\},$$

where m is uneven. We thus find

$$2(m+2)_2\,\{\sigma_m(n)-(2^2-1^2)\,\sigma_m(n-1)+(3^2-2^2)\,\sigma_m(n-3)-\dots\}$$
$$+2(m+2)_4\,\{\sigma_{m-2}(n)-(2^4-1^4)\,\sigma_{m-2}(n-1)+(3^4-2^4)\,\sigma_{m-2}(n-3)-\dots\}$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$+2(m+2)\,\{\sigma(n)-(2^{m+1}-1^{m+1})\,\sigma(n-1)+(3^{m+1}-2^{m+1})\,\sigma(n-3)-\dots\}$$
$$=[(-1)^{g-1}\,\{g(g+1)^{m+2}-(g+1)\,g^{m+2}\}].$$

We may also write the theorem in the form

$$\tfrac{1}{2}(m+1)\{\sigma_m(n)-(2^2-1^2)\,\sigma_m(n-1)+(3^2-2^2)\,\sigma_m(n-3)-\ldots\}$$
$$+\tfrac{1}{4}(m+1)_3\{\sigma_{m-2}(n)-(2^4-1^4)\,\sigma_{m-2}(n-1)+(3^4-2^4)\,\sigma_{m-2}(n-3)-\ldots\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+\sigma(n)-(2^{m+1}-1^{m+1})\,\sigma(n-1)+(3^{m+1}-2^{m+1})\,\sigma(n-3)-\ldots = 0,$$

where $\sigma(0)$ is to have the value $\dfrac{n}{m+2}$, the other quantities $\sigma_3(0)$, $\sigma_5(0)$, ... all retaining their zero values.

45. As particular cases, we have, putting $m = 1$,

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+\ldots = 0,$$

where $\sigma(0) = \tfrac{1}{3}n$, and, putting $m = 3$,

$$\sigma_3(n)-3\sigma_3(n-1)+5\sigma_3(n-3)-7\sigma_3(n-6)+\ldots$$
$$+\tfrac{1}{2}\{\sigma\;(n)-3(1^2+2^2)\,\sigma(n-1)+5(2^2+3^2)\,\sigma(n-3)-\ldots\} = 0,$$

where $\sigma(0) = \tfrac{1}{5}n$.

PART II.—THEOREMS RELATING TO DIVISORS WHICH HAVE UNEVEN CONJUGATES. §§ 46–71.

Definitions. § 46.

46. The preceding portion of this paper was concerned with all the divisors of a number, each divisor having the positive or negative sign, according as it was uneven or even. In this portion of the paper only those divisors are involved which have uneven conjugates.

In accordance with the notation employed in the previous paper, I use

$$G_n\{\phi(\delta'),\ \psi(\delta'),\ \chi(\delta'),\ \ldots\}$$

to denote the group of numbers

$$\begin{gathered}\phi(\delta'_1),\ \phi(\delta'_2),\ \ldots,\ \phi(\delta'_f),\\ \psi(\delta'_1),\ \psi(\delta'_2),\ \ldots,\ \psi(\delta'),\\ \chi(\delta'_1),\ \chi(\delta'_2),\ \ldots,\ \chi(\delta'_f),\\ \ldots \quad \ldots \quad \ldots\end{gathered}$$

where δ'_1, δ'_2, ..., δ'_f are all the divisors of n which have uneven conjugates.

It is evident that, if n is uneven, $\delta'_1, \delta'_2, \ldots, \delta'$ are all the divisors of n; and that, if n is even, $\delta'_1, \delta'_2, \ldots, \delta'$ are all even; in fact, if $n = 2^r m$, where m is uneven, $\delta'_1, \delta'_2, \ldots, \delta'$ consist of the divisors of m each multiplied by 2^r.

Formulæ derived from the Function zd. § 47.

47. For the function zd x, we have the two following q-formulæ

$$\rho \,\mathrm{zd}\, \rho x = \frac{4q \sin x - 8q^4 \sin 2x + 12q^9 \sin 3x - \ldots}{1 - 2q \cos x + 2q^4 \cos 2x - 2q^9 \cos 3x + \ldots},$$

$$\rho \,\mathrm{zd}\, \rho x = \frac{4q}{1-q^2} \sin x + \frac{4q^2}{1-q^4} \sin 2x + \frac{4q^3}{1-q^6} \sin 3x + \ldots.$$

Expanding the coefficients in the second equation, we find

$$\rho \,\mathrm{zd}\, \rho x = 4\Sigma_1^\infty \Delta'(\sin nx)\, q^n,$$

where $\Delta'\phi(n)$ denotes the sum $\phi(\delta'_1) + \phi(\delta'_2) + \ldots + \phi(\delta'_f)$.

Thus, equating the values of $\rho \,\mathrm{zd}\, \rho x$,

$$\frac{q \sin x - 2q^4 \sin 2x + 3q^9 \sin 3x - \ldots}{1 - 2q \cos x + 2q^4 \cos 2x - 2q^9 \cos 3x + \ldots} = \Sigma_1^\infty \Delta'(\sin nx)\, q^n.$$

Theorems relating to Actual Divisors. § 48.

48. This last formula may be written

$$(1 - 2q \cos x + 2q^4 \cos 2x - 2q^9 \cos 3x + \ldots)\, \Sigma_1^\infty \Delta'(\sin nx)\, q^n$$
$$= q \sin x - 2q^4 \sin 2x + 3q^9 \sin 3x - \ldots,$$

whence, equating the coefficients of q^n,

$$\Sigma_n \sin \delta' x - \Sigma_{n-1} \{\sin(\delta'+1)x + \sin(\delta'-1)x\}$$
$$+ \Sigma_{n-4} \{\sin(\delta'+2)x + \sin(\delta'-2)x\} + \ldots$$
$$= 0 \;\text{ or }\; (-1)^{n-1} \sqrt{n} \sin(\sqrt{n}\, x),$$

according as n is not or is a square number.

This equation gives the following theorem connecting actual divisors of $n, n-1, n-4, \ldots$:—

The numbers given by the formula

$$G_n(\delta') - G_{n-1}(\delta' \pm 1) + G_{n-4}(\delta' \pm 2) - G_{n-9}(\delta' \pm 3) + \ldots$$

all cancel one another, unless n is a square number r^2, in which case r r's or r $(-r)$'s remain uncancelled, according as r is uneven or even.

As an example, take $n = 15$. We form the scheme

$$\begin{array}{l} 3, 15, \quad 3, 13, \quad 5, 9, \\ 1, 3, 5, 15, -(2, 14) + (1, 11) - (2, 6), \\ 1, 13, -1, \quad 9, -1, 3 \end{array}$$

(the numbers in brackets being the values of δ' for $n-1$, $n-4$, ...), whence we derive the numbers given by the formula, viz.,

$$1, 3, 5, 15, \begin{array}{l} -3, -15, \quad 3, 13, -5, -9, \\ -1, -13, -1, \quad 9, \quad 1, -3, \end{array}$$

all of which cancel one another.

Taking $n = 16$, the scheme is

$$\begin{array}{l} 2, 4, 6, 16, \quad 6, 14, \quad 4, 10, \\ 16, -(1, 3, 5, 15) + (4, 12) - (1, \ 7), \\ 0, 2, 4, 14, \quad 2, 10, -2, \ 4, \end{array}$$

giving

$$16, \begin{array}{l} -2, -4, -6, -16, 6, 14, -4, -10, \\ 0, -2, -4, -14, 2, 10, \quad 2, -4, \end{array}$$

in which four (-4)'s are left uncancelled.

Taking $n = 25$, the scheme is

$$\begin{array}{l} 9, 25, \quad 3, 5, 9, 23, \quad 19, \quad 5, \quad 7, 13 \\ 1, 5, 25, -(8, 24) + (1, 3, 7, 21) - (16) + (1, \quad 3, \ 9) \\ 7, 23, -1, 1, 5, 19, \quad 13, -3, -1, \ 5, \end{array}$$

giving

$$1, 5, 25, \begin{array}{l} -9, -25, \quad 3, 5, 9, 23, -19, \quad 5, \quad 7, 13, \\ -7, -23, -1, 1, 5, 19, -13, -3, -1, \ 5, \end{array}$$

in which five 5's are left uncancelled.

General Formula connecting $\Delta'_m, \Delta'_{m-2}, \ldots$ *of* $n, n-1, n-4, \ldots$.
§§ 49, 50.

49. By putting $G_n(\delta') = \Sigma_n \delta'^m, \ldots, m$ being uneven, we find that

$$\begin{aligned}
&\Delta'_m(n)-2\Delta'_m(n-1)+2\Delta'_m(n-4)-2\Delta'_m(n-9)+\ldots \\
&-2(m)_2\{\Delta'_{m-2}(n-1)-2^2\Delta'_{m-2}(n-4)+3^2\Delta'_{m-2}(n-9)-\ldots\} \\
&-2(m)_4\{\Delta'_{m-4}(n-1)-2^4\Delta'_{m-4}(n-4)+3^4\Delta'_{m-4}(n-9)-\ldots\} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&-2m\{\Delta'(n-1)-2^{m-1}\Delta'(n-4)+3^{m-1}\Delta'(n-9)-\ldots\} \\
&\qquad\qquad\qquad\qquad = [(-1)^{n-1} n^{\frac{1}{2}(m+1)}],
\end{aligned}$$

where the square brackets indicate that the quantity included in them occurs only when n is a square number; otherwise the right-hand side is zero.

The right-hand side is always zero if we assign to $\Delta'(0)$ the value $n/2m$, the other quantities $\Delta'_3(0), \Delta'_5(0), \ldots, \Delta'_m(0)$ all retaining their zero values.

50. As the particular case of $m = 1$, we have

$$\Delta'(n)-2\Delta'(n-1)+2\Delta'(n-4)-2\Delta'(n-9)-\ldots = 0,$$

where $\Delta'(0) = \frac{1}{2}n$. This formula was given on p. 68 of the previous paper.

Putting $m = 3$, we have

$$\begin{aligned}
&\Delta'_3(n)-2\Delta'_3(n-1)+2\Delta'_3(n-4)-2\Delta'_3(n-9)+\ldots \\
&-6\{\Delta'(n-1)-2^2\Delta'(n-4)+3^2\Delta'(n-9)-\ldots\} = 0,
\end{aligned}$$

where $\Delta'_3(0) = \frac{1}{6}n$; and, putting $m = 5$,

$$\begin{aligned}
&\Delta'_5(n)-2\Delta'_5(n-1)+2\Delta'_5(n-4)-2\Delta'_5(n-9)+\ldots \\
&-20\{\Delta'_3(n-1)-2^2\Delta'_3(n-4)+3^2\Delta'_3(n-9)-\ldots\} \\
&-10\{\Delta'(n-1)-2^4\Delta'(n-4)+3^4\Delta'(n-9)-\ldots\} = 0,
\end{aligned}$$

where $\Delta'_5(0) = \frac{1}{10}n$.

Theorem relating to Actual Divisors of all Numbers from unity to n.
§§ 51, 52.

51. By multiplying the first formula in § 48 by $1+q+q^2+q^3+\ldots$, we find that

$$\{1+(1-2\cos x)(q+q^2+q^3)+(1-2\cos x+2\cos 2x)(q^4+\ldots+q^8)+\ldots\}$$
$$\times \Sigma_1^\infty \Delta'(\sin nx)\, q^n$$
$$= \sin x(q+q^2+q^3)+(\sin x-2\sin 2x)(q^4+\ldots+q^8)$$
$$+(\sin x-2\sin 2x+3\sin 3x)(q^9+\ldots+q^{15})+\ldots.$$

This equation shows that the numbers given by the formula

$$G_n(\delta')+(G_{n-1}+G_{n-2}+G_{n-3})\{\delta',\ -(\delta'\pm 1)\}$$
$$+(G_{n-4}+G_{n-5}+\ldots+G_{n-8})\{\delta',\ -(\delta'\pm 1),\ \delta'\pm 2\}+\ldots,$$

all cancel one another with the exception of

one 1, two (-2)'s, three 3's, four (-4)'s, ..., $r(-1)^{r-1}\,r$'s,

where r^2 is the square next inferior to, or equal to, n.

For example, taking $n=8$, the formula gives the numbers

$$\begin{array}{c|cccccc|ccccc} & & & & & & & 6, & 3, & 5, & 4, & 3 \\ & -2, & -8, & -3, & -7, & -2, & -6 & -5, & -2, & -4, & -3, & -2 \\ 8 & 1, & 7; & 2, & 6; & 1, & 5 & 4; & 1, & 3; & 2; & 1 \\ & 0, & -6, & -1, & -5, & 0, & -4 & -3, & 0, & -2, & -1, & 0 \\ & & & & & & & 2, & -1, & 1, & 0, & -1 \end{array}$$

which cancel one another except $1,\ -2,\ -2$.

To obtain the numbers given by the formula, we write in the central line the δ''s of n, $n-1$, $n-2$, ... divided into sets of one, three, five, ... (so that the first numbers in the sets are n, $n-1$, $n-4$, ...). Beginning with the second set we write, in lines above and below, the values of $\delta'+1$ and $\delta'-1$. Beginning with the third set, we write, in lines above and below, the values of $\delta'+2$ and $\delta'-2$, and so on. We then change the signs of the numbers in the two rows next, above and below, to the central line, in the rows next but two, above and below, to the central line, and so on.

As another example, take $n = 9$. The formula gives the numbers

$$\begin{array}{r|l|l}
 & & 3,\ 7,\ 6,\ 3,\ 5,\ 4,\ 3 \\
 & -9,\ -2,\ -8,\ -3,\ -7 & -2,\ -6,\ -5,\ -2,\ -4,\ -3,\ -2 \\
1,\ 3,\ 9 & 8;\ 1,\ 7;\ 2,\ 6 & 1,\ 5;\ 4;\ 1,\ 3;\ 2,\ 1 \\
 & -7,\ 0,\ -6,\ -1,\ -5 & 0,\ -4,\ -3,\ 0,\ -2,\ -1,\ 0 \\
 & & -1,\ 3,\ 2,\ -1,\ 1,\ 0,\ -1
\end{array}$$

which cancel one another except 1, -2, -2, 3, 3, 3.

52. We may enunciate the theorem in a somewhat different form as follows.*

Write down the numbers given by

$$G_n(\delta')+(G_{n-1}+G_{n-2}+G_{n-3})(\delta',\ \delta'\pm 1)$$

$$+(G_{n-4}+\ldots+G_{n-8})(\delta',\ \delta'\pm 1,\ \delta'\pm 2)+\ldots,$$

as before, but instead of changing the signs of the rows next, next but two, ..., to the central line, change the signs of the columns which have even numbers in the central line. The numbers remaining uncancelled are then one 1, two 2's, three 3's, ..., r r's.

As an example, taking $n = 8$, this process gives the numbers

$$\begin{array}{r|l l}
 & & -6,\ 3,\ 5,\ -4,\ 3 \\
 & 2,\ 8,\ -3,\ -7,\ 2,\ 6 & -5,\ 2,\ 4,\ -3,\ 2 \\
-8 & 1,\ 7;\ -2,\ -6;\ 1,\ 5 & -4;\ 1,\ 3;\ -2;\ 1 \\
 & 0,\ 6,\ -1,\ -5,\ 0,\ 4 & -3,\ 0,\ 2,\ -1,\ 0 \\
 & & -2,\ -1,\ 1,\ 0,\ -1
\end{array}$$

which cancel one another except 1, 2, 2.

---

* The theorem in this form may be deduced from the trigonometrical formula in § 51, by putting $\pi - x$ for x. We thus find that the numbers given by the formula

$$(-1)^{n-1}\{G_n(\delta')+(-G_{n-1}+G_{n-2}-G_{n-3})(\delta',\ \delta'\pm 1)$$
$$+(G_{n-4}-G_{n-5}+\ldots+G_{n-8})(\delta',\ \delta'\pm 1,\ \delta'\pm 2)+\ldots\}$$

cancel one another, with the exception of one 1, two 2's, three 3's, ..., r r's.

General Formula connecting Δ'_m, Δ'_{m-2}, ... of all Numbers from unity to n. §§ 53, 54.

53. By putting $G_n(\delta') = \Sigma_n \delta'^m$, ..., m being uneven, in the theorem of § 51, we find that

$$\Delta'_m(n)-\Delta'_m(n-1)-\Delta'_m(n-2)-\Delta'_m(n-3)+\text{next five}-\text{next seven}+\dots$$

$$-2(m)_2\{\Delta'_{m-2}(n-1)+\Delta'_{m-2}(n-2)+\Delta'_{m-2}(n-3)-(2^2-1^2)(\text{next five})$$
$$+(3^2-2^2+1^2)(\text{next seven})-\dots\}$$

$$-2(m)_4\{\Delta'_{m-4}(n-1)+\dots+\Delta'_{m-4}(n-3)-(2^4-1^4)(\text{next five})$$
$$+(3^4-2^4+1^4)(\text{next seven})-\dots\}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$-2m\{\Delta'(n-1)+\dots+\Delta'(n-3)-(2^{m-1}-1^{m-1})(\text{next five})$$
$$+(3^{m-1}-2^{m-1}+1^{m-1})(\text{next seven})-\dots\}$$

$$= (-1)^{r-1}\{r^{m+1}-(r-1)^{m+1}+(r-2)^{m+1}-\dots\pm 1^{m+1}\},$$

where r^2 is the square next inferior to, or equal to, n.

54. As a particular case, putting $m = 1$, this formula gives

$$\Delta'(n)-\Delta'(n-1)-\Delta'(n-2)-\Delta'(n-3)+\text{next five}-\text{next seven}+\dots$$
$$= (-1)^{r-1}\tfrac{1}{2}r(r+1),^*$$

and, putting $m = 3$,

$$\Delta'_3(n)-\Delta'_3(n-1)-\Delta'_3(n-2)-\Delta'_3(n-3)+\text{next five}-\text{next seven}+\dots$$
$$-6\{\Delta'(n-1)+\Delta'(n-2)+\Delta'(n-3)-3(\text{next five})+6(\text{next seven})-\dots\}$$
$$= (-1)^{r-1}\tfrac{1}{2}(r+r)(r^2+r-1).$$

Other Theorems relating to Actual Divisors of all Numbers from unity to n. §§ 55–57.

55. Putting $2x$ for x in the trigonometrical formula of § 51, and noticing that

$$1-2\cos 2x = -\frac{\cos 3x}{\cos x},$$

$$1-2\cos 2x+2\cos 4x = \frac{\cos 5x}{\cos x}, \ \dots,$$

---

* This formula was given in the note to § 68 of the previous paper (Vol. XXII., p. 409). In that paper the right-hand side of the formula is put equal to zero, the quantity $\Delta'(0)$ having the value $\frac{1}{2}r(r+1)$, where $2r+1$ is the number of terms in the completed group in which $\Delta'(0)$ falls. It is easy to see that this definition of r agrees with that used in the text.

we find that

$$2\{\cos x-\cos 3x\,(q+q^2+q^3)+\cos 5x(q^4+\ldots+q^8)-\ldots\}\,\Sigma_1^\infty\,\Delta'(\sin 2nx)\,q^n$$
$$=(\sin x+\sin 3x)(q+q^2+q^3)+(\sin x-\sin 3x-2\sin 5x)(q^4+\ldots+q^8)+\ldots,$$

the coefficient of $q^{r^2}+q^{r^2+1}+\ldots+q^{r^2+2r}$ on the right-hand side being

$$\sin x-\sin 3x+\sin 5x-\ldots+(-1)^{r-1}\sin(2r-1)\,x+(-1)^{r-1}\,r\sin(2r+1)\,x.$$

From this formula we may deduce the following theorem relating to the actual divisors of the numbers n, $n-1$, ..., 1.

The numbers given by the formula

$$G_n(2\delta'\pm1)-(G_{n-1}+G_{n-2}+G_{n-3})(2\delta'\pm3)+(G_{n-4}+\ldots+G_{n-8})(2\delta'\pm5)-\ldots$$

cancel one another with the exception of

one 1, one -3, one 5, ..., one $(-1)^{r-1}(2r-1)$ and $r\,\{(-1)^{r-1}(2r+1)\}$'s,

where r has the same meaning as before, viz., r^2 is the square next inferior to, or equal to, n.

Taking 8 as an example, we form the scheme

$$\begin{array}{r|l}
17 & 5,\ 17,\quad 7,\ 15,\quad 5,\ 13\quad 13,\quad 7,\ 11,\quad 9,\quad 7 \\
(8) & (1,\ \ 7);\ (2,\ \ 6);\ (1,\ \ 5)\ \ (4);\ \ (1,\ \ 3);\ \ (2);\ \ (1) \\
15 & -1,\ 11,\quad 1,\ \ 9,\ -1,\ \ 7)\quad 3,\ -3,\ \ 1,\ \ -1,\ -3.
\end{array}$$

The formula therefore gives the numbers

$$\begin{array}{l}
17,\ -5,\ -17,\ -7,\ -15,\ -5,\ -13,\ 13,\quad 7,\ 11,\quad 9,\quad 7 \\
15,\quad 1,\ -11,\ -1,\ -\ 9,\quad 1,\ -\ 7,\ \ 3,\ -3,\ \ 1,\ -1,\ -3
\end{array}$$

which cancel one another except 1, -3, -5, -5.

56. Putting $\frac{1}{2}\pi-x$ for x in the trigonometrical formula of the preceding section, we obtain the equation

$$2\{\sin x+\sin 3x(q+q^2+q^3)+\sin 5x\,(q^4+\ldots+q^8)+\ldots\}$$
$$\times\,\Sigma_1^\infty\,(-1)^{n-1}\,\Delta'(\sin 2nx)\,q^n$$
$$=(\cos x-\cos 3x)(q+q^2+q^3)+(\cos x+\cos 3x-2\cos 5x)(q^4+\ldots+q^8)+\ldots,$$

the coefficient of $q^{r^2}+\ldots+q^{r^2+2r}$ on the right-hand side being

$$\cos x+\cos 3x+\ldots+\cos(2r-1)\,x-r\cos(2r+1)\,x.$$

This equation shows that the numbers given by the formula

$$
\begin{aligned}
&(-1)^n G_n \left\{ \begin{matrix} 2\delta'+1 \\ -[2\delta'-1] \end{matrix} \right\} \\
&+\{(-1)^{n-1}G_{n-1}+(-1)^{n-2}G_{n-2}+(-1)^{n-3}G_{n-3}\} \left\{ \begin{matrix} 2\delta'+3 \\ -[2\delta'-3] \end{matrix} \right\} \\
&+\{(-1)^{n-4}G_{n-4}+\ldots+(-1)^{n-8}G_{n-8}\} \left\{ \begin{matrix} 2\delta'+5 \\ -[2\delta'-5] \end{matrix} \right\} +\ldots,
\end{aligned}
$$

where $[2\delta'-s]$ denotes the absolute value of $2\delta'-s$, cancel one another with the exception of

$$\text{one } 1, \text{ one } 3, \ldots, \text{ one } (2r-1), \text{ and } r\{-(2r+1)\}\text{'s},$$

where r has the same meaning as in the preceding section.

In order to obtain the numbers given by this formula, we place, as before, the δ''s of $n, n-1, n-2, \ldots$ in a central line. We write $2\delta'+1$ and $2\delta'-1$ above and below in the first set (of one), $2\delta'+3$ and $2\delta'-3$ above and below in the second set (of three), $2\delta'+5$ and $2\delta'-5$ above and below in the third set (of five), and so on. We then make all the signs in the lower line negative by changing the signs of those that are positive. Finally, we change the signs of the numbers which are above and below uneven numbers in the central line.*

Taking $n = 8$, the process is therefore as follows. We form the scheme

$$
\begin{array}{r|rrrrrr|rrrrr}
17 & 5, & 17, & 7, & 15, & 5, & 13 & 13, & 7, & 11, & 9, & 7 \\
(8) & (1, & 7); & (2, & 6); & (1, & 5) & (4); & (1, & 3); & (2); & (1) \\
-15 & -1, & -11, & -1, & -9, & -1, & -7 & -3, & -3, & -1, & -1, & -3
\end{array}
$$

whence, by changing the signs of the numbers above and below uneven numbers in the central line, we obtain the numbers given by the formula, viz.,

$$
\begin{array}{rrrrrrrrrrrr}
17, & -5, & -17, & 7, & 15, & -5, & -13, & 13, & -7, & -11, & 9, & -7 \\
-15, & 1, & 11, & -1, & -9, & 1, & 7, & -3, & 3, & 1, & -1, & 3
\end{array}
$$

which cancel one another except $1, 3, -5, -5$.

* If we change the signs of the numbers above and below the even numbers, the numbers left uncancelled are $-1, -3, \ldots, -(2r-1)$, and $r(2r+1)$'s.

57. The numbers given by the formulæ in §§ 55 and 56 differ only by the reversal of the sign of those of the form $4k+3$. This can be explained by similar reasoning to that employed in § 15. Thus taking the upper line, and, supposing n even, the forms of the numbers are

$$\begin{array}{c|ccc|cc} 4k+1 & 4k+1, & 4k+3, & 4k+1 & 4k+1, & 4k+3, \ \dots \\ (n) & (n-1), & (n-2), & (n-3) & (n-4), & (n-5), \ \dots \end{array}$$

By the process of § 55 we obtain numbers of the form

$$4k+1, \ -(4k+1), \ -(4k+3), \ -(4k+1), \ 4k+1, \ 4k+3, \ \dots,$$

and, by the process of § 56, numbers of the form

$$4k+1, \ -(4k+1), \ 4k+3, \ -(4k+1), \ 4k+1, \ -(4k+3), \ \dots,$$

which differ by a change of sign in the numbers of the form $4k+3$.

This reasoning also gives the same result when applied to the negative numbers in the lower line, but in the case of the positive numbers it shows that the numbers obtained by changing the signs of the numbers in the alternate sets differ from those obtained by changing the signs of the numbers derived from the uneven divisors, only by the reversal of sign in the numbers of the form $4k+1$. If, therefore, we first change the sign of the positive numbers in the lower line, as is done in the process of § 56, the two processes yield numbers in which those of the form $4k+1$ have the same sign, and those of the form $4k+3$ have opposite signs.

A similar argument holds good when n is uneven.

General Formulæ connecting Δ'_m, Δ'_{m-2}, ... of all Numbers from unity to n.
§§ 58–62.

58. By putting $G_n(2\delta'\pm 1) = \Sigma_n \{(2\delta'+1)^m+(2\delta'-1)^m\}$, ..., where m is uneven, in the theorem of § 55, we find that

$$\begin{aligned} & 2^{m+1}\{\Delta'_m(n)-\Delta'_m(n-1)-\Delta'_m(n-2)-\Delta'_m(n-3)+\text{next five} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\text{next seven}+\dots\} \\ & +2^{m-1}(m)_2\{\Delta'_{m-2}(n)-3^2(\text{next three})+5^2(\text{next five})-\dots\} \\ & +2^{m-3}(m)_4\{\Delta'_{m-4}(n)-3^4(\text{next three})+5^4(\text{next five})-\dots\} \\ & \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & +2^2 m\{\Delta'(n)-3^{m-1}(\text{next three})+5^4(\text{next five})-\dots\} \\ & \qquad\qquad = 1^m-3^m+5^m+\dots+(-1)^{r-1}(2r-1)^m+(-1)^{r-1}r(2r+1)^m, \end{aligned}$$

where r^2 is the square next inferior to, or equal to, n.

59. As a particular case, putting $m = 1$, this formula gives

$$4\{\Delta'(n)-\Delta'(n-1)-\Delta'(n-2)-\Delta'(n-3)+\text{next five}-\dots\}$$
$$=(-1)^{r-1}r+(-1)^{r-1}r(2r+1)=(-1)^{r-1}2(r^2+r),$$

which agrees with the result in § 54.

Putting $m = 3$, we find

$$16\{\Delta_3'(n)-\Delta_3'(n-1)-\Delta_3'(n-2)-\Delta_3'(n-3)+\text{next five}-\dots$$
$$+12\{\Delta'(n)-3^2\,(\text{next three})+5^2\,(\text{next five})-\dots\}$$
$$=(-1)^{r-1}(4r^3-3r)+(-1)^{r-1}r(2r+1)^3,$$

that is,

$$4\{\Delta_3'(n)-\Delta_3'(n-1)-\Delta_3'(n-2)-\Delta_3'(n-3)+\text{next five}-\dots\}$$
$$+\ 3\{\Delta'(n)-3^2\,(\text{next three})+5^2\,(\text{next five})-\dots\}$$
$$=(-1)^{r-1}\tfrac{1}{2}(r^2+r)(4r^2+4r-1).$$

60. Similarly, by putting

$$G_n\left\{\begin{matrix}2\delta'+1\\-[2\delta'-1]\end{matrix}\right\}=\Sigma_n\{(2\delta'+1)^{m+1}-(2\delta'-1)^{m+1}\},\ \dots,$$

in the formula of § 56, we find that

$$2^{m+1}(m+1)[\Delta_m'(n)+3\{-\Delta_m'(n-1)+\Delta_m'(n-2)-\Delta_m'(n-3)\}$$
$$+5(\text{next five})+7(\text{next seven})+\dots]$$
$$+2^{m-1}(m+1)_3[\Delta_{m-2}'(n)+3^3\{-\Delta_{m-2}'(n-1)+\Delta_{m-2}'(n-2)-\Delta_{m-2}'(n-3)\}$$
$$+5^3(\text{next five})+\dots]$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$+2^2(m+1)[\Delta'(n)+3^{m-1}\{-\Delta'(n-1)+\Delta'(n-2)-\Delta'(n-3)\}$$
$$+5^{m-1}(\text{next five})+\dots]$$
$$=(-1)^n\{1^{m+1}+3^{m+1}+\dots+(2r-1)^{m+1}-r(2r+1)^{m+1}\}.$$

The terms in the series on the left-hand side are alternately positive and negative. The definition of r is the same as in § 58.

61. As particular cases, putting $m = 1$, we have

$$\Delta'(n)+3\{-\Delta'(n-1)+\Delta'(n-2)-\Delta'(n-3)\}+5\,(\text{next five})+\dots$$
$$=(-1)^{n-1}\tfrac{1}{6}r(r+1)(2r+1),$$

and, putting $m = 3$,

$$4\left[\Delta_3'(n)+3\ \{-\Delta_3'(n-1)+\Delta_3'(n-2)-\Delta_3'(n-3)\}+5\ (\text{next five})+\ldots\right]$$

$$+\Delta'(n)+3^3\{-\Delta'(n-1)+\Delta_3'(n-2)-\Delta_3'(n-3)\}+5^3\,(\text{next five})+\ldots$$

$$= (-1)^{n-1}\,\tfrac{1}{30}r(r+1)(2r+1)(12r^2+12r+1).$$

62. It is noticeable that the expression which occurs on the right-hand side of the general formula in § 60 is the same as that which occurs in the general formula of § 33 of the previous paper, and that the coefficients of the series on the left-hand side in the two formulæ are identical. This arises from the similarity in form between the trigonometrical formulæ of § 56 of this paper and § 28 of the previous paper.

Other Theorems relating to Actual Divisors of all Numbers from unity to n. §§ 63–65.

63. By multiplying the trigonometrical formula in § 55 by $\sin x$, and replacing x by $\frac{1}{2}x$, we obtain the formula

$$2\{1+\cos x-(\cos x+\cos 2x)(q+q^2+q^3)+(\cos 2x+\cos 3x)(q^4+\ldots+q^8)-\ldots\}$$

$$\times\,\Sigma_1^\infty\,\Delta'(\sin nx)\,q^n$$

$$= (2\sin x+\sin 3x)(q+q^2+q^3)-(3\sin 2x+2\sin 2x)(q^4+\ldots+q^8)+\ldots,$$

the coefficient of $q^{r^2}+\ldots+q^{r^2+2r}$ on the right-hand side being

$$(-1)^{r-1}\{(r+1)\sin rx+r\sin(r+1)x\}.$$

This equation shows that the numbers given by the formula

$$G_n(\delta', \delta', \delta'\pm1)-(G_{n-1}+G_{n-2}+G_{n-3})(\delta'\pm1, \delta'\pm2)$$

$$+(G_{n-4}+\ldots+G_{n-8})(\delta'\pm2, \delta'\pm3)-\ldots,$$

all cancel one another with the exception of $(r+1)\,r$'s and $r(r+1)$'s or $(r+1)(-r)$'s and $r\{-(r+1)\}$'s, according as r is uneven or even; where, as before, r^2 is the square next inferior to, or equal to, n.

Taking, as an example, $n = 8$, for which $r = 2$, we form the scheme

9	3, 9,	4, 8,	3, 7	7,	4, 6,	5,	4
8	2, 8,	3, 7,	2, 6	6,	3, 5,	4,	3
(8)	(1, 7),	(2, 6),	(1, 5)	(4),	(1, 3),	(2),	(1)
8	0, 6,	1, 5,	0, 4	2,	−1, 1,	0,	−1
7	−1, 5,	0, 4,	−1, 3	1,	−2, 0,	−1,	−2

whence we derive the numbers given by the formula, viz.,

$$\begin{array}{cccccccccccc}
9, & -3, & -9, & -4, & -8, & -3, & -7, & 7, & 4, & 6, & 5, & 4 \\
8, & -2, & -8, & -3, & -7, & -2, & -6, & 6, & 3, & 5, & 4, & 3 \\
8, & 0, & -6, & -1, & -5, & 0, & -4, & 2, & -1, & 1, & 0, & -1 \\
7, & 1, & -5, & 0, & -4, & 1, & -3, & 1, & -2, & 0, & -1, & -2
\end{array}$$

which cancel one another except -2, -2, -2, -3, -3.

64. By putting $\pi - x$ for x in the trigonometrical equation of the preceding section, we obtain another trigonometrical equation which shows that the numbers given by the formula

$$\begin{aligned}
&(-1)^n G_n \{\delta', \delta', -(\delta' \pm 1)\} \\
&+\{(-1)^{n-1} G_{n-1} + (-1)^{n-2} G_{n-2} + (-1)^{n-3} G_{n-3}\} \{\delta' \pm 1, -(\delta' \pm 2)\} \\
&+\{(-1)^{n-4} G_{n-4} + \ldots + (-1)^{n-8} G_{n-8}\} \{\delta' \pm 2, -(\delta' \pm 3)\} + \ldots,
\end{aligned}$$

all cancel each other with the exception of $r(r+1)$'s and $(r+1)(-r)$'s, where r has the same meaning as in the preceding section.

To obtain the numbers given by this formula we may begin by forming the same scheme as in the preceding section; we then change all the signs in the top and bottom rows, and finally change all the signs in the columns in which the values of δ' in the central line are even.

Thus in the case of $n = 8$ the numbers given by the formula are

$$\begin{array}{cccccccccccc}
-9, & 3, & 9, & -4, & -8, & 3, & 7, & -7, & 4, & 6, & -5, & 4 \\
8, & -2, & -8, & 3, & 7, & -2, & -6, & 6, & -3, & -5, & 4, & -3 \\
8, & 0, & -6, & 1, & 5, & 0, & -4, & 2, & 1, & -1, & 0, & 1 \\
-7, & -1, & 5, & 0, & -4, & -1, & 3, & -1, & -2, & 0, & 1, & -2
\end{array}$$

which cancel one another except 3, 3, -2, -2, -2.

The numbers given by this formula differ from those derived from the formula in the preceding section only by the reversal of sign in the uneven numbers. The first form of the theorem (*i.e.*, that in the preceding section) is the simpler of the two.

The results in this and the preceding section have a general resemblance in form to the d-theorems of § 36 of this paper and § 65 of the previous paper.

65. By multiplying the trigonometrical formula in § 56 by $\cos x$ and replacing x by $\frac{1}{2}x$, we obtain the equation

$$2\{\sin x+(\sin x+\sin 2x)(q+q^2+q^3)+(\sin 2x+\sin 3x)(q^4+\ldots+q^8)+\ldots\}$$
$$\times \Sigma_1^\infty (-1)^{n-1}\,\Delta'(\sin nx)\,q^n$$
$$= (1-\cos 2x)(q+q^2+q^3)+(1+2\cos x-\cos 2x-2\cos 3x)(q^4+\ldots+q^8)+\ldots,$$

the coefficient of $q^{r^2}+\ldots+q^{r^2+2r}$ on the right-hand side being

$$1+2\cos x+2\cos 2x+2\cos(r-1)\,x-(r-1)\cos rx-r\cos(r+1)\,x.$$

This equation shows that the numbers given by the formula

$$(-1)^n G_n \begin{Bmatrix} \delta'+1 \\ -[\delta'-1] \end{Bmatrix}$$
$$+\{(-1)^{n-1}G_{n-1}+(-1)^{n-2}G_{n-2}+(-1)^{n-3}G_{n-3}\}\begin{Bmatrix} \delta'+1, & \delta'+2 \\ -[\delta'-1], & -[\delta'-2] \end{Bmatrix}$$
$$+\{(-1)^{n-4}G_{n-4}+\ldots+(-1)^{n-8}G_{n-8}\}\begin{Bmatrix} \delta'+2, & \delta'+3 \\ -[\delta'-2], & -[\delta'-3] \end{Bmatrix}+\ldots,$$

all cancel one another with the exception of

one 0, two 1's, two 2's, ..., two $(r+1)$'s, $(r-1)(-r)$'s, and $r\{-(r+1)\}$'s,

where, as before, r^2 is the square next inferior to, or equal to, n. The square brackets denote that the absolute value of the quantity included in them is to be taken. Zeros are to be taken account of and treated in the same way as other numbers.

As an example, taking $n = 8$, we form the scheme

$$\begin{array}{r|cccccccccccc}
 & 3, & 9, & 4, & 8, & 3, & 7 & 7, & 4, & 6, & 5, & 4 \\
9 & 2, & 8, & 3, & 7, & 2, & 6 & 6, & 3, & 5, & 4, & 3 \\
(8) & (1, & 7), & (2, & 6), & (1, & 5) & (4), & (1, & 3), & (2) & (1) \\
-7 & -0, & -6, & -1, & -5, & -0, & -4 & -2, & -1, & -1, & -0, & -1 \\
 & -1, & -5, & -0, & -4, & -1, & -3 & -1, & -2, & -0, & -1, & -2
\end{array}$$

in which, after writing the numbers of the form $\delta'+r$ above the central line and those of the form $\delta'-r$ below, we change the sign from positive to negative of all the latter which are not already negative. We then change the signs in all the columns in which the central number is uneven.

The numbers thus obtained are

$$\begin{array}{cccccccccccc} & -3, & -9, & 4, & 8, & -3, & -7, & 7, & -4, & -6, & 5, & -4 \\ 9, & -2, & -8, & 3, & 7, & -2, & -6, & 6, & -3, & -5, & 4, & -3 \\ -7, & 0, & 6, & -1, & -5, & 0, & 4, & -2, & 1, & 1, & -0, & 1 \\ & 1, & 5, & 0, & -4, & 1, & 3, & -1, & 2, & 0, & -1, & 2 \end{array}$$

which cancel one another, except 0, 1, 1, -2, -3, -3.

We may also derive another form of the theorem from the equation obtained by multiplying the trigonometrical formula in § 55 by $\sin x$.

General Formulæ connecting Δ'_m, Δ'_{m-2}, ... of all Numbers from unity to n.
§§ 66, 67.

66. It seems worth while to notice the general formulæ involving Δ'_m, Δ'_{m-2}, ... which may be derived from the theorems of §§ 63 and 64.

By putting $G_n(\delta', \delta', \delta' \pm 1) = \delta'^m + \delta'^m + (\delta'-1)^m + (\delta'+1)^m, \dots$ (m being uneven) in the theorem of § 63, we obtain the formula

$$\begin{aligned} &2\{\Delta'_m(n) - \Delta'_m(n-1) - \Delta'_m(n-2) - \Delta'_m(n-3) + \text{next five} - \dots\} \\ &+(m)_2\{\Delta'_{m-2}(n) - (1^2+2^2)(\text{next three}) + (2^2+3^2)(\text{next five}) - \dots\} \\ &+(m)_4\{\Delta'_{m-4}(n) - (1^4+2^4)(\text{next three}) + (2^4+3^4)(\text{next five}) - \dots\} \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &+(m)_1\{\Delta'(n) - (1^{m-1}+2^{m-1})(\text{next three}) + (2^{m-1}+3^{m-1})(\text{next five}) - \dots\} \\ &\qquad = (-1)^{r-1}\tfrac{1}{2}r(r+1)\{r^{m-1}+(r+1)^{m-1}\}, \end{aligned}$$

where r has the same meaning as in § 63.

Putting $m = 1$ in the general formula we obtain the formula of § 54, and putting $m = 3$, we find

$$\begin{aligned} &2\{\Delta'_3(n) - \Delta'_3(n-1) - \Delta'_3(n-2) - \Delta'_3(n-3) + \text{next five} - \dots\} \\ &+3\{\Delta'(n) - (1^2+2^2)(\text{next three}) + (2^2+3^2)(\text{next five}) - \dots\} \\ &\qquad = (-1)^{r-1}\tfrac{1}{2}r(r+1)(2r^2+2r+1), \end{aligned}$$

which is also deducible from the formulæ of § 59.

67. Putting $G_n\{\delta', \delta', -(\delta'\pm 1)\} = \delta'^m+\delta'^m-(\delta'-1)^m-(\delta'-1)^m, \ldots$ (m being uneven) in the theorem of § 64, we find

$$(m)_2\left[\Delta'_{m-2}(n)+(2^2-1^2)\{-\Delta'_{m-2}(n-1)+\Delta'_{m-2}(n-2)-\Delta'_{m-2}(n-3)\}\right.$$
$$\left.+(3^2-2^2)(\text{next five})+\ldots\right]$$
$$+(m)_4\{\Delta'_{m-4}(n)+(2^4-1^4)(\text{next three})+(3^4-2^4)(\text{next five})+\ldots\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+(m)_1\{\Delta'(n)+(2^{m-1}-1^{m-1})(\text{next three})+(3^{m-1}-2^{m-1})(\text{next five})+\ldots\}$$
$$=(-1)^{n-1}\tfrac{1}{2}r(r+1)\{(r+1)^{m-1}-r^{m-1}\}.$$

The signs of the terms in all the series are alternately positive and negative.

Putting $m=3$, we obtain the first formula of § 61, and putting $m=5$, we find

$$2\left[\Delta'_3(n)+3\{-\Delta'_3(n-1)+\Delta'_3(n-2)-\Delta'_3(n-3)\}+5(\text{next five})-\ldots\right]$$
$$+\Delta'(n)+3(2^2+1)(\text{next three})+5(3^2+2^2)(\text{next five})+\ldots$$
$$=(-1)^{n-1}\tfrac{1}{10}r(r+1)(4r^3+6r^2+4r+1),$$

which is also deducible from the formulæ of § 61.

Other Theorems relating to Actual Divisors of n, $n-1$, $n-4$, ...
General remarks. §§ 68–71.

68. We may obtain other divisor theorems by multiplying the trigonometrical equations by $\cos x$, $\cos 2x$, ..., or making any other changes which lead to new equations of the requisite form.

Taking the original trigonometrical equation of § 47, viz.,

$$(1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\ldots)\,\Sigma_1^\infty\,\Delta'(\sin nx)\,q^n$$
$$=q\sin x-2q^4\sin 2x+3q^9\sin 3x-\ldots,$$

and putting $2x$ for x, and multiplying by $\cos x$, we find

$$2\{\cos x-q(\cos x+\cos 3x)+q^4(\cos 3x+\cos 5x)-\ldots\}\,\Sigma_1^\infty\,\Delta'(\sin 2nx)\,q^n$$
$$=q(\sin x+\sin 3x)-2q^4(\sin 3x+\sin 5x)+3q^9(\sin 5x+\sin 7x)-\ldots,$$

whence it follows that the numbers given by the formula

$$G_n(2\delta'\pm 1)-G_{n-1}(2\delta'\pm 1,\ 2\delta'\pm 3)+G_{n-4}(2\delta'\pm 3,\ 2\delta'\pm 5)-\ldots,$$

all cancel one another, unless n is a square $= r^2$ in which case the numbers left uncancelled are $r\,\{\pm(2r-1)\}$'s and $r\,\{\pm(2r+1)\}$'s, the upper or lower signs being taken according as r is uneven or even.

69. If we multiply the original equation as it stands by $\cos x$, we obtain the equation

$$2\,\{\cos x - q\,(1+\cos 2x) + q^4(\cos x + \cos 3x) - q^9(\cos 2x + \cos 2x) + \dots\}$$
$$\times \Sigma_1^\infty \Delta'(\sin nx)\, q^n$$
$$= q \sin 2x - 2q^4(\sin x + \sin 3x) + 3q^9(\sin 2x + \sin 4x) - \dots,$$

showing that the numbers given by the formula

$$G_n(\delta' \pm 1) - G_n(\delta' \pm 0,\ \delta' \pm 2) + G_{n-4}(\delta' \pm 1,\ \delta' \pm 3) - \dots,$$

all cancel one another, unless n is a square $= r^2$, in which case the numbers left uncancelled are $r\,\{\pm(r-1)\}$'s and $r\,\{\pm(r+1)\}$'s, the upper or lower signs being taken according as r is uneven or even.

70. More generally, if instead of multiplying by $\cos x$, we multiply by $\cos tx$, we obtain the equation

$$2[\cos tx - q\,\{\cos(t-1)\,x + \cos(t+1)x\} + q^4\,\{\cos(t-2)x + \cos(t+2)\,x\} - \dots$$
$$\times \Sigma_1^\infty \Delta'(\sin nx)\, q^n$$
$$= q\,\{\sin(t+1)x - \sin(t-1)\,x\} - 2q^4\,\{\sin(t+2)x - \sin(t-2)x\}$$
$$+ 3q^9\,\{\sin(t+3)\,x - \sin(t-3)x\} - \dots,$$

showing that the numbers given by the formula

$$G_n(\delta' \pm t) - G_{n-1}\,\{\delta' \pm (t-1),\ \delta' \pm (t+1)\}$$
$$+ G_{n-4}\,\{\delta' \pm (t-2),\ \delta' \pm (t+2)\} - \dots$$

all cancel one another, unless n is a square $= r^2$ in which case the numbers left uncancelled are $r\,\{\pm(r-t)\}$'s and $r\,\{\pm(r+t)\}$'s, the upper or lower signs being taken according as r is uneven or even.

71. In this theorem t may be fractional, and it is evident therefore that it may be decomposed into two theorems, one containing the numbers in which t occurs with the positive sign and one containing those in which the sign of t is negative, viz., we see that the numbers given by

the formula

$$G_n(\delta'+t)-G_{n-1}(\delta'+t\pm 1)+G_{n-4}(\delta'+t\pm 2)-\ldots$$

all cancel one another, unless n is a square $= r^2$, in which case the numbers left uncancelled are $r\{\pm(r+t)\}$'s.

This result is equivalent to the original theorem of § 48, being derivable from it by merely adding t to every number (irrespective of sign) given by that formula.

It thus appears that, in general, by multiplying a trigonometrical formula by $\cos tx$, $\sin tx$, ... for various values of t, we obtain new theorems relating to divisors. These are more complicated and less interesting than the theorems obtained from the trigonometrical equations in their simplest forms, and it would seem that they can generally be derived arithmetically from the simpler theorems (by adding the same quantity to each number, combining two or more theorems, &c.).*

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* The theorem of § 68 is equivalent to the case $t = \frac{1}{2}$ of the theorem of § 70. If in the original trigonometrical equation quoted in § 68, we put kx for x, and then multiply by $\cos tx$, we do not increase the generality of the derived arithmetical theorem, since t may be fractional.

PART II.—THEOREMS RELATING TO DIVISORS WHICH HAVE UNEVEN CONJUGATES.
§§ 46–71.

ON A FORM OF THE ELIMINANT OF TWO QUANTICS

By A. L. DIXON.

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1. I wish to prove, in the first place, that the eliminant of two equations of the form

$$\sum_1^{n+1} \frac{A_s}{x-r_s} = 0, \qquad \sum_1^{n+1} \frac{B_s}{x-r_s} = 0$$

is given by the symmetrical determinant of the nth order

$$\begin{vmatrix} \Sigma(1, s), & -(2, 1), & -(3, 1), & \ldots, & -(n, 1) \\ -(1, 2), & \Sigma(2, s), & -(3, 2), & \ldots, & -(n, 2) \\ -(1, 3), & -(2, 3), & \Sigma(3, s), & \ldots, & -(n, 3) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -(1, n), & -(2, n), & -(3, n), & \ldots, & \Sigma(n, s) \end{vmatrix},$$

multiplied by the discriminant of $\prod_{s=1}^{s=n+1} (x-r_s)$, where

$$(s, t) = (t, s) = (A_s B_t - A_t B_s)/(r_s - r_t),$$

and $\sum_s (t, s) = (t, 1)+(t, 2)+\ldots+(t, t-1)+(t, t+1)+\ldots+(t, n+1),$

the term (t, t) being omitted.

This determinant is symmetrical with respect to all the numbers $1, 2, \ldots, n+1$, and may be written as the sum of $(n+1)^{n-1}$ terms, each term of the sum having a positive sign and being the product of n different quantities (rs). Each of the numbers $1, 2, 3, \ldots, (n+1)$ occurs at least once in each term, and the sum may be written down by taking n of the numbers, say $2, 3, \ldots, (n+1)$, and distributing them, one in each bracket, and then filling in the other number in each bracket in all possible ways, so that 1 shall occur at least once, and that in any one term no two brackets shall occur which contain the same pair of numbers.

This result, which is of practical value as a simple expression for the eliminant of two equations of a particular form, is a particular case of a

more general theorem, that, if $r_1, r_2, \ldots, r_n$ and $a_1, \ldots, a_n$ are two sets of arbitrary quantities, and we write (a_s, r_t) for $[\phi(a_s)\psi(r_t)-\phi(r_t)\psi(a_s)]/(a_s-r_t)$, where $\phi(x)$ and $\psi(x)$ are two quantics, each of the nth degree, then the eliminant of $\phi(x)$ and $\psi(x)$ is given by

$$\begin{vmatrix} (a_1, r_1), & (a_1, r_2), & (a_1, r_3), & \ldots, & (a_1, r_n) \\ (a_2, r_1), & (a_2, r_2), & (a_2, r_3), & \ldots, & (a_2, r_n) \\ (a_3, r_1), & (a_3, r_2), & (a_3, r_3), & \ldots, & (a_3, r_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_n, r_1), & (a_n, r_2), & (a_n, r_3), & \ldots, & (a_n, r_n) \end{vmatrix},$$

divided by the product of differences

$$\Pi(r_s-r_t)\,\Pi(a_s-a_t).$$

The eliminant in the form first given is, in fact, the result of putting $a_s = r_s$, for all values of s, in this last expression.

For, writing
$$f(x) = \Pi(x-r_s),$$
and
$$\frac{\phi(x)}{f(x)} = \Sigma\frac{\phi(r_s)}{f'(r_s)}\frac{1}{x-r_s} = \Sigma\frac{A_s}{x-r_s},$$
$$\frac{\psi(x)}{f(x)} = \Sigma\frac{\psi(r_s)}{f'(r_s)}\frac{1}{x-r_s} = \Sigma\frac{B_s}{x-r_s},$$
we get for a term of the principal diagonal, $\Sigma(1, s)$ for example,

$$\Sigma(1, s) = \frac{\phi(r_1)}{f'(r_1)}\sum_2^{n+1}\frac{\psi(r_s)}{f'(r_s)(r_1-r_s)} - \frac{\psi(r_1)}{f'(r_1)}\sum_2^{n+1}\frac{\phi(r_s)}{f'(r_s)(r_1-r_s)}$$
$$= \frac{\phi(r_1)}{f'(r_1)}\left\{\frac{\psi'(r_1)}{f'(r_1)} - \tfrac{1}{2}\frac{f''(r_1)\psi(r_1)}{[f'(r_1)]^2}\right\} - \frac{\psi(r_1)}{f'(r_1)}\left\{\frac{\phi'(r_1)}{f'(r_1)} - \tfrac{1}{2}\frac{f''(r_1)\phi(r_1)}{[f'(r_1)]^2}\right\}$$
$$= \frac{\phi(r_1)\psi'(r_1)-\phi'(r_1)\psi(r_1)}{[f'(r_1)]^2},$$

which is the limit of $-(1, s)$ when r_s is made equal to r_1.

If we go further and take the limit when all the a's and all the r's are made equal to infinity (or zero), we get finally Bezout's determinant.

2. I take then two equations of the nth degree

$$\sum_1^{n+1}\frac{A_s}{x-r_s} = 0, \tag{A}$$

$$\sum_1^{n+1}\frac{B_s}{x-r_s} = 0. \tag{B}$$

Then
$$(s, t) = (t, s) = (A_sB_t - A_tB_s)/(r_s-r_t).$$

Also I write $D(r_1, r_2, \ldots, r_{n+1})$ for the discriminant of $\prod_1^{n+1}(x-r_s)=0$, and $E(r_1, r_2, \ldots, r_{n+1})$ for the eliminant of the equations (A) and (B).

If we increase the roots of each equation by r_1, and write $\rho_s \equiv r_s - r_1$, we have

$$\frac{A_1}{x}+\frac{A_2}{x-\rho_2}+\frac{A_3}{x-\rho_3}+\ldots=0,$$

$$\frac{B_1}{x}+\frac{B_2}{x-\rho_2}+\frac{B_3}{x-\rho_3}+\ldots=0.$$

If these equations be multiplied up, and arranged in ascending powers of x, we get

$$\rho_2\rho_3\ldots\rho_{n+1}A_1 - x\left[A_1\beta + A_2\rho_3\rho_4\ldots\rho_{n+1}+A_3\rho_2\rho_4\ldots\rho_{n+1}+\ldots\right]+\ldots=0,$$

$$\rho_2\rho_3\ldots\rho_{n+1}B_1 - x\left[B_1\beta + B_2\rho_3\rho_4\ldots\rho_{n+1}+B_3\rho_2\rho_4\ldots\rho_{n+1}+\ldots\right]+\ldots=0,$$

where β is the sum of the products of $\rho_2, \rho_3, \ldots, \rho_{n+1}$, $n-1$ at a time.

Again, it is a well known and immediate deduction from Bezout's expression for the eliminant as a determinant,* that for two equations of the forms

$$a - a_1x + \ldots = 0,$$

$$b - b_1x + \ldots = 0,$$

the terms of the eliminant which do not contain either a^2, ab, or b^2 are $(ab_1 - a_1b)$, multiplied by the eliminant of the equations, of the next lower degree, obtained by putting $a=0$, $b=0$, and dividing by x.

Thus, taking our equations in the form just obtained, (ab_1-a_1b) becomes

$$\rho_2^2\rho_3^2\rho_4^2\ldots\rho_{n+1}^2\sum_{s=2}^{s=n+1}(A_1B_s - B_1A_s)/\rho_s,$$

or $$(r_2-r_1)^2(r_3-r_1)^2\ldots(r_{n+1}-r_1)^2\,\Sigma(A_1B_s-B_1A_s)/(r_1-r_s).$$

3. It is obvious that each term in the eliminant must contain either A_s or B_s for every value of s, since, when $A_s=0$, $B_s=0$, the two equations

$$\sum_1^{n+1}A_s/(x-r_s)=0, \qquad \sum_1^{n+1}B_s/(x-r_s)=0,$$

have a common root $x=r_s$.

* Salmon, *Higher Algebra*, p. 82, § 85.

Also for each term in the eliminant there must be at least two values of s, say t is one, such that A_t or B_t occurs to the first power only, that is to say, that the term does not contain A_t^2, A_tB_t or B_t^2. For, if the term contained A_s^2, A_sB_s or B_s^2 for all but one, that is, for n values of s, its degree would be $2n+1$, at least, in the A's and B's together, and the eliminant is only of degree n in the A's and B's respectively.

Thus, from the necessary symmetry in respect to the sets of quantities A_s, B_s, r_s, the eliminant is completely determined when the terms containing A_1 or B_1 to the first power only are known.

When $n = 1$, we have at once

$$E(r_1, r_2) = (r_1-r_2)(A_1B_2-A_2B_1) = D(r_1, r_2)(1, 2).$$

Thus the terms in $E(r_1, r_2, r_3)$ which contain A_1 or B_1 to the first power, are

$$(r_2-r_1)^2(r_3-r_1)^2\{(1, 2)+(1, 3)\}(r_2-r_3)^2(2, 3),$$

and we get

$$E(r_1, r_2, r_3) = D(r_1, r_2, r_3)\{(1, 2)(1, 3)+(2, 1)(2, 3)+(3, 1)(3, 2)\}.$$

From this again, the terms in $E(r_1, r_2, r_3, r_4)$, of the first power in A_1 or B_1, will be

$$D(r_1, r_2, r_3, r_4)\{(1, 2)+(1, 3)+(1, 4)\}\{(2, 3)(2, 4)+(3, 2)(3, 4)+(4, 2)(4, 3)\},$$

and the complete expression is

$$\begin{aligned}E(r_1, r_2, r_3, r_4)/D(r_1, r_2, r_3, r_4)\\ = \quad & (1, 2)(1, 3)(1, 4)+(2, 1)(2, 3)(1, 4)+(3, 1)(3, 2)(1, 4)\\ & +(1, 2)(1, 3)(2, 4)+(2, 1)(2, 3)(2, 4)+(3, 1)(3, 2)(2, 4)\\ & +(1, 2)(1, 3)(3, 4)+(2, 1)(2, 3)(3, 4)+(3, 1)(3, 2)(3, 4)\\ & +(4, 1)(4, 2)(1, 3)+(4, 1)(4, 2)(2, 3)+(4, 1)(4, 3)(1, 2)+(4, 1)(4, 3)(3, 2)\\ & +(4, 2)(4, 3)(2, 1)+(4, 2)(4, 3)(3, 1)+(4, 1)(4, 2)(4, 3).\end{aligned}$$

It is seen, on inspection, that the rules given in § 1, for writing down the eliminant as a sum of terms, apply to $E(r_1, r_2, r_3)$ and $E(r_1, r_2, r_3, r_4)$, and therefore by induction must apply universally. For, if they apply to

$E(r_2, r_3, \ldots, r_{n+1})$, they must apply to the terms in $E(r_1, r_2, \ldots, r_{n+1})$, which only contain the number 1 once, since these terms are

$$\{(1, 2)+(1, 3)+\ldots+(1, n+1)\}\, E(r_2, r_3, \ldots, r_{n+1}),$$

and so, by symmetry, must apply to all the terms of $E(r_1, r_2, \ldots, r_{n+1})$.

4. Again $E(r_1, r_2, r_3)/D(r_1, r_2, r_3)$, which is

$$(1, 2)(1, 3)+(2, 1)(2, 3)+(3, 1)(3, 2),$$

may obviously be written in the form

$$\begin{vmatrix} (1, 2)+(1, 3), & -(2, 1) \\ -(1, 2), & (2, 1)+(2, 3) \end{vmatrix},$$

and then, again by induction,

$$\begin{vmatrix} \Sigma(1, s), & -(2, 1), & -(3, 1), & \ldots, & -(n, 1) \\ -(1, 2), & \Sigma(2, s), & -(3, 2), & \ldots, & -(n, 2) \\ -(1, 3), & -(2, 3), & \Sigma(3, s), & \ldots, & -(n, 3) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -(1, n), & -(2, n), & -(3, n), & \ldots, & \Sigma(n, s) \end{vmatrix}$$

may be identified with $E(r_1, r_2, \ldots, r_{n+1})/D(r_1, r_2, \ldots, r_{n+1})$.

For, by adding all the other rows to the last row, and then all the other columns to the last column, the determinant is easily seen to be symmetrical with regard to all the numbers 1, 2, 3, ..., $(n+1)$. Also the terms in it which contain 1 once only, are $(1, 2)+(1, 3)+\ldots+(1, n+1)$ multiplied by the corresponding determinant got by erasing the first row and column. Then since the expression is right for $E(r_1, r_2, r_3)/D(r_1, r_2, r_3)$, it is also right for $E(r_1, r_2, r_3, r_4)/D(r_1, r_2, r_3, r_4)$, and so by induction for all values of n.

5. I have found the number of terms in the eliminant when expressed as a sum of products of the quantities (r, s) as follows.

Disregarding altogether the factor $D(r_1, \ldots, r_{n+1})$, suppose the terms of $E(r_1, \ldots, r_{n+1})$ arranged in sets, according as the term contains two of the numbers 1, 2, 3, ..., $n+1$, once only, or three numbers once only, and so on, and put

$$E(r_1, \ldots, r_{n+1}) = S_2+S_3+S_4+\ldots+S_n.$$

Then we shall also have

$$\sum_{s=1}^{s=n+1}\left\{\sum_{t=1}^{t=n+1}(t,s)\,E(r_1,\ldots,r_{s-1},r_{s+1},\ldots,r_n)\right\}=2S_2+3S_3+\ldots+nS_n.$$

Similarly, $(t\neq s',\ t'\neq s)$,

$$\sum_{s'=1}^{s'=n+1}\sum_{s=1}^{s=n+1}\left\{\sum_{t=1}^{t=n+1}\sum_{t'=1}^{t'=n+1}(t,s)(t',s')\,E(r_1,\ldots,r_{s-1},r_{s+1},\ldots,r_{s'-1},r_{s'+1},\ldots,r_n)\right\}$$

$$=S_2+\frac{3\,.\,2}{1\,.\,2}S_3+\frac{4\,.\,3}{1\,.\,2}S_4+\ldots+\frac{n(n-1)}{1\,.\,2}S_n.$$

A similar sum obtained by taking the product of three brackets $(t,s)(t',s')(t'',s'')$, into the eliminant of the equations obtained by crossing out s, s', s'' is equal to

$$S_3+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}S_4+\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}S_5+\ldots+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}S_n,$$

and we may continue the series until we finally arrive at $n!\,S_n/n!$ on the right-hand side, and $\sum\limits_{s=1}^{s=n+1}\ \prod\limits_{t=1}^{t=n+1}(s,t)$, $(t\neq s)$, on the left.

From these equations we eliminate S_2, S_3, ..., by taking the left-hand sides with alternately positive and negative signs, and arrive at an identity which may be written

$$E_{n+1}-\sum_s\sum_t(s,t)\,E_n[s]+\sum_{ss'}\sum_{tt'}(s,t)(s',t')\,E_{n-1}[s,s']-\ldots=0,$$

where, for example, $E_{n-1}[s,s']$ means the eliminant of the equations obtained by striking out the terms with s or s' as suffix.

Now, write N_{n+1} for the number of terms in E_{n+1}, N_n for the number in E_n, and so on.

In $\sum\limits_s\sum\limits_t(s,t)$ we have $n+1$ choices for s and n for t, so that the number of terms is $(n+1)\,n$. In $\sum\limits_{ss'}\sum\limits_{tt'}(s,t)(s',t')$ we have $\frac{1}{2}n\,(n+1)$ pairs s, s', and for each pair $n-1$ choices for t or t', so that the number of terms is $\frac{1}{2}n\,(n+1)\,(n-1)^2$.

Determining in this way the number of terms in each constituent, we have, since all the terms in the eliminants have positive signs, the relation

$$N_{n+1}-(n+1)\,nN_n+\frac{(n+1)\,n}{1\,.\,2}(n-1)^2N_{n-1}$$

$$-\frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}(n-2)^3N_{n-2}+\ldots+(-1)^n(n+1)N_1=0,$$

where N_1 is to be taken as 1. Now we know that $N_2 = 1$, $N_3 = 3$, $N_4 = 16$, and the above relation is satisfied by putting

$$N_{n+1} = (n+1)^{n-1},$$

becoming in this case

$$(n+1)^{n-1} - (n+1)\,n^{n-1} + \frac{(n+1)\,n}{1.2}(n-1)^{n-1} - \ldots = 0.$$

Therefore the number of terms in the eliminant is $(n+1)^{n-1}$.

6. This expression for the eliminant as a determinant, once obtained, may be easily verified in a different manner.

Returning to the equations

$$\sum_1^{n+1} \frac{A_s}{x-r_s} = 0, \qquad \sum_1^{n+1} \frac{B_s}{x-r_s} = 0,$$

multiply the first by $B_1/(x-r_1)$, and the second by $A_1/(x-r_1)$, and subtract.

The term $(A_sB_1 - A_1B_s)/(x-r_s)(x-r_1)$ on the left-hand side of the resulting equation may be written

$$\frac{(A_sB_1 - A_1B_s)}{r_s - r_1}\left\{\frac{1}{x-r_s} - \frac{1}{x-r_1}\right\},$$

and we obviously get

$$-\frac{1}{x-r_1}\sum_{s=2}^{s=n+1}(1,\,s) + \sum_{s=2}^{s=n+1}\frac{(1,\,s)}{x-r_s} = 0.$$

There will be $n-1$ similar equations got by replacing 1 by $2, 3, \ldots, n$, and we may add to these the equation

$$\Sigma\, A_s/(x-r_s) = 0,$$

and eliminate the $(n+1)$ quantities $1/(x-r_s)$. The result is

$$\begin{vmatrix} \Sigma(1,\,s), & -(2,\,1), & -(3,\,1), & \ldots, & -(n+1,\,1) \\ -(1,\,2), & \Sigma(2,\,s), & -(3,\,2), & \ldots, & -(n+1,\,2) \\ -(1,\,3), & -(2,\,3), & \Sigma(3,\,s), & \ldots, & -(n+1,\,3) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_1, & A_2, & A_3, & \ldots, & A_{n+1} \end{vmatrix} = 0.$$

By adding all the other columns to the last column, all its constituents

become zero except the last, which is ΣA_s, and the result is (ΣA_s) multiplied by the determinant of the nth order already obtained (§ 4). That by this method of elimination, the extraneous factor ΣA_s is introduced, is easily seen, if we notice that all the equations of the type

$$\Sigma\{(1, s)/(x-r_s)\} - \Sigma(1, s)/(x-r_1) = 0,$$

have a common root $x = \infty$, since the sum of the coefficients is zero.

7. By what is really the same method we obtain the eliminant in the form of the determinant whose elements are (a_s, r_t) (§ 1).

If $\phi(x)$, $\psi(x)$ have a common root x, this is also a root of

$$[\phi(x)\,\psi(a) - \phi(a)\,\psi(x)]/(x-a).^*$$

Putting
$$f(x) \equiv \prod_{s=1}^{s=n} (x-r_s),$$

we have, by partial fractions,

$$\frac{\phi(x)\,\psi(a) - \phi(a)\,\psi(x)}{f(x)(x-a)} = \sum_{s=1}^{s=n} \frac{(a, r_s)}{f'(r_s)(x-r_s)},$$

and now putting the left-hand side equal to zero, and giving to a the n values, $a_1, a_2, \ldots, a_n$, we have n equations linear in $1/f'(r_s)(x-r_s)$, and the determinant is at once obtained.

It is not difficult to show directly that this determinant, which I will call D, whose constituents are (a_s, r_t) is equal to $\Pi(r_s-r_t)\,\Pi(a_s-a_t)\,B$, where B is Bezout's determinant.

If we write $f(a, r)$ for (a, r), it is obvious, by expanding $f(a_s, r_t)$ in ascending powers of (a_s-a) by Taylor's theorem and applying the formula for the multiplication of two determinants, that D is equal to the product of two determinants, the rows (or columns) of which are respectively,

$$f(a, r_s), \quad \frac{\partial f(a, r_s)}{\partial a}, \quad \frac{1}{2!}\frac{\partial^2 f(a, r_s)}{\partial a^2}, \quad \ldots, \quad \frac{1}{(n-1)!}\frac{\partial^{n-1} f(a, r_s)}{\partial a^{n-1}},$$

and
$$1, \quad a_s-a, \quad (a_s-a)^2, \quad \ldots, \quad (a_s-a)^{n-1},$$

the latter determinant being $\Pi(a_s-a_t)$.

Again, applying the same method and expanding in ascending powers

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* Salmon, *loc. cit.*, p. 83, § 87, Cayley's "Statement of Bezout's method."

of $(r_s - r)$, the first of these determinants becomes $\Pi(r_s - r_t)$ multiplied by the determinant whose constituents are $\dfrac{1}{s!\,t!}\,\dfrac{\partial^{s+t} f(a, r)}{\partial a^s \partial r^t}$. It remains to show, therefore, that this last determinant is equal to B; it is, in fact, identical with it when $a = 0$, $r = 0$. For, if

$$\phi(x) = A_0 + A_1 x + \ldots + A_r x^r + \ldots + A_n x^n,$$

$$\psi(x) = B_0 + B_1 x + \ldots + B_r x^r + \ldots + B_n x^n,$$

$$\begin{vmatrix} \phi(a), & \phi(r) \\ \psi(a), & \psi(r) \end{vmatrix} = \sum_{p,\,q} \begin{vmatrix} A_p, & A_q \\ B_p, & B_q \end{vmatrix} \begin{vmatrix} a^p, & a^q \\ r^p, & r^q \end{vmatrix};$$

and therefore, assuming $p > q$,

$$f(a, r) = \Sigma \begin{vmatrix} A_p, & A_q \\ B_p, & B_q \end{vmatrix} (a^{p-1} r_q + a^{p-2} r^{q-1} + \ldots + a^{q-1} r^{p-2} + a^q r^{p-1}).$$

Thus $\left[\dfrac{1}{s!\,t!}\,\dfrac{\partial^{s+t} f(a, r)}{\partial a^s \partial r^t}\right]_{a=0,\; r=0}$, which is the coefficient of $a^s r^t$ in $f(a, r)$, is equal to $\Sigma(A_p B_q - A_q B_p)$, where $p + q = s + t + 1$, and, taking $p > q$, $q \not> s$ or t, $p - 1 \not< s$ or t. Thus we have finally Bezout's determinant, viz.,

$$\begin{vmatrix} (A_0, B_1), & (A_0, B_2), & (A_0, B_3), & \ldots \\ (A_0, B_2), & (A_0, B_3) + (A_1, B_2), & (A_0, B_4) + (A_1, B_3), & \ldots \\ (A_0, B_3), & (A_0, B_4) + (A_1, B_3), & (A_0, B_5) + (A_1, B_4) + (A_2, B_3), & \ldots \\ \ldots & \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \end{vmatrix}.$$

8. There remains the consideration of the modifications to be introduced when the degrees of the given equations are different, say n and $n - m$.

Consider first the determinant whose constituents are (r, s), the equations being given in the forms

$$\Sigma A_s/(x - r_s) = 0, \qquad \text{(A)}$$

$$\Sigma B_s/(x - r_s) = 0; \qquad \text{(B)}$$

and suppose that the first m coefficients of (B), when it is multiplied up and arranged in descending powers of x, vanish identically. The equation (B) may be considered as having m infinite roots, and the expression

for the eliminant, as already found, will have an extraneous factor $(\Sigma A_s)^m$.

If, however, $B_1 = B_2 = B_3 = \ldots = B_m = 0$, we have

$$(s, t) = 0, \text{ if } s < m+1, \ t < m+1,$$

$$(s, t) = A_s B_t/(r_s - r_t), \text{ if } s < m+1, \ t > m.$$

Then A_s can be divided out from either the sth row or the sth column, where $s < m+1$, and we have left the pure eliminant, in the form of a determinant, whose first principal minor of the mth order consists of a principal diagonal only, all the other elements being zero.

Next consider the determinant whose constituents are (a_s, r_t). To find the eliminant of $\phi'(x)$, $\psi(x)$, where $\psi(x)$ is of degree n, and $\phi'(x)$ of degree $n-m$, let us write

$$\phi(x) = \phi'(x) \prod_{t=1}^{t=m} (x-r_t).$$

Then $$\phi(r_t) = 0 \quad (t = 1, 2, \ldots, m);$$

and therefore $$(a_s, r_t) = \phi(a_s)\psi(r_t)/(a_s - r_t) \quad (t < m+1),$$

so that the first m columns of the determinant have each a common factor $\psi(r_t)$, which can be divided out, leaving as a result the pure eliminant.

9. By a method similar to that used in § 7, we may also find the eliminant of two quantics $\phi(x)$, $\psi(x)$ of degrees n and m respectively, as a determinant of order $n+m$, which is equivalent to the result obtained by Sylvester's dialytic method. Put

$$f(x) = \prod_{s=1}^{s=n} (x-a_s), \qquad F(x) = \prod_{s=1}^{s=m} (x-b_s).$$

Then $$\frac{\phi(x)}{f(x)(x-b_t)} = \frac{\phi(b_t)}{f(b_t)}\frac{1}{x-b_t} + \sum_{s=1}^{s=n} \frac{\phi(a_s)}{f'(a_s)(a_s-b_t)}\frac{1}{x-a_s},$$

and $$\frac{\psi(x)}{F(x)(x-a_t)} = \frac{\psi(a_t)}{F(a_t)}\frac{1}{x-a_t} + \sum_{s=1}^{s=m} \frac{\psi(b_s)}{F'(b_s)(b_s-a_t)}\frac{1}{x-b_s}.$$

Putting $\phi(x) = 0$ and $\psi(x) = 0$, we have $m+n$ equations linear in the $m+n$ quantities, $1/(x-a_s)$, $1/(x-b_s)$, and the eliminant of $\phi(x)$ and $\psi(x)$

appears as the determinant

$$\begin{vmatrix} \frac{\phi(b_1)}{f(b_1)}, & 0, & \dots, & 0, & \frac{\phi(a_1)}{f'(a_1)(a_1-b_1)}, & \frac{\phi(a_2)}{f(a_2)(a_2-b_1)}, & \dots, & \frac{\phi(a_n)}{f'(a_n)(a_n-b_1)} \\ 0, & \frac{\phi(b_2)}{f(b_2)}, & \dots, & 0, & \frac{\phi(a_1)}{f'(a_1)(a_1-b_2)}, & \frac{\phi(a_2)}{f(a_2)(a_2-b_2)}, & \dots, & \frac{\phi(a_n)}{f'(a_n)(a_n-b_2)} \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\ 0, & 0, & \dots, & \frac{\phi(b_m)}{f(b_m)}, & \frac{\phi(a_1)}{f'(a_1)(a_1-b_m)}, & \frac{\phi(a_2)}{f(a_2)(a_2-b_m)}, & \dots, & \frac{\phi(a_n)}{f'(a_n)(a_n-b_m)} \\ \frac{\psi(b_1)}{F'(b_1)(b_1-a_1)}, & \frac{\psi(b_2)}{F'(b_2)(b_2-a_1)}, & \dots, & \frac{\psi(b_m)}{F'(b_m)(b_m-a_1)}, & \frac{\psi(a_1)}{F(a_1)}, & 0, & \dots, & 0 \\ \frac{\psi(b_1)}{F'(b_1)(b_1-a_2)}, & \frac{\psi(b_2)}{F'(b_2)(b_2-a_2)}, & \dots, & \frac{\psi(b_m)}{F'(b_m)(b_m-a_2)}, & 0, & \frac{\psi(a_2)}{F(a_2)}, & \dots, & 0 \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\ \frac{\psi(b_1)}{F'(b_1)(b_1-a_n)}, & \frac{\psi(b_2)}{F'(b_2)(b_2-a_n)}, & \dots, & \frac{\psi(b_m)}{F'(b_m)(b_m-a_n)}, & 0, & 0, & \dots, & \frac{\psi(a_n)}{F(a_n)} \end{vmatrix}$$

multiplied by the eliminant of $f(x)$ and $F(x)$.

Note on § 5.

It has been pointed out to me by a referee that, when we know that the eliminant is the sum of products of (r, s), each product having a positive sign, the number of terms is at once found by writing 1 for (r, s) throughout the determinant of § 4. In this way we get

$$\begin{vmatrix} n, & -1, & -1, & \dots \\ -1, & n, & -1, & \dots \\ -1, & -1, & n, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} n, & -n-1, & -n-1, & \dots \\ -1, & n+1, & 0, & \dots \\ -1, & 0, & n+1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & \dots \\ -1, & n+1, & 0, & \dots \\ -1, & 0, & n+1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = (n+1)^{n-1}.$$

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